ANNALES SCIENTIFIQUES DE L'É.N.S.

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Annales scientifiques de l'É.N.S. 4e série, tome 33, nº 2 (2000), p. 181-209
<http://www.numdam.org/item?id=ASENS_2000_4_33_2_181_0>

Ann. Scient. Éc. Norm. Sup.,
4^{e} série, t. 33, 2000, p. 181 à 209.

THE LENGTH SPECTRUM OF RIEMANNIAN TWO-STEP NILMANIFOLDS *

RUTH GORNET AND MAURA B. MAST

ABSTRACT. – This paper has three main theorems. First, we express the length spectrum of a Riemannian two-step nilmanifold in terms of metric Lie algebra data. We use the length spectrum to motivate the definition of a new family of two-step nilpotent metric Lie algebras, which we call Heisenberg-like. This leads to our next result, the explicit computation of the length spectrum of all Heisenberg-like manifolds. Using a variety of characterizations of Lie algebras of Heisenberg type, we show that Heisenberg-like Lie algebras are their natural generalization. Finally, as an application in spectral geometry, we show that all known examples of two-step nilmanifolds that have the same Laplace spectrum on functions must also have the same length spectrum. © 2000 Éditions scientifiques et médicales Elsevier SAS

Keywords: Length spectrum; Laplace spectrum; Heisenberg groups; Heisenberg type algebras; Nilpotent Lie algebras

RÉSUMÉ. – Cet article contient trois théorèmes. Tout d'abord, nous exprimons le spectre des longueurs d'une nilvariété Riemannienne de rang deux au moyen de son algèbre de Lie munie d'une métrique. Nous utilisons le spectre des longueurs pour motiver la définition d'une nouvelle famille des algèbres de Lie nilpotentes de rang deux, munies d'une métrique, que nous appelons "Heisenberg-like." Ensuite, nous calculons d'une manière explicite le spectre des longueurs de toutes les nilvariétés "Heisenberg-like." En utilisant plusieurs caractérisations des algèbres de Lie de type de Heisenberg, nous prouvons que les algèbres "Heisenberg-like" sont une généralisation naturelle des algèbres de Lie de type de Heisenberg. Finalement, une application de notre théorie à la géométrie spectrale révèle que tous les exemples connus de nilvariétés de rang deux qui ont le même spectre du Laplacien pour les fonctions, doivent aussi avoir le même spectre des longueurs. © 2000 Éditions scientifiques et médicales Elsevier SAS

Introduction

The *length spectrum* of a Riemannian manifold (M, g) is the collection of lengths of smoothly closed geodesics in M. In much of the literature [17,23,25] a multiplicity is attached to each length in the length spectrum, but here, unless otherwise stated, this issue is completely ignored. All manifolds considered here are closed, i.e., compact and without boundary.

The length spectrum, in addition to being geometrically interesting in its own right, is also relevant to the Laplace spectrum. The *Laplace spectrum* of a closed Riemannian manifold (M, g) is the collection of eigenvalues of the Laplace–Beltrami operator Δ, counted with multiplicity. Two manifolds are *isospectral* if their Laplace spectra coincide. A major open question in spectral

* The first author was supported in part by NSF Grant No. DMS-9409209 and by the Texas Advanced Research Program under Grant No. 03644-002. The second author was supported in part by NSF Grant GER-9450114.

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE. – 0012-9593/00/02/

geometry is whether there can exist examples of isospectral manifolds with different periods in the length spectrum.

Colin de Verdière [9] used the heat kernel to show that generically, the Laplace spectrum determines the length spectrum. Duistermaat and Guillemin [11] have shown that the singularities of the wave trace $e^{-t\sqrt{\Delta}}$, a spectrally determined, tempered distributional operator, are contained in the length spectrum.

Two closed Riemann surfaces are isospectral if and only if they have the same length spectrum, counting multiplicities [26,27,8]; in this case a purely analytic notion, the Laplace spectrum, is equivalent to a geometric notion, the length spectrum. (Here, the multiplicity of a length is the number of free homotopy classes containing a closed geodesic of that length.) In contrast, the Zöll and standard spheres have the same length spectrum (the length spectrum in both cases consists of all integer multiples of 2π) but are not isospectral. Indeed, standard spheres of dimension less than 6 are known to be spectrally determined [3,43]. (See [6] for details on Zöll spheres.)

This paper focuses on Riemannian nilmanifolds. A Riemannian nilmanifold is a closed manifold of the form $(\Gamma\backslash G, g)$, where G is a simply connected nilpotent Lie group, Γ is a cocompact (i.e., $\Gamma\backslash G$ compact) discrete subgroup of G, and g arises from a left invariant metric on G. Nilmanifolds do not satisfy the genericity assumptions of Colin de Verdière and Duistermaat–Guillemin, as closed geodesics of any length come in large-dimensional families; thus we must use other means to compare the length spectrum of isospectral nilmanifolds.

The simplest example of a Riemannian nilmanifold is the flat torus $\Gamma\backslash\mathbb{R}^n$, where Γ is a lattice of rank n. In fact, the first examples of isospectral, nonisometric manifolds were pairs of 16-dimensional isospectral tori, constructed by Milnor [35]. The flat torus case is well understood: The Poisson summation formula shows that two flat tori are isospectral if and only if they have the same length spectrum, counting multiplicities (with multiplicity defined the same as for the Riemann surface case above). Moreover, both the length spectrum and the Laplace spectrum of a flat torus are explicitly computable. See [2] or [4] for details.

Two-step nilpotent Lie groups are the Lie groups that come as close as possible to being abelian without actually being so; from the point of view of both the length spectrum and the Laplace spectrum, this case is more intricate. The best known example here is the Heisenberg Lie group, which has a one-dimensional center. Pesce [39] developed a Poisson-type formula for Heisenberg manifolds and used it to show that continuous families of Heisenberg manifolds with the same length spectrum must be isospectral. In contrast, Gordon [17] has constructed pairs of isospectral Heisenberg manifolds with different multiplicities in the length spectrum (with multiplicity defined the same as for the Riemann surface case above).

Two-step Riemannian nilmanifolds are of particular importance when considering these questions, as they have provided a rich source of examples of isospectral manifolds, hence a natural environment in which to study the length spectrum. Pesce [38] has explicitly computed the Laplace spectrum of a Riemannian two-step nilmanifold. In Section 2, we establish the following result (see Theorems 2.4 and 2.8).

THEOREM. – *The length spectrum of a Riemannian two-step nilmanifold $(\Gamma\backslash N, g)$ can be expressed in terms of metric Lie algebra data.*

Until recently, all known examples of isospectral manifolds were also locally isometric, i.e., shared a common Riemannian cover. The locally isometric examples are known to have the same length spectrum (ignoring multiplicities). See [23,25] for references and more details.

The first examples of closed isospectral manifolds with different local geometry were pairs of isospectral nilmanifolds, constructed by Gordon [18,19]. Further examples using her method as well as a generalization of the method were provided by Gordon and Wilson in [22]. We apply our length spectrum calculations to their examples and show the following (see Theorem 4.11).

THEOREM. – *The examples of isospectral two-step nilmanifolds constructed by Gordon and Wilson in* [19,22] *must have the same length spectrum.*

The only other examples of closed isospectral manifolds that are not locally isometric are large-dimensional continuous families constructed in [20] and simply connected continuous families constructed by Schueth [41]. The calculation of the length spectrum in these new continuous examples appears very difficult at present, but since they are constructed using an adaptation of the general method used by Gordon and Wilson in [19,22], understanding the length spectrum in the nilmanifold examples may lend insight into the other examples.

Two-step nilpotent Lie groups are interesting in and of themselves. Eberlein [12,13] has studied their geometry extensively and we generalize many of his results. As others have done, we study Lie algebras to obtain information about simply connected Lie groups. Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra with inner product $\langle\ ,\ \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{V}$ be the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$. We write $\mathfrak{n} = \mathfrak{V} \oplus \mathfrak{z}$.

The main tool we use to study two-step nilpotent Lie groups and Lie algebras is the j operator, first introduced by Kaplan [28]. Given the decomposition $\mathfrak{n} = \mathfrak{V} \oplus \mathfrak{z}$ for a metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$, define a linear transformation $j : \mathfrak{z} \to \mathrm{so}(\mathfrak{V})$ by $\langle j(Z)X, Y\rangle = \langle [X,Y], Z\rangle$ for all $X, Y \in \mathfrak{V}$ and $Z \in \mathfrak{z}$. The mappings $\{j(Z)\colon\ Z \in \mathfrak{z}\}$ contain the geometry of $(N, \langle\ ,\ \rangle)$ in the sense that the covariant derivative, curvature tensor [45], and Ricci tensor [36], can be formulated entirely using j, $\mathfrak{V}$, and $\mathfrak{z}$. One purpose of this paper is to describe the length spectrum entirely in terms of $\mathfrak{n}$, $\langle\ ,\ \rangle$, and j.

A two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ is said to be of *Heisenberg type* if for every $Z \in \mathfrak{z}$, $j(Z)^2 = -|Z|^2 Id_{|\mathfrak{V}}$. Heisenberg type Lie groups and Lie algebras were originally defined by Kaplan in [28,29]. Since then, the Heisenberg type Lie algebras have played an important role in such areas as spectral geometry, producing the first examples of isospectral manifolds that are not locally isometric [42,18], and harmonic analysis, playing a critical role in the construction of a counterexample [10] to the Lichnerowicz conjecture on harmonic manifolds. See [5] for an introduction to generalized Heisenberg groups and Damek–Ricci spaces.

Eberlein calculated the length spectrum of all nilmanifolds of Heisenberg type. In Section 3, we use our formulation of the length spectrum in terms of metric Lie algebra data to motivate the definition of a new family of two-step nilpotent Lie groups and Lie algebras that we call *Heisenberg-like*, and we prove the following (see Theorem 3.10).

THEOREM. – *The length spectrum of Heisenberg-like nilmanifolds can be explicitly calculated.*

Heisenberg-like Lie groups generalize naturally from those of Heisenberg type in a variety of ways: by the definition, by the formulation of the length spectrum of resulting nilmanifolds, and in terms of the prevalence of periodic geodesics contained in three-dimensional totally geodesic submanifolds. In particular, Lie groups of Heisenberg type are characterized by the fact that periodic geodesics are always contained in three-dimensional totally geodesic submanifolds. We prove that Lie groups that are Heisenberg-like are characterized by the fact that every central period in the length spectrum may be realized by a periodic geodesic that is contained in a three-dimensional totally geodesic submanifold (see Theorem 3.13).

Because of the importance of algebras of Heisenberg type in geometric analysis, Lie groups, and mathematical physics, the authors suggest that the most general setting for many results in these areas is that of Heisenberg-like Lie algebras, rather than those of Heisenberg type.

This paper is organized as follows. In Section 1 we review background material; in particular, we define two-step nilpotent metric Lie groups and Lie algebras. In Section 2 we establish a primary result of this paper, an expression of the length spectrum of a Riemannian two-step nilmanifold $(\Gamma\backslash N, g)$ entirely in terms of the Lie algebra $\mathfrak{n}$ of N, the inner product $\langle\, ,\, \rangle$ on $\mathfrak{n}$, and the inverse image of the subgroup Γ under the Lie group exponential mapping. The formulation of the length spectrum motivates the definition of a new family of Riemannian nilmanifolds, the *Heisenberg-like* nilmanifolds. In Section 3 we explicitly compute the length spectrum of all Heisenberg-like nilmanifolds. In addition, we study the geometry of these nilmanifolds and, using a variety of characterizations of Heisenberg type nilmanifolds, we show that Heisenberg-like nilmanifolds are their natural generalization. As an application of the length spectrum formula from Section 2, we prove in Section 4 that examples of isospectral two-step nilmanifolds constructed by Gordon and Wilson necessarily have the same length spectrum.

The authors would like to thank Carolyn Gordon and Pat Eberlein for many helpful and useful comments, John McCarthy for providing references in operator theory, and the referee for detailed and helpful suggestions and comments.

1. Two-step nilmanifolds and two-step nilpotent metric Lie algebras

Let $\mathfrak{n}$ denote a two-step nilpotent, finite-dimensional, real Lie algebra with nontrivial center $\mathfrak{z}$ and Lie bracket $[\, ,\,]$. Recall that a Lie algebra $\mathfrak{n}$ is *two-step nilpotent* if $\mathfrak{n}$ is nonabelian and $[X,Y] \in \mathfrak{z}$ for all $X, Y \in \mathfrak{n}$; a Lie group is two-step nilpotent if its Lie algebra is. Let N denote the unique, simply connected Lie group with Lie algebra $\mathfrak{n}$.

The Lie group exponential $\exp : \mathfrak{n} \to N$ is a diffeomorphism [40], so N is diffeomorphic to $\mathbb{R}^n$ where $n = \dim \mathfrak{n}$. By the Campbell–Baker–Hausdorff formula [44], we may write the group operation of N in terms of the Lie algebra $\mathfrak{n}$ by

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y]\right), \tag{1.1}$$

where $X, Y \in \mathfrak{n}$. Thus,

$$\begin{aligned} \exp(X)^{-1} &= \exp(-X), \\ \exp(X)\exp(Y)\exp(X)^{-1} &= \exp(Y + [X,Y]). \end{aligned} \tag{1.2}$$

A Riemannian metric g on N is *left invariant* if left translations L_p are isometries for all $p \in N$. Note that a left invariant metric on N determines an inner product on $\mathfrak{n} = T_eN$ and an inner product $\langle\, ,\, \rangle$ on $\mathfrak{n}$ induces a left invariant metric on N. A Lie algebra together with an inner product $(\mathfrak{n}, \langle\, ,\, \rangle)$ is called a *metric Lie algebra*. We will use $\langle\, ,\, \rangle$ to denote both the inner product on $\mathfrak{n}$ and the corresponding left invariant metric on N. We denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ by $\mathfrak{V}$ and write $\mathfrak{n} = \mathfrak{V} \oplus \mathfrak{z}$.

Our objects of study are Riemannian nilmanifolds. Let Γ denote a cocompact (i.e., $\Gamma\backslash N$ is compact), discrete subgroup of N. The quotient manifold $\Gamma\backslash N$ obtained by letting Γ act by left translations on N is a *two-step nilmanifold*, and the left invariant metric $\langle\, ,\, \rangle$ on N descends to a Riemannian metric on $\Gamma\backslash N$, also denoted by $\langle\, ,\, \rangle$.

Example 1.3 (*The Heisenberg Lie group and Lie algebra*). – The Heisenberg group is, up to isomorphism, the only two-step nilpotent Lie group with a one-dimensional center. As such, it is the nilpotent Lie group that is as close as possible to being abelian. The $(2n+1)$-dimensional

Heisenberg group H_n is the set of all real $(n+2)\times(n+2)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ & 1 & 0 & \cdots & 0 & y_1 \\ & & 1 & \ddots & \vdots & y_2 \\ & & & \ddots & 0 & \vdots \\ & 0 & & & 1 & y_n \\ & & & & & 1 \end{pmatrix}$$

for $x_i,\ y_i,\ z \in \mathbb{R},\ i=1,\dots,n$. The group operation is matrix multiplication. The Lie algebra $\mathfrak{h}_n$ of H_n is the $(2n+1)$-dimensional vector space with basis $\{X_1,\dots,X_n,Y_1,\dots,Y_n,Z\}$ and Lie bracket $[X_i,Y_i]=-[Y_i,X_i]=Z$ for $1\leqslant i\leqslant n$, and all other basis brackets zero.

The center $\mathfrak{z}$ of $\mathfrak{h}_n$ equals $\mathfrak{z}=\operatorname{span}_{\mathbb{R}}\{Z\}$. One determines a left invariant metric on H_n by specifying an inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}_n$.

Definition 1.4. – The *length spectrum* of a Riemannian manifold (M,g) is the set of lengths of smoothly closed geodesics.

Note that we are not assuming that the closed geodesics are simply closed, thus if ω is in the length spectrum of (M,g), so is $m\omega$ for all $m\in\mathbb{Z}^+$.

Remark 1.5. – In much of the literature [17,23,25] a multiplicity is attached to each length in the length spectrum. Note that there are several definitions of multiplicity in the literature. For example, one definition of the multiplicity of a length ω is the number of distinct free homotopy classes containing a smoothly closed geodesic of length ω. In this paper, unless otherwise stated, multiplicity is completely ignored and we concern ourselves exclusively with existence.

We study closed geodesics on $(\Gamma\backslash N,\langle\ ,\ \rangle)$ by lifting them to the universal cover $(N,\langle\ ,\ \rangle)$.

Definition 1.6. – Let σ be a unit speed geodesic in $(N,\langle\ ,\ \rangle)$. A non-identity element $\gamma\in N$ *translates σ by an amount* $\omega>0$ if $\gamma\sigma(t)=\sigma(t+\omega)$ for all $t\in\mathbb{R}$. The number ω is called a *period* of γ.

As N is simply connected, the free homotopy classes of $\Gamma\backslash N$ correspond to the conjugacy classes in the fundamental group Γ. Note that $(N,\langle\ ,\ \rangle)\to(\Gamma\backslash N,\langle\ ,\ \rangle)$ is a Riemannian covering. Thus there exists a closed geodesic of length ω in the free homotopy class represented by $\gamma\in\Gamma$ if and only if there exists a unit speed geodesic $\sigma(s)$ on $(N,\langle\ ,\ \rangle)$ such that γ translates σ with period ω. The geodesic σ then projects to a smoothly closed geodesic of length ω on $(\Gamma\backslash N,\langle\ ,\ \rangle)$ in the free homotopy class represented by γ.

We use the following information about two-step nilpotent metric Lie algebras to study periodic geodesics on two-step nilpotent Lie groups equipped with a left invariant metric. See Kaplan [28] for further information.

Definition 1.7. – Let $(\mathfrak{n},\langle\ ,\ \rangle)$ be a two-step nilpotent metric Lie algebra, $\mathfrak{n}=\mathfrak{V}\oplus\mathfrak{z}$. Define a linear transformation $j:\mathfrak{z}\to so(\mathfrak{V})$ by $j(Z)X=(\operatorname{ad}X)^*Z$ for $Z\in\mathfrak{z}$ and $X\in\mathfrak{V}$. Equivalently, for each $Z\in\mathfrak{z}$, $j(Z):\mathfrak{V}\to\mathfrak{V}$ is the skew-symmetric linear transformation defined by

$$\langle j(Z)X,Y\rangle=\langle Z,[X,Y]\rangle, \tag{1.8}$$

for all $X,\ Y$ in $\mathfrak{V}$. Here $\operatorname{ad}X(Y)=[X,Y]$ for all $X,Y\in\mathfrak{n}$, and $(\operatorname{ad}X)^*$ denotes the (metric) adjoint of $\operatorname{ad}X$.

By skew-symmetry, $j(Z)$ has $\dim_{\mathbb{R}}(\mathfrak{V})$ purely complex eigenvalues counting (algebraic) multiplicities and the nonzero eigenvalues occur in complex conjugate pairs; the eigenvalues of $j(Z)^2$ are then real and nonpositive. If $j(Z)$ is nonsingular for some Z in $\mathfrak{z}-\{0\}$, then the (real) dimension of $\mathfrak{V}$ is even.

Thus each two-step nilpotent metric Lie algebra carries with it the j operator. On the other hand, given inner product spaces $\mathfrak{V}$ and $\mathfrak{z}$ and a linear transformation $j:\mathfrak{z}\to so(\mathfrak{V})$, one can define a two-step nilpotent metric Lie algebra $(\mathfrak{V}\oplus\mathfrak{z},\langle\ ,\ \rangle)$ by requiring that $\mathfrak{z}$ be central, that $\oplus$ be orthogonal direct sum, and by defining the Lie bracket $[\ ,\ ]$ via Eq. (1.8). All two-step nilpotent metric Lie algebras are determined this way.

Definition 1.9. – A two-step nilpotent metric Lie algebra $(\mathfrak{n},\langle\ ,\ \rangle)$ is of *Heisenberg type* if $j(Z)^2=-|Z|^2 Id_{|\mathfrak{V}}$ for all $Z\in\mathfrak{z}$.

Note that if $\mathfrak{n}$ is of Heisenberg type, the distinct eigenvalues of $j(Z)$ are $\pm|Z|i$.

Example 1.10. – Let $\mathfrak{n}=\mathfrak{V}\oplus\mathfrak{z}$ be a 6-dimensional real vector space with inner product such that $\{X_1,X_2,X_3,X_4\}$ is an orthonormal basis for $\mathfrak{V}$ and $\{Z_1,Z_2\}$ is an orthonormal basis for $\mathfrak{z}$. Define a bracket operation on $\mathfrak{n}$ by

$$[X_1,X_2]=-[X_2,X_1]=Z_1,\qquad [X_1,X_3]=-[X_3,X_1]=Z_2,$$
$$[X_2,X_4]=-[X_4,X_2]=-Z_2,\qquad [X_3,X_4]=-[X_4,X_3]=Z_1,$$

with all other brackets of basis elements equal to zero. With this bracket, $(\mathfrak{n},\langle\ ,\ \rangle)$ is a two-step nilpotent metric Lie algebra with center $\mathfrak{z}$. Using the relationship $\langle j(Z)X,Y\rangle=\langle[X,Y],Z\rangle$ and the basis $\{X_1,X_2,X_3,X_4,Z_1,Z_2\}$, we obtain the following matrix representations for $j(Z_1)$ and $j(Z_2)$:

$$j(Z_1)=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\qquad j(Z_2)=\begin{pmatrix}0&0&-1&0\\0&0&0&1\\1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

It follows that $Z=\alpha Z_1+\beta Z_2$ has distinct eigenvalues $\{\pm i\sqrt{\alpha^2+\beta^2}\}$ and $\mathfrak{n}$ is of Heisenberg type.

Remarks 1.11. –

(a) It is known [28] that there exist Lie algebras of Heisenberg type with centers of arbitrary dimension.

(b) The $(2n+1)$-dimensional Heisenberg Lie algebra with the natural inner product (that is, the inner product making the basis an orthonormal basis) is of Heisenberg type. In general, however, a Heisenberg metric Lie algebra $(\mathfrak{h}_n,\langle\ ,\ \rangle)$ is not of Heisenberg type. (See [21,17] for more details.)

Definitions 1.12. –

(1) A two-step nilpotent Lie algebra $\mathfrak{n}$ is *strictly nonsingular* if there exists an inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$ such that $j(Z)$ is nonsingular for all $Z\in\mathfrak{z}-\{0\}$.

(2) A two-step nilpotent metric Lie algebra $(\mathfrak{n},\langle\ ,\ \rangle)$ is *almost nonsingular* if $j(Z)$ is nonsingular for all $Z\in\mathcal{D}$, where $\mathcal{D}$ is a dense open subset of $\mathfrak{z}$.

(3) A two-step nilpotent metric Lie algebra $(\mathfrak{n},\langle\ ,\ \rangle)$ is *strictly singular* if $j(Z)$ is singular for all $Z\in\mathfrak{z}$.

Remark 1.13. – Eberlein showed that strict nonsingularity, which he called nonsingularity, may be defined as follows: A Lie algebra $\mathfrak{n}$ is strictly nonsingular if for every $Z\in\mathfrak{z}-\{0\}$ and for every $X\in\mathfrak{n}-\mathfrak{z}$ there exists $Y\in\mathfrak{n}$ such that $[X,Y]=Z$. (See [12, Lemma 1.8].) Thus strict nonsingularity is a purely algebraic notion; i.e., it is independent of the choice of inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$. Likewise, by comparing the structure constants with respect to different bases of $\mathfrak{n}$, one may show that strict singularity and almost nonsingularity are also independent of the choice of inner product; i.e., strict singularity and almost nonsingularity are algebraic notions.

Note that all Lie algebras of Heisenberg type are strictly nonsingular. The almost nonsingular and strictly singular conditions are illustrated in the following examples.

Example 1.14. – Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a 6-dimensional Lie algebra such that $\mathfrak{V}$ has orthonormal basis $\{X_1, X_2, X_3, X_4\}$, $\mathfrak{z}$ has orthonormal basis $\{Z_1, Z_2\}$, and $j(Z_1)$ and $j(Z_2)$ have the following matrix representations:

$$j(Z_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad j(Z_2) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $Z = \alpha Z_1 + \beta Z_2$, the eigenvalues of $j(Z)$ are $\{\pm i\alpha, \pm i\beta\}$. It follows that $\mathfrak{n}$ is almost nonsingular.

Example 1.15 ([12]). – Let $\mathfrak{n} = \mathfrak{V} \oplus \mathfrak{z}$ be a 5-dimensional Lie algebra with inner product such that $\{X_1, X_2, X_3\}$ is an orthonormal basis for $\mathfrak{V}$ and $\{Z_1, Z_2\}$ is an orthonormal basis for $\mathfrak{z}$. Set

$$j(Z_1) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad j(Z_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and extend the definition of j to $\mathfrak{z}$ by linearity. For $Z = \alpha Z_1 + \beta Z_2$, $j(Z)$ has eigenvalues $\{0, \pm i\sqrt{\alpha^2 + \beta^2}\}$, hence $\mathfrak{n}$ is strictly singular.

LEMMA 1.16. – *A two-step nilpotent metric Lie algebra* $(\mathfrak{n}, \langle\ ,\ \rangle)$ *is exactly one of strictly nonsingular, almost nonsingular, or strictly singular.*

Proof. – The result follows from properties of the map $\det \circ j : \mathfrak{z} \to \mathbb{R}$. See [33] for further details. □

Definition 1.17. – Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent metric Lie algebra and let $Z \in \mathfrak{z}$.

(1) Let $-\theta_1(Z)^2, -\theta_2(Z)^2, \ldots, -\theta_n(Z)^2$ denote the $n = \dim(\mathfrak{V})$ eigenvalues of $j(Z)^2$, with the assumption that $0 \leqslant \theta_1(Z) \leqslant \theta_2(Z) \leqslant \cdots \leqslant \theta_n(Z)$.

(2) Let $\mu(Z)$ denote the number of distinct eigenvalues of $j(Z)^2$. For ease of notation, we write μ rather than $\mu(Z)$ when Z is understood.

(3) Let $-\vartheta_1(Z)^2, -\vartheta_2(Z)^2, \ldots, -\vartheta_\mu(Z)^2$ denote the μ *distinct* eigenvalues of $j(Z)^2$, with the assumption that $0 \leqslant \vartheta_1(Z) < \vartheta_2(Z) < \cdots < \vartheta_\mu(Z)$. The distinct eigenvalues of $j(Z)$ are then $\{\pm\vartheta_1(Z)i, \ldots, \pm\vartheta_\mu(Z)i\}$.

(4) Let $W_m(Z)$ denote the invariant subspace of $j(Z)$ corresponding to $\vartheta_m(Z)$, $m = 1, \ldots, \mu$. Then $j(Z)^2_{|W_m(Z)} = -\vartheta_m(Z)^2 Id_{|W_m(Z)}$; i.e., $W_m(Z)$ is the eigenspace of $j(Z)^2$ with eigenvalue $-\vartheta_m(Z)^2$. By the skew-symmetry of $j(Z)$, $\mathfrak{V}$ is the orthogonal direct sum of the invariant subspaces $W_m(Z)$. Note that if $\vartheta_m(Z) \neq 0$, then $j(Z)^{-1}$ is well defined on $W_m(Z)$.

It is straightforward to construct two-step nilpotent metric Lie algebras for which μ is nonconstant; the Lie algebra in (1.14) is one such example. See Section 3 for examples of two-step nilpotent metric Lie algebras such that μ is constant on $\mathfrak{z} - \{0\}$.

Because of the possibility that the number of distinct eigenvalues of j can change, we need the following.

Definition 1.18. – For a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$, define $\mathcal{U} = \{Z \in \mathfrak{z}$: there exists an open neighborhood $\mathcal{O}$ of Z such that μ is constant on $\mathcal{O}\}$. We call $\mathcal{U}$ the *simple subdomain* of $\mathfrak{z}$.

PROPOSITION 1.19. – *Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent metric Lie algebra. The following hold for* $\mathfrak{n}$.

(1) *The simple subdomain $\mathcal{U}$ is open and dense in $\mathfrak{z}$.*
(2) *The function $\theta_m : \mathfrak{z} - \{0\} \to \mathbb{R}$ is continuous on $\mathfrak{z} - \{0\}$, $m = 1, \ldots, n$.*
(3) *The function μ is constant on $\mathcal{U}$.*
(4) *If Z is a limit point of $\mathcal{U}$ then $\mu(Z) \leqslant \mu(\mathcal{U})$.*
(5) *The function $\vartheta_m : \mathcal{U} \to \mathbb{R}$ is smooth on $\mathcal{U}$, $m = 1, \ldots, \mu(\mathcal{U})$.*
(6) *The function ϑ_μ is continuous on $\mathfrak{z} - \{0\}$, where $\mu = \mu(Z)$.*

Proof. –

(1) See Theorem 4.1, claim (1) in [34]. Note that the nonsingularity condition used there is not needed for this proof.
(2) This follows from Theorem 5.2 in Chapter 2 of [31].
(3) Let $Z_1, Z_2 \in \mathcal{U}$. Define $A(t) = j((1-t)Z_1 + tZ_2)$ for all $t \in \mathbb{R}$. Then $A(t)^2$ is a real analytic family of self adjoint operators on the n-dimensional space $\mathfrak{V}$. By [16, Theorem S6.3], there exist eigenvalue curves $\phi_1(t), \ldots, \phi_n(t)$ for $A(t)^2$ that are real analytic in t. Note that we cannot specify an order on the ϕ_k, or analyticity may be lost. Now at $t = 0$ the ϕ_k's group into $\mu(Z_1)$ sets $S_1, \ldots, S_\mu$; i.e., ϕ_k is in S_m if and only if $\phi_k(0) = -\vartheta_m(Z_1)^2$. Since $Z_1 \in \mathcal{U}$, there exists an open neighborhood of Z_1 such that $\mu(Z_1)$ does not change. So there exists an open neighborhood $(-\varepsilon, \varepsilon)$ of 0 in $\mathbb{R}$ such that if $\phi_k \in S_m$ and $\phi_{k'} \in S_m$, then $\phi_k(t) = \phi_{k'}(t)$ for all $t \in (-\varepsilon, \varepsilon)$. By real analyticity, $\phi_k(t) = \phi_{k'}(t)$ for all $t \in \mathbb{R}$, in particular, $\phi_k(1) = \phi_{k'}(1)$. Thus there can be no more than $\mu(Z_1)$ distinct eigenvalues of $A(t)^2$ at $t = 1$, i.e., $\mu(Z_2) \leqslant \mu(Z_1)$. Reversing the roles of Z_1 and Z_2, we conclude that $\mu(Z_1) = \mu(Z_2)$.
(4) If $\lim_k Z_k = Z^*$ for some $Z_k, Z^* \in \mathfrak{z} - \{0\}$, then by (2) the eigenvalues of $j(Z^*)^2$ are contained in the limit of the eigenvalues of $j(Z_k)^2$.
(5) See pp. 568–569 of [31].
(6) This follows from (2) and the fact that $\vartheta_\mu(Z) = \max\limits_m\{\theta_m(Z)\}$ for all Z. □

PROPOSITION 1.20. – *Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent metric Lie algebra and let $Z \in \mathcal{U}$. Fix $m \in \{1, \ldots, \mu(Z)\}$ and let $X_m \in W_m(Z)$ be a unit vector. Then $[X_m, j(Z)X_m] = \vartheta_m(Z)\nabla\vartheta_m(Z)$, where ∇ is the gradient.*

Proof. – The result follows from Lemma 3.2 of [33]. □

2. The length spectrum of a Riemannian two-step nilmanifold

In this section, we calculate the length spectrum of an arbitrary two-step nilmanifold by calculating the periods of elements in an arbitrary two-step nilpotent metric Lie group $(N, \langle\ ,\ \rangle)$. (See Definition 1.6 ff.)

Notation 2.1. – Let $X_0 + Z_0$ be a vector in $\mathfrak{n}$ with $Z_0 \in \mathfrak{z}$ and $X_0 \in \mathfrak{V}$.

(1) Define X_1 and X_2 by $X_0 = X_1 + X_2$ such that $X_1 \in \ker j(Z_0)$, and $X_2 \perp \ker j(Z_0)$.

(2) The index m will range from 1 to $\mu(Z_0)$ if $j(Z_0)$ is nonsingular, and from 2 to $\mu(Z_0)$ if $j(Z_0)$ is singular. Recall that $\mu(Z_0)$ denotes the number of distinct eigenvalues of $j(Z_0)^2$.
(3) Let ξ_m denote the component of X_2 in $W_m(Z_0)$, for each m. We write $X_2 = \sum_m \xi_m$. Note that if $\ker j(Z_0)$ is not empty, then ξ_1 is not defined.

Definitions 2.2. –
(1) For V in $\mathfrak{V}$, define $P_V : \mathfrak{z} \to [V,\mathfrak{n}]$ as orthogonal projection onto $[V,\mathfrak{n}]$. Define $P_V^{\perp} : \mathfrak{z} \to [V,\mathfrak{n}]^{\perp}$ as projection onto $[V,\mathfrak{n}]^{\perp}$, the orthogonal complement of $[V,\mathfrak{n}]$ in $\mathfrak{z}$.
(2) For $V \in \mathfrak{V}$ and $Z \in \mathfrak{z}$, define $Z_V = P_V(Z)$ and $Z_V^{\perp} = P_V^{\perp}(Z)$. We write $Z = Z_V + Z_V^{\perp}$.
(3) With notation as in (2.1), define $K : \mathfrak{n} \to \mathfrak{z}$ by

$$K(X_0 + Z_0) = Z_0 + \frac{1}{2}\sum_m \left[j(Z_0)^{-1}\xi_m, \xi_m\right].$$

(4) Set $K_V = P_V \circ K$ and $K_V^{\perp} = P_V^{\perp} \circ K$.

By definition, $\xi_m \notin \ker j(Z_0)$, thus $j(Z_0)^{-1}\xi_m$ is defined. Furthermore,

$$\left\langle Z_0, K(X_2 + Z_0)\right\rangle = |Z_0|^2 + \frac{1}{2}\sum_m \left\langle Z_0, \left[j(Z_0)^{-1}\xi_m, \xi_m\right]\right\rangle = |Z_0|^2 + \frac{1}{2}|X_2|^2. \tag{2.3}$$

Thus $K(X_2 + Z_0) = 0$ if and only if $X_2 + Z_0 = 0$, and $K(X_2 + Z_0) = Z_0$ if and only if $X_2 = 0$. Note also that if $Z_0 \perp [V,\mathfrak{n}]$, $Z_0 \neq 0$, then $K_V^{\perp}(X_2 + Z_0) \neq 0$.

THEOREM 2.4. – *Let N be a simply connected two-step nilpotent Lie group with left invariant metric $\langle\ ,\ \rangle$. Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be the associated metric Lie algebra. With notation as above, let $\gamma = \exp(V + Z_V + Z_V^{\perp})$ be an element of N. The periods of γ are precisely*

$$\left\{\sqrt{|V|^2 + \frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(X_2 + Z_0)|^2}}:\ X_2 + Z_0 \textit{ satisfy } \text{(i)–(iv)} \textit{ below}\right\}, \tag{†}$$

where

$$\frac{|Z_V^{\perp}|}{|K_V^{\perp}(X_2 + Z_0)|} = 0 \quad \textit{if } Z_V^{\perp} = 0.$$

Given $X_2 \in \mathfrak{V}$ and $Z_0 \in \mathfrak{z}$, the conditions referred to in (†) *are the following*:
(i) $|X_2 + Z_0| = 1$ *or* $X_2 + Z_0 = 0$.
(ii) $V \in \ker j(Z_0)$ *and* $X_2 \perp \ker j(Z_0)$.
(iii) $Z_V^{\perp} \in \operatorname{span}_{\mathbb{R}^+}\{K_V^{\perp}(X_2 + Z_0)\}$.
(iv) *for all m such that $\xi_m \neq 0$,*

$$\frac{|Z_V^{\perp}|\vartheta_m(Z_0)}{2\pi|K_V^{\perp}(X_2 + Z_0)|} \in \mathbb{Z}.$$

Let σ be a geodesic in N such that $\sigma(0) = e$ and $\dot\sigma(0) = X_0 + Z_0$, where $X_0 \in \mathfrak{V}$ and $Z_0 \in \mathfrak{z}$. We write $\sigma(s) = \exp(\mathbf{X}(s) + \mathbf{Z}(s))$, $\mathbf{X}(s) \in \mathfrak{V}$, $\mathbf{Z}(s) \in \mathfrak{z}$; note that $\dot{\mathbf{X}}(0) = X_0$ and $\dot{\mathbf{Z}}(0) = Z_0$, where $\dot{\mathbf{X}}(s)$ and $\dot{\mathbf{Z}}(s)$ denote the derivatives of $\mathbf{X}$ and $\mathbf{Z}$, respectively, with respect to s. Geodesics in two-step nilpotent Lie groups were studied extensively by Kaplan [29,30] and Eberlein [12]. See Sections 3 and 4 of [12] for proofs of the following properties of geodesics, which are used in the sequel.

THEOREM 2.5 ([12]). – *Let N be a simply connected two-step nilpotent Lie group with left invariant metric $\langle\ ,\ \rangle$ and metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$. Let $\phi = \exp(V + Z)$ be an element of N with $V \in \mathfrak{V}$, $Z \in \mathfrak{z}$. Let $X_0 + Z_0$ be a unit vector in $\mathfrak{n}$ with $X_0 \in \mathfrak{V}$, $Z_0 \in \mathfrak{z}$. We write $X_0 = X_1 + X_2$, $X_2 = \sum_m \xi_m$ as in (2.1). Given a unit speed geodesic σ with $\sigma(0) = e$ and $\dot\sigma(0) = X_0 + Z_0$, the following statements hold.*

(1) $\mathbf{X}(s) = sX_1 + \sum_m \mathbf{X}_m(s)$, *where* $\mathbf{X}_m(s) \in W_m(Z_0)$ *and* $\mathbf{X}_m(s) \perp \ker j(Z_0)$ *for all s and for all m.*

(2)

$$\mathbf{X}_m(s) = \big(\cos\big(s\vartheta_m(Z_0)\big) - 1\big) j(Z_0)^{-1}\xi_m + \frac{1}{\vartheta_m(Z_0)} \sin\big(s\vartheta_m(Z_0)\big)\xi_m.$$

(3) *If there exists $\omega > 0$ such that $\phi\sigma(s) = \sigma(s + \omega)$ for all s, then $\mathbf{X}(\omega) = V = \omega X_1$ and $\mathbf{X}_m(\omega) = 0$ for all m. Thus, $\xi_m \neq 0$ implies $\omega\vartheta_m(Z_0) \in 2\pi\mathbb{Z}^+$. Furthermore,*

$$\mathbf{Z}(\omega) = Z = \omega K(X_2 + Z_0) + \big[V, j(Z_0)^{-1}X_2\big].$$

(4) *Suppose $\phi = \sigma(\omega)$. Then $\phi\sigma(s) = \sigma(s + \omega)$ for all s if and only if $Z_0 \perp [V, \mathfrak{n}]$.*

Proof of Theorem 2.4. – We first show that all periods of γ are contained in (†).

Let $\alpha(s)$ be a unit speed geodesic in $(N, \langle\ ,\ \rangle)$ such that $\alpha(0) = p$ and γ translates α with period $\omega > 0$. Then $\sigma(s) = L_{p^{-1}}(\alpha(s))$ is also a unit speed geodesic, since left translations are isometries, and $\sigma(0) = e$. Set $\phi = p^{-1}\gamma p$. Then ϕ translates σ with period ω, since

$$\phi\sigma(s) = p^{-1}\gamma\alpha(s) = p^{-1}\alpha(s + \omega) = \sigma(s + \omega).$$

Using (1.2),

$$\phi = p^{-1}\gamma p = \exp\big(\log\gamma - [\log p, \log\gamma]\big) = \exp\big(V + Z_V + Z_V^\perp - [\log p, V]\big).$$

Since $\gamma = \exp(V + Z_V + Z_V^\perp)$, it follows that $\phi = \exp(V + \widetilde{Z}_V + Z_V^\perp)$, where $\widetilde{Z}_V = Z_V - [\log p, V] \in [V, \mathfrak{n}]$.

Let $X_0 \in \mathfrak{V}$ and $Z_0 \in \mathfrak{z}$ be chosen such that $\dot\sigma(0) = X_0 + Z_0$. Write $\sigma(s) = \exp(\mathbf{X}(s) + \mathbf{Z}(s))$, where $\mathbf{X}(s) \in \mathfrak{V}$, $\mathbf{Z}(s) \in \mathfrak{z}$. Let $X_0 = X_1 + X_2$ and $X_2 = \sum_m \xi_m$, as in (2.1).

If $X_2 + Z_0 = 0$, then $X_2 = 0$ (and thus $\xi_m = 0$ for all m). Thus (i), (ii), and (iv) are trivially satisfied. By (2.5.3),

$$\widetilde{Z}_V + Z_V^\perp = \omega K(X_2 + Z_0) + \big[V, j(Z_0)^{-1}X_2\big] = 0.$$

It follows that $Z_V^\perp = 0$ and (iii) is satisfied. Finally, by (2.5.3), $V = \omega X_1$. Since σ is unit speed, $\omega = |V|$, which is contained in (†) in the case that $Z_V^\perp = 0$.

If $X_2 + Z_0 \neq 0$, define

$$\overline{X}_2 + \overline{Z}_0 = \frac{X_2 + Z_0}{|X_2 + Z_0|}.$$

We show that $\overline{X}_2 + \overline{Z}_0$ satisfies conditions (i)–(iv).

Clearly $|\overline{X}_2 + \overline{Z}_0| = 1$, and (i) follows. To prove the remaining statements we first note that

$$K_V^\perp(cX_2 + cZ_0) = cK_V^\perp(X_2 + Z_0)$$

for all real c. This follows from the fact that $j(cZ_0)^{-1} = \frac{1}{c} j(Z_0)^{-1}$ for all nonzero c.

Recall that $X_1 \in \ker j(Z_0) = \ker j(\overline{Z}_0)$. By (2.5.3), $\omega X_1 = V$. This and (2.5.1) imply condition (ii) is satisfied. Using (2.5.3) again, we obtain

$$Z_V^{\perp} = \omega K_V^{\perp}(X_2 + Z_0) = \omega |X_2 + Z_0| K_V^{\perp}(\overline{X}_2 + \overline{Z}_0),$$

hence condition (iii) holds.

Now $|Z_V^{\perp}| = \omega |X_2 + Z_0||K_V^{\perp}(\overline{X}_2 + \overline{Z}_0)|$. By (2.5.3), for all m such that $\xi_m \neq 0$ there exists $k_m \in \mathbb{Z}^+$ such that

$$k_m = \frac{\omega \vartheta_m(Z_0)}{2\pi} = \frac{\omega |X_2 + Z_0| \vartheta_m(\overline{Z}_0)}{2\pi} = \frac{|Z_V^{\perp}| \vartheta_m(\overline{Z}_0)}{2\pi |K_V^{\perp}(\overline{X}_2 + \overline{Z}_0)|}.$$

This implies condition (iv). Finally, by (2.5.3), $|V| = \omega |X_1|$, hence

$$|V|^2 + \frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(\overline{X}_2 + \overline{Z}_0)|^2} = \omega^2 \big(|X_1|^2 + |X_2 + Z_0|^2\big) = \omega^2,$$

and ω is an element of (†).

We now show that all the elements of (†) must arise as periods of γ.

Let X_2 and Z_0 satisfy (i) through (iv) of the hypothesis and let

$$\omega = \sqrt{|V|^2 + \frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(X_2 + Z_0)|^2}}.$$

Define $\overline{X}_1 = V/\omega$. Define $\overline{X}_2 + \overline{Z}_0 = (1 - |\overline{X}_1|^2)^{1/2}(X_2 + Z_0)$, so by (i), $\overline{X}_1 + \overline{X}_2 + \overline{Z}_0$ is a unit vector.

Let $\sigma(s) = \exp(\mathbf{X}(s) + \mathbf{Z}(s))$ be the unique geodesic through $\sigma(0) = e$ with initial velocity $\overline{X}_1 + \overline{X}_2 + \overline{Z}_0$. Recall that since left translations are isometries, $L_p(\sigma(s))$ is also a geodesic. To complete the proof, we show that there exists an element $p = \exp(A)$ such that for all s, $p\sigma(s + \omega) = \gamma p \sigma(s)$.

By condition (iv),

$$\begin{aligned}\omega^2 \vartheta_m(\overline{Z}_0)^2 &= \omega^2 \big(1 - |\overline{X}_1|^2\big) \vartheta_m(Z_0)^2 = \big(\omega^2 - |V|^2\big) \vartheta_m(Z_0)^2 \\ &= \frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(X_2 + Z_0)|^2} \vartheta_m(Z_0)^2 = (2\pi k_m)^2\end{aligned}$$

for some $k_m \in \mathbb{Z}^+$. By condition (ii) and parts (2.5.1) and (2.5.2) of Theorem 2.5, $\mathbf{X}(\omega) = V \in \ker j(\overline{Z}_0)$, which implies $Z_0 \perp [V, \mathfrak{n}]$. Let $\phi = \sigma(\omega)$. By (2.5.4), $\phi\sigma(s) = \sigma(s + \omega)$ for all s. It remains to show that there exists p such that $\phi = p^{-1}\gamma p$.

Now

$$\begin{aligned}\omega^2 \big|K_V^{\perp}(\overline{X}_2 + \overline{Z}_0)\big|^2 &= \omega^2 \big(1 - |\overline{X}_1|^2\big) \big|K_V^{\perp}(X_2 + Z_0)\big|^2 \\ &= \big(\omega^2 - |V|^2\big) \big|K_V^{\perp}(X_2 + Z_0)\big|^2 = |Z_V^{\perp}|^2.\end{aligned}$$

So $\omega K_V^{\perp}(\overline{X}_2 + \overline{Z}_0) = Z_V^{\perp}$ by (iii). By (2.5.3),

$$\begin{aligned}\mathbf{Z}(\omega) &= \omega \big(K_V^{\perp}(\overline{X}_2 + \overline{Z}_0) + K_V(\overline{X}_2 + \overline{Z}_0)\big) + \big[V, j(\overline{Z}_0)^{-1}\overline{X}_2\big] \\ &= Z_V^{\perp} + \big[V, j(\overline{Z}_0)^{-1}\overline{X}_2\big] + \omega K_V(\overline{X}_2 + \overline{Z}_0).\end{aligned}$$

Now $[V, j(\overline{Z}_0)^{-1}\overline{X}_2] + \omega K_V(\overline{X}_2 + \overline{Z}_0) - Z_V = [V, A]$ for some $A \in \mathfrak{V}$. Let $p = \exp(A)$. Then by the equations in (1.2),

$$\begin{aligned} p^{-1}\gamma p\sigma(0) &= \exp\big(V + Z_V + Z_V^\perp + [-A, V]\big) \\ &= \exp\big(V + Z_V^\perp + \big[V, j(\overline{Z}_0)^{-1}\overline{X}_2\big] + \omega K_V(\overline{X}_2 + \overline{Z}_0)\big) \\ &= \exp\big(\mathbf{X}(\omega) + \mathbf{Z}(\omega)\big) = \sigma(\omega) = \phi, \end{aligned}$$

as desired. □

Remarks 2.6. –

(a) The value $\omega = |V + Z_V^\perp|$ is always a period of $\gamma = \exp(V + Z)$. If $Z_V^\perp = 0$, this is obvious. If $Z_V^\perp \neq 0$, set $Z_0 = Z_V^\perp/|Z_V^\perp|$ and $X_2 = 0$, then $|K_V^\perp(X_2 + Z_0)| = |Z_0| = 1$ and conditions (i)–(iv) of Theorem 2.4 are clearly satisfied.

(b) Theorem 2.4 explains a phenomenon previously discovered by Eberlein. In [12], Eberlein studied the uniqueness of periods in certain free homotopy classes. In particular, he noted that all but finitely many central elements must have more than one period. In the strictly nonsingular case, noncentral elements always have a unique period. In contrast, he constructed a strictly singular example (Example 1.15 here) that exhibited noncentral elements with nonunique periods. Theorem 2.4 shows that for noncentral elements, a necessary condition for nonunique periods to occur is $|Z_V^\perp| \neq 0$. See also Remark 3.12(b).

Notation 2.7. – Given $X_0 + Z_0$ as in (2.1) and $\gamma = \exp(V + Z)$ with $V \in \mathfrak{V}$ and $Z \in \mathfrak{z}$, let β denote the angle between Z_0 and $K_V^\perp(X_2 + Z_0)$.

The following theorem is a restatement of Theorem 2.4 that incorporates the geometric data $\cos(\beta)$ to describe the periods. Clearly, if $K_V^\perp(X_2 + Z_0) \in \operatorname{span}_{\mathbb{R}^+}\{Z_0\}$ then $\cos(\beta) = 1$. In Section 3, we study the case $\cos(\beta) = 1$ for all $Z_0 \in \mathfrak{z}$.

THEOREM 2.8. – *Let N be a simply connected two-step nilpotent Lie group with left invariant metric $\langle\ ,\ \rangle$ and associated metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$. With notation as above, let $\gamma = \exp(V + Z_V + Z_V^\perp)$ be an element of N. The periods of γ are precisely*

$$(\ddagger)\quad \left\{|V + Z_V^\perp|, \sqrt{|V|^2 + \frac{4\pi k_m(\vartheta_m(\overline{Z}_0)\cos\beta|Z_V^\perp| - \pi k_m)}{\vartheta_m(\overline{Z}_0)^2}}:\ X_2 + Z_0 \text{ satisfy (i)–(iv) below}\right\},$$

where $\overline{Z}_0 = Z_0/|Z_0|$. Given $X_2 \in \mathfrak{V}$ and $Z_0 \in \mathfrak{z}$, the conditions referred to in ($\ddagger$) are the following:

(i) $|X_2 + Z_0| = 1$,

(ii) $V \in \ker j(Z_0)$ *and* $X_2 \perp \ker j(Z_0)$,

(iii) $Z_V^\perp \in \operatorname{span}_{\mathbb{R}^+}\{K_V^\perp(X_2 + Z_0)\}$,

(iv) *for all m such that $\xi_m \neq 0$, there exists $k_m \in \mathbb{Z}^+$ such that*

$$k_m = \frac{|Z_V^\perp|\vartheta_m(Z_0)}{2\pi|K_V^\perp(X_2 + Z_0)|}.$$

Remark 2.9. – For each $X_2 + Z_0$ satisfying (i)–(iv) above there is only one corresponding period; note that by condition (iv), $\vartheta_m(Z_0)/k_m$ has the same value for all m such that $\xi_m \neq 0$.

Proof of Theorem 2.8. – If $X_2 + Z_0$ satisfy conditions (i)–(iv) and $X_2 = 0$, then $\omega = |V + Z_V^\perp|$, by Remark 2.6(a). Using Theorem 2.4, it suffices to show that if $X_2 + Z_0$ and k_m satisfy (i)–(iv) and $X_2 \neq 0$, then

$$\frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(X_2+Z_0)|^2} = \frac{4\pi k_m(\vartheta_m(\overline{Z}_0)\cos\beta|Z_V^{\perp}| - \pi k_m)}{\vartheta_m(\overline{Z}_0)^2}.$$

Since $V \in \ker j(Z_0)$, $\langle Z_0, K(X_2+Z_0)\rangle = \langle Z_0, K_V^{\perp}(X_2+Z_0)\rangle$. Then

$$\langle Z_0, K(X_2+Z_0)\rangle = |K_V^{\perp}(X_2+Z_0)||Z_0|\cos\beta = |Z_0|^2 + \frac{1}{2}|X_2|^2 = \frac{1+|Z_0|^2}{2}.$$

The second equality follows from (2.3).

Hence

$$|K_V^{\perp}(X_2+Z_0)| = \frac{1+|Z_0|^2}{2|Z_0|\cos\beta}. \tag{2.10}$$

Since $|Z_0| \leqslant 1$, we obtain

$$|Z_0| = \cos\beta|K_V^{\perp}(X_2+Z_0)| - \sqrt{\cos^2\beta|K_V^{\perp}(X_2+Z_0)|^2 - 1}.$$

By condition (iv), we know that there exists $k_m \in \mathbb{Z}^+$ such that

$$2\pi k_m|K_V^{\perp}(X_2+Z_0)| = \vartheta_m(Z_0)|Z_V^{\perp}|,$$

so

$$|Z_0| = \frac{2\pi k_m|K_V^{\perp}(X_2+Z_0)|}{|Z_V^{\perp}|\vartheta_m(\overline{Z}_0)}.$$

Setting the two values of $|Z_0|$ equal and solving for $|K_V^{\perp}(X_2+Z_0)|$, we obtain

$$|K_V^{\perp}(X_2+Z_0)| = \frac{\vartheta_m(\overline{Z}_0)|Z_V^{\perp}|}{\sqrt{4\pi k_m(\vartheta_m(\overline{Z}_0)|Z_V^{\perp}|\cos\beta - \pi k_m)}},$$

as desired. □

3. Heisenberg-like two-step nilmanifolds

Lie groups and Lie algebras of Heisenberg type were first introduced by Kaplan [28,29,30] as natural generalizations of the (classical) Heisenberg groups. Since then, the Heisenberg type algebras have played an important role in such areas as spectral geometry, producing the first examples of isospectral manifolds that are not locally isometric [42,18], and harmonic analysis, playing a critical role in the construction of a counterexample [10] to the Lichnerowicz conjecture on harmonic manifolds.

In this section, we introduce a new family of two-step nilpotent Lie algebras and Lie groups that we call *Heisenberg-like*. The related class of nilmanifolds generalizes naturally from those of Heisenberg type in a variety of ways: by the definition, by the formulation of the length spectrum, and in terms of the prevalence of simply closed geodesics contained in three-dimensional totally geodesic subgroups. Throughout this section, we give numerous examples showing that manifolds that are Heisenberg-like need not be of Heisenberg type.

Because of the importance of algebras of Heisenberg type in geometric analysis, Lie groups, and mathematical physics, the authors suggest that the most general setting for many results in these areas is that of Heisenberg-like Lie algebras, rather than those of Heisenberg type.

Definition 3.1. – Let N be a two-step nilpotent Lie group with left-invariant metric $\langle\ ,\ \rangle$ and metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$. A simply connected subgroup N^* of N is *totally geodesic* if every geodesic that starts in N^* remains in N^*; equivalently, $\nabla_X Y \in \mathfrak{n}^*$ for all $X, Y \in \mathfrak{n}^*$, the Lie algebra of N^*.

If $\mathfrak{n}$ is of Heisenberg type, then for all $X, Z \in \mathfrak{n}$ with $X \in \mathfrak{V}$ and $Z \in \mathfrak{z}$, the subalgebra $\operatorname{span}_{\mathbb{R}}\{X, j(Z)X, Z\}$ is totally geodesic [13]. Using notation established in Section 1, we generalize this property to define a Heisenberg-like Lie algebra.

Definition 3.2. – A two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ is *Heisenberg-like* if the subalgebra $\operatorname{span}_{\mathbb{R}}\{X_m, j(Z)X_m, Z\}$ is totally geodesic for every $Z \in \mathfrak{z}$ and every $X_m \in W_m(Z)$, $m = 1, \ldots, \mu(Z)$. A two-step nilpotent metric Lie group is Heisenberg-like if and only if its Lie algebra is. We say that a Riemannian two-step nilmanifold is Heisenberg-like if and only if its simply connected Riemannian cover is Heisenberg-like.

We now establish an alternate definition of Heisenberg-like, using properties of the covariant derivative.

LEMMA 3.3. – *A two-step nilpotent metric Lie algebra* $(\mathfrak{n}, \langle\ ,\ \rangle)$ *is Heisenberg-like if and only if* $[j(Z)X_m, X_m] \in \operatorname{span}_{\mathbb{R}}\{Z\}$ *for all* $Z \in \mathfrak{z}$ *and all* $X_m \in W_m(Z)$, $m = 1, \ldots, \mu(Z)$.

Proof. – The covariant derivative on $\mathfrak{n} = \mathfrak{V} \oplus \mathfrak{z}$ has the following properties:

$$\begin{aligned}
\nabla_X Y &= \frac{1}{2}[X, Y],\\
\nabla_X Z &= \nabla_Z X = -\frac{1}{2} j(Z)X,\\
\nabla_Z Z^* &= 0,
\end{aligned}$$

where $X, Y \in \mathfrak{V}$ and $Z, Z^* \in \mathfrak{z}$ (see [12]). Therefore, for $X \in \mathfrak{V}$ and $Z \in \mathfrak{z}$, the subalgebra $\operatorname{span}_{\mathbb{R}}\{X, j(Z)X, Z\}$ is totally geodesic if and only if $[j(Z)X, X] = cZ$ for some $c \in \mathbb{R}$. □

Examples 3.4. –

(a) By Lemma 3.3, any two-step nilpotent metric Lie algebra with a 1-dimensional center must be Heisenberg-like; in particular, this holds for all examples of Heisenberg Lie algebras (see Example 1.3). Note that for an arbitrary choice of inner product, $(\mathfrak{h}_n, \langle\ ,\ \rangle)$ need not be of Heisenberg type.

(b) By Remark 3.8 of [12], a metric Lie algebra that is of Heisenberg type is necessarily Heisenberg-like. In particular, Example 1.10 is Heisenberg-like.

If $\mathfrak{n}$ is of Heisenberg type, then for all $Z \in \mathfrak{z}$ and $X \in \mathfrak{V}$,

$$\big[X, j(Z)X\big] = |X|^2 Z.$$

If $\mathfrak{n}$ is Heisenberg-like, then for all nonzero $Z \in \mathfrak{z}$ and every $X_m \in W_m(Z)$, $m = 1, \ldots, \mu(Z)$,

$$\big[X_m, j(Z)X_m\big] = \left(\frac{\vartheta_m(Z)|X_m|}{|Z|}\right)^2 Z.$$

Recall that a simply connected Riemannian manifold (M, g) has a Euclidean deRham factor if it can be factored as $(N, h) \times \mathbb{R}^k$. In the two-step nilpotent case, this can be determined at the

Lie algebra level using the j operator. In particular, a simply connected two-step nilpotent metric Lie group has a Euclidean deRham factor if and only if there exists nonzero Z in $\mathfrak{z}$ such that $j(Z) \equiv 0$. (See [12].)

LEMMA 3.5. – *A Heisenberg-like two-step nilpotent metric Lie algebra has no Euclidean deRham factors.*

Proof. – Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a Heisenberg-like Lie algebra. We show that the set $\mathcal{E} = \{Z \in \mathfrak{z}\colon\ j(Z) \equiv 0\}$ contains only the zero element. Suppose $Z^* \in \mathcal{E}$, $Z^* \neq 0$. Then for all $Z \in \mathfrak{z}$, $j(Z + Z^*) = j(Z)$. Hence $\mu(Z) = \mu(Z + Z^*)$ and, for each $m \in \{1, \ldots, \mu(Z)\}$, $W_m(Z) = W_m(Z + Z^*)$. Let $X \in W_m(Z + Z^*)$. Since $\mathfrak{n}$ is Heisenberg-like, $[j(Z + Z^*)X, X] = c^*(Z + Z^*)$ and $[j(Z)X, X] = cZ$ for some $c^*, c \in \mathbb{R}$. But $Z^* \neq 0$, so $[j(Z + Z^*)X, X] = 0$. That is,

$$|j(Z)X|^2 = \langle Z, [X, j(Z)X] \rangle = 0$$

for all $Z \in \mathfrak{z}$ and for all $X \in W_m(Z)$, $m = 1, \ldots, \mu(Z)$. Thus $j(Z) \equiv 0$ for all $Z \in \mathfrak{z}$, contradicting $\mathfrak{n}$ two-step nilpotent. □

Note that a Heisenberg type metric Lie algebra is strictly nonsingular. That is, $j(Z)$ is nonsingular for all $Z \in \mathfrak{z} - \{0\}$.

THEOREM 3.6. – *A Heisenberg-like metric Lie algebra is either strictly nonsingular or strictly singular.*

See Examples 3.9 below for Heisenberg-like Lie algebras that are strictly nonsingular and strictly singular.

Theorem 3.6 follows directly from the following characterization of Heisenberg-like Lie algebras, which was obtained independently by Hugues Blanchard [7] and the authors.

THEOREM 3.7. – *The two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ is Heisenberg-like if and only if for every $m = 1, \ldots, \mu$ there exists a constant c_m such that for every $Z \in \mathfrak{z}$, $\vartheta_m(Z) = c_m|Z|$. That is, the eigenvalues of j depend only on the norm of Z.*

Remarks 3.8. –

(a) Blanchard [7] uses this characterization when he compares the eigenvalues of the Laplace operator and the lengths of closed geodesics on manifolds that are Heisenberg-like.

(b) Theorem 3.7 shows again that Lie algebras of Heisenberg type must by Heisenberg-like, since by definition, a metric Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ is of Heisenberg type up to scaling if and only if there exists a constant c such that the only eigenvalues of $j(Z)$ are $\pm c|Z|$.

(c) In [14,15], H. Fanaï shows that nilmanifolds that are in strong resonance and satisfy the condition $j^*j = c\,Id_{|\mathfrak{z}}$ must be Heisenberg-like, via the equivalent formulation in Theorem 3.7. He uses this to prove his main result: Any two-step nilmanifold in this class is C^0-geodesically rigid within the class of all Riemannian nilmanifolds.
Conversely, nilmanifolds that are Heisenberg-like necessarily satisfy the condition $j^*j = c\,Id_{|\mathfrak{z}}$, as follows. Let $\mathfrak{n}$ be Heisenberg-like and let Z_1, Z_2 be orthogonal vectors in $\mathfrak{z} - \{0\}$. Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{V}$ composed of invariant vectors of $j(Z_2)$. Then

$$\begin{aligned}\langle j^*j(Z_1), Z_2 \rangle &= \langle j(Z_1), j(Z_2) \rangle = \operatorname{trace}\big(j(Z_1)j(Z_2)^{\mathrm{T}}\big)\\ &= -\sum_{i=1}^{n} \langle j(Z_1)j(Z_2)E_i, E_i \rangle = -\sum_{i=1}^{n} \langle Z_1, [j(Z_2)E_i, E_i] \rangle = 0\end{aligned}$$

by Lemma 3.3. Thus $j^*j(Z) \in \mathrm{span}_{\mathbb{R}}\{Z\}$ for all $Z \in \mathfrak{z}$, from which one easily concludes that j^*j is a constant multiple of $Id_{|\mathfrak{z}}$.

Nilmanifolds that are Heisenberg-like need not be in strong resonance.

Proof of Theorem 3.7. – Assume $(\mathfrak{n}, \langle\ ,\ \rangle)$ is Heisenberg-like.

Let $Z \in \mathcal{U}$, and let $m \in \{1, \ldots, \mu\}$. Let $\xi \in W_m(Z)$ be a unit vector. For the nonzero ϑ_m, we show that the function $\vartheta_m(Z)/|Z|$ is constant on connected components of $\mathcal{U}$ by showing that $\nabla\left(\vartheta_m(Z)/|Z|\right) = 0$. We then show that $\mathcal{U} = \mathfrak{z} - \{0\}$; i.e., $\mathcal{U}$ has only one connected component, and we are done.

Now

$$\nabla\left(\frac{\vartheta_m(Z)}{|Z|}\right) = \frac{|Z|\nabla\vartheta_m(Z) - \vartheta_m(Z)\nabla|Z|}{|Z|^2}.$$

Since $\mathfrak{n}$ is Heisenberg-like, $[j(Z)\xi, \xi] = -\vartheta_m(Z)^2 Z/|Z|^2$. On the other hand, by Proposition 1.20, $[j(Z)\xi, \xi] = -\vartheta_m(Z)\nabla\vartheta_m(Z)$. So if $\vartheta_m(Z) \neq 0$, then $\nabla\vartheta_m(Z) = \vartheta_m(Z)Z/|Z|^2$ for all $Z \neq 0$. One easily calculates that $\nabla|Z| = Z/|Z|$. Substituting, we obtain $\nabla(\vartheta_m(Z)/|Z|) = 0$.

We now show that $\mathcal{U} = \mathfrak{z} - \{0\}$; that is, we show that μ is constant on all of $\mathfrak{z} - \{0\}$. Let $Z^* \in \mathcal{U}$, the simple subdomain of $\mathfrak{z} - \{0\}$, and let $\mu = \mu(Z^*) = \mu(\mathcal{U})$. Let Z be any nonzero element in $\mathfrak{z} - \{0\}$. Note that $\vartheta_m(cZ) = c\vartheta_m(Z)$ for all nonzero real c, so we may assume Z and Z^* are linearly independent.

Define $A(t) = j((1-t)Z^* + tZ)$ for all $t \in \mathbb{R}$. Now $A(t)^2$ is a real analytic family of self adjoint operators on the n-dimensional space $\mathfrak{V}$. By [16, Theorem S6.3], there exist eigenvalue curves $\phi_1(t), \ldots, \phi_n(t)$ for $A(t)^2$ that are real analytic in t. From Lemma 3.5 we may assume that one of the eigenvalue curves, say $\phi_n(t)$, is nonzero in a neighborhood of 0. From above, there exist $c_k \in \mathbb{R}$, $k = 1, \ldots, n$, such that for all t in a neighborhood of 0, $\phi_k(t) = c_k\phi_n(t)$. If $c_k \neq 0$, then the two real analytic curves $\phi_k(t)/c_k$ and $\phi_n(t)$ agree in a neighborhood of 0, hence agree for all $t \in \mathbb{R}$. That is, $\phi_k(t) = c_k\phi_n(t)$ for all $t \in \mathbb{R}$. If $c_k = 0$, then the two real analytic functions $\phi_k(t)$ and $\sigma(t) \equiv 0$ agree in a neighborhood of 0, hence $\phi_k(t) = 0$ for all real t.

Hence the number of distinct eigenvalues of $A(t)^2$ remains constant for all t, except possibly if $\phi_n(t) = 0$ at some point. But if $\phi_n(t) = 0$, then all of the eigenvalues are zero, and by Lemma 3.5, this can only happen if Z and Z^* are linearly dependent. Thus $\mu(Z) = \mu$, and the number of eigenvalues remains constant on $\mathfrak{z} - \{0\}$, i.e., $\mathcal{U} = \mathfrak{z} - \{0\}$.

It now follows that if $\vartheta_1(Z) \neq 0$ for some Z in $\mathcal{U}$, then ϑ_1 is nonzero on all of $\mathfrak{z} - \{0\}$, and the case $\vartheta_1(Z) = 0$ now follows.

For the converse direction, we assume that for every $m = 1, \ldots, \mu$ there exists a constant c_m such that for every $Z \in \mathfrak{z}$, $\vartheta_m(Z) = c_m|Z|$.

Let $Z \in \mathfrak{z} - \{0\}$, and let $m \in \{1, \ldots, \mu\}$. Let $\xi \in W_m(Z)$ be a unit vector. By Proposition 1.20, $[j(Z)\xi, \xi] = -\vartheta_m(Z)\nabla\vartheta_m(Z)$. But by hypothesis, $\nabla\vartheta_m(Z) = c_m\nabla|Z| = (c_m/|Z|)Z$, which is clearly an element of $\mathrm{span}_{\mathbb{R}}\{Z\}$, and $\mathfrak{n}$ is Heisenberg-like by Lemma 3.3. □

Proof of Theorem 3.6. – Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a Heisenberg-like Lie algebra. From Theorem 3.7, if $\vartheta_1(Z) = 0$ for some nonzero Z in $\mathfrak{z} - \{0\}$, then $\vartheta_1(Z) \equiv 0$ on all of $\mathfrak{z} - \{0\}$. Thus $\mathfrak{n}$ cannot be almost nonsingular. The result follows by Lemma 1.16. □

We now present examples of Lie algebras that are Heisenberg-like, but not of Heisenberg type. We also examine the Heisenberg-like condition on examples presented in Section 1.

Examples 3.9. –

(a) Example 1.14 is not Heisenberg-like by Theorem 3.6, since it is almost nonsingular. In addition, the number of distinct eigenvalues changes.

(b) Example 1.15 is Heisenberg-like by Theorem 3.7, with $\vartheta_1(Z)=0$, $\vartheta_2(Z)=|Z|$.

(c) Let $\mathfrak{n}=\mathfrak{V}\oplus\mathfrak{z}$ be a 6-dimensional Lie algebra with inner product such that $\{X_1,X_2,X_3,X_4\}$ is an orthonormal basis for $\mathfrak{V}$ and $\{Z_1,Z_2\}$ is an orthonormal basis for $\mathfrak{z}$. Let a,b,c,d be real numbers, and set

$$j(Z_1)=\begin{pmatrix}0&a&0&0\\-a&0&0&0\\0&0&0&b\\0&0&-b&0\end{pmatrix},\qquad j(Z_2)=\begin{pmatrix}0&0&c&0\\0&0&0&d\\-c&0&0&0\\0&-d&0&0\end{pmatrix}.$$

The matrix $j(\alpha Z_1+\beta Z_2)$ has eigenvalues

$$\left\{\pm\frac{i}{2}\left(\sqrt{(a-b)^2\alpha^2+(c+d)^2\beta^2}\pm\sqrt{(a+b)^2\alpha^2+(c-d)^2\beta^2}\right)\right\}.$$

A straightforward calculation shows that $\mathfrak{n}$ is Heisenberg-like if and only if $a^2+b^2=c^2+d^2$ and $ab+cd=0$. A sufficient but not necessary condition is that

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

be a scalar multiple of an orthogonal matrix.

If N is of Heisenberg type, then geodesics in N are well-behaved. Let σ denote a unit speed geodesic with initial velocity X_0+Z_0, $X_0\in\mathfrak{V}$ and $Z_0\in\mathfrak{V}$, and initial position $\sigma(0)=e$. We write $\sigma(s)=\exp(\mathbf{X}(s)+\mathbf{Z}(s))$. If N is of Heisenberg type, then $\mathbf{Z}(s)$ is a multiple of Z_0 for all s (see [12]). In the Heisenberg-like case, such a geodesic will have the property that $\mathbf{Z}(s)$ periodically lands on a multiple of Z_0. We use Theorem 2.8 to determine the lengths of closed geodesics for a two-step nilpotent Heisenberg-like nilmanifold.

THEOREM 3.10. – *Let $(N,\langle\ ,\ \rangle)$ be a Riemannian two-step nilpotent metric Lie group that is Heisenberg-like. Let $\gamma\in N$, $\gamma\neq e$. With notation as in* (2.2), *write $\gamma=\exp(V+Z_V+Z_V^{\perp})$. Then the periods of γ are precisely*

$$\left\{|V+Z_V^{\perp}|,\ \sqrt{|V|^2+\frac{4\pi k_m|Z_V^{\perp}|^2(\vartheta_m(Z_V^{\perp})-\pi k_m)}{\vartheta_m(Z_V^{\perp})^2}}:\ k_m\in\mathbb{Z},\ 1\leqslant k_m<\frac{\vartheta_m(Z_V^{\perp})}{2\pi},\ m=1,\dots,\mu\right\}.$$

Note that if $(N,\langle\ ,\ \rangle)$ is of Heisenberg type, then N is nonsingular. It follows from Theorem 3.10 that in this case the periods of $\gamma=\exp(Z)$, $Z\in\mathfrak{z}-\{0\}$, are

$$\left\{|Z|,\ \sqrt{4\pi k(|Z|-\pi k)}:\ k\in\mathbb{Z},\ 1\leqslant k<\frac{|Z|}{2\pi}\right\}.$$

This result was obtained originally by Eberlein in [12].

Proof of Theorem 3.10. – Let $X_0+Z_0\in\mathfrak{n}$, $X_0=X_1+X_2$, such that $X_1\in\ker j(Z_0)$, $X_2\perp\ker j(Z_0)$ and X_2+Z_0 satisfy conditions (i) to (iv) of Theorem 2.8. Let σ be a geodesic such that $\sigma(0)=e$, $\dot\sigma(0)=X_0+Z_0$, and γ translates σ with period ω. Recall from Theorem 2.5 that $Z_0\perp[V,\mathfrak{n}]$ in this case.

If N is Heisenberg-like, then $K_V^{\perp}(X_2+Z_0)=cZ_V^{\perp}$ for some $c>0$ if and only if $Z_0=c'Z_V^{\perp}$ for some $c'>0$. Hence β, the angle between Z_0 and $K_V^{\perp}(X_2+Z_0)$, equals 0. It follows that

$$\vartheta_m(Z_V^{\perp})=|Z_V^{\perp}|\vartheta_m(\overline{Z}_0),\quad \overline{Z}_0=\frac{Z_0}{|Z_0|},$$

and the periods take on the desired form, provided we show that $1\leqslant k<\vartheta_m(Z_V^{\perp})/2\pi$.

If $|Z_0|=1$, then $|X_2|=0$ and $\omega=|V+Z_V^{\perp}|$. If $|Z_0|=0$, then $\ker j(Z_0)=\mathfrak{V}$ so $X_2=0$, which implies $Z_V^{\perp}=0$, and $\omega=|V|$ is the unique period of γ.

So we may assume $0<|Z_0|<1$. Using $Z_0\perp[V,\mathfrak{n}]$, $|X_2+Z_0|=1$, and (2.3), we obtain

$$\langle Z_0,K(X_2+Z_0)\rangle=\langle Z_0,K_V^{\perp}(X_2+Z_0)\rangle=\frac{1}{2}\left(1+|Z_0|^2\right).$$

Thus

$$\langle\overline{Z}_0,K_V^{\perp}(X_2+Z_0)\rangle=\frac{1}{2}\left(\frac{1}{|Z_0|}+|Z_0|\right).$$

One easily checks that for $0<|Z_0|<1$, $\frac{1}{|Z_0|}+|Z_0|>2$, so $|K_V^{\perp}(X_2+Z_0)|>1$. By (iv),

$$k_m=\frac{\vartheta_m(Z_V^{\perp})}{2\pi|K_V^{\perp}(X_2+Z_0)|}<\frac{\vartheta_m(Z_V^{\perp})}{2\pi},$$

as desired.

It remains to prove that if $1\leqslant k<\vartheta_m(Z_V^{\perp})/2\pi$, then there exists X_2+Z_0 satisfying conditions (i) through (iv) of Theorem 2.8. Let

$$Z_0=\frac{1}{|Z_V^{\perp}|}\sqrt{\frac{\pi k}{(\vartheta_m(Z_V^{\perp})-\pi k)}}Z_V^{\perp}.$$

Note that $k<\vartheta_m(Z_V^{\perp})/2\pi$ implies $|Z_0|<1$. Let $\xi_m\in W_m(Z_V^{\perp})$ be a unit vector. Set $X_2=\sqrt{1-|Z_0|^2}\xi_m$, so X_2+Z_0 is a unit vector. Note that by the assumption on k, $\vartheta_m(Z_V^{\perp})>0$, hence ξ_m and X_2 are orthogonal to $\ker j(Z_V^{\perp})$. Similarly, $Z_V^{\perp}\perp[V,\mathfrak{n}]$ implies $V\in\ker j(Z_0)$, and conditions (i) through (iii) of Theorem 2.4 are satisfied. To show condition (iv), by (2.10),

$$|K_V^{\perp}(X_2+Z_0)|=\frac{1+|Z_0|^2}{2|Z_0|}=\frac{\vartheta_m(Z_V^{\perp})}{2\sqrt{\pi k(\vartheta_m(Z_V^{\perp})-\pi k)}}=\frac{|Z_V^{\perp}|\vartheta_m(Z_0)}{2\pi k},$$

and we are done. □

Example 3.11. – Consider the 5-dimensional Lie algebra of Example 1.15. We briefly review the construction. Let $\mathfrak{n}=\mathfrak{V}\oplus\mathfrak{z}$ be a 5-dimensional Lie algebra with inner product such that $\{X_1,X_2,X_3\}$ is an orthonormal basis for $\mathfrak{V}$ and $\{Z_1,Z_2\}$ is an orthonormal basis for $\mathfrak{z}$. Give $\mathfrak{n}$ a Lie bracket such that

$$j(Z_1)=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},\qquad j(Z_2)=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}.$$

For $Z=\alpha Z_1+\beta Z_2$, $j(Z)$ has eigenvalues $\{0,\pm i\sqrt{\alpha^2+\beta^2}\}$, that is, $\vartheta_1(Z)=0$ and $\vartheta_2(Z)=|Z|$. This Lie algebra is strictly singular and (from Theorem 3.7) is Heisenberg-like. The invariant subspaces of $j(\alpha Z_1+\beta Z_2)$ are $W_1(Z)=\operatorname{span}_{\mathbb{R}}\{\beta X_2-\alpha X_3\}$ and $W_2(Z)=\operatorname{span}_{\mathbb{R}}\{X_1,\alpha X_2+\beta X_3\}$.

We compute the periods of various elements in $\mathfrak{n}$. Let $\gamma=\exp(V+Z)$, where $V=v_1X_1+v_2X_2+v_3X_3$ and $Z=z_1Z_1+z_2Z_2$.

One easily checks that $[V,\mathfrak{n}]=\mathfrak{z}$ if and only if $v_1\neq 0$. In this case, $Z_V^\perp=0$ and the unique period of γ is $\omega=|V|$.

If $v_1=0$ and $|V|\neq 0$, then $[V,\mathfrak{n}]=\operatorname{span}_{\mathbb{R}}\{v_2Z_1+v_3Z_2\}$, and $[V,\mathfrak{n}]^\perp=\operatorname{span}_{\mathbb{R}}\{v_3Z_1-v_2Z_2\}$. So

$$Z_V^\perp=\frac{z_1v_3-z_2v_2}{|V|^2}(v_3Z_1-v_2Z_2)$$

and

$$|Z_V^\perp|=\vartheta_2\big(Z_V^\perp\big)=\frac{|z_1v_3-z_2v_2|}{|V|}.$$

By Theorem 3.10, the periods of γ are

$$\left\{\sqrt{|V|^2+|Z_V^\perp|^2},\sqrt{|V|^2+4\pi k\big(|Z_V^\perp|-\pi k\big)}\colon\ k\in\mathbb{Z},\ 1\leqslant k<\frac{|Z_V^\perp|}{2\pi}\right\}.$$

Note that in this case, γ will have more than one period if and only if $|Z_V^\perp|>2\pi$.

Finally, if $V=0, Z\neq 0$, then the periods of $\gamma=\exp(Z)$ are

$$\left\{|Z|,\sqrt{4\pi k(|Z|-\pi k)}\colon\ k\in\mathbb{Z},\ 1\leqslant k<\frac{|Z|}{2\pi}\right\}.$$

Remarks 3.12. –

(a) Note that the length spectrum result for central periods of Example 3.11 mimics the Heisenberg type result, since the Lie algebra in 3.11 is, up to nonsingularity, Heisenberg type; that is, the only *nonzero* eigenvalues are $\pm i|Z|$.

(b) Eberlein used Example 3.11 to demonstrate the phenomenon of nonunique, noncentral periods (see (4.8) of [12]). Let α and β be nonzero and set $\beta^*=1+\frac{1}{2}\beta^2$. Let $\gamma=\exp(2\pi\alpha X_3+2\pi\beta^*Z_1)$. By the calculation in Example 3.11, with $V=2\pi\alpha X_3$ and $Z=Z_V^\perp=2\pi\beta^*Z_1$, it follows that γ has at least two periods,

$$\omega=\sqrt{|V|^2+4\pi\big(|Z_V^\perp|-\pi\big)}=2\pi\sqrt{1+\alpha^2+\beta^2}$$

and

$$\omega^*=\sqrt{|V|^2+|Z_V^\perp|^2}=2\pi\sqrt{\alpha^2+\beta^{*2}}.$$

For the existence of the first period, note that $1<|Z_V^\perp|/2\pi=\beta^*$.

We denote the center of a Lie group N by $Z(N)$. Note that for simply connected two-step nilpotent Lie groups, $Z(N)=\exp(\mathfrak{z})$.

THEOREM 3.13. – *Let $(N,\langle\ ,\ \rangle)$ be a simply connected two-step nilpotent metric Lie group. Then N is Heisenberg-like if and only if for every $\gamma\in Z(N)$ and every period ω of γ there exists a γ-periodic geodesic σ of period ω that is contained in a three-dimensional totally geodesic subgroup of N.*

Theorem 3.13 is a natural generalization of the following characterization of Heisenberg-type metric Lie groups.

THEOREM 3.14 ([13]). – *Let $(N, \langle\ ,\ \rangle)$ be a simply connected, strictly nonsingular two-step nilpotent metric Lie group. If N is of Heisenberg type, then every γ-periodic geodesic of N is contained in a three-dimensional totally geodesic submanifold H. On the other hand, if for every geodesic γ with $\gamma(0) = e$ there exists a connected, three-dimensional totally geodesic submanifold H such that $\gamma'(0) \in \mathcal{H} = T_e H$ and $\mathcal{H} \cap \mathfrak{z} \neq \{0\}$, then N is of Heisenberg type.*

Proof of Theorem 3.13. – Let N be Heisenberg-like and let $\gamma = \exp(Z)$ for some $Z \in \mathfrak{z} - \{0\}$. Suppose that γ has period ω. If $\omega = |Z|$, then $\sigma(s) = \exp(sZ/|Z|)$ is a unit speed γ-periodic geodesic with period $|Z|$, and by the Heisenberg-like assumption, $\sigma(s)$ is contained in the totally geodesic subgroup corresponding to the subalgebra $\operatorname{span}_{\mathbb{R}}\{X_m, j(Z)X_m, Z\}$ for any X_m in $W_m(Z)$, and any $m \in \{1, \ldots, \mu(Z)\}$.

If $\omega \neq |Z|$, by Theorem 3.10,

$$\omega = \frac{2\pi k|Z|\sqrt{\vartheta_m(Z) - \pi k}}{\vartheta_m(Z)}$$

for appropriate choices of m and k. Note that this implies $\vartheta_m(Z) \neq 0$. Let $\widehat{\sigma}(s)$ be a unit speed geodesic that is translated by γ with period ω such that $\widehat{\sigma}(0) = e$ and $\widehat{\sigma}'(0) = \widehat{X_0} + Z_0$. Since N is Heisenberg-like, $Z_0 \in \operatorname{span}_{\mathbb{R}}\{Z\}$. If $\operatorname{span}_{\mathbb{R}}\{\widehat{X_0}, Z_0, j(Z_0)\widehat{X_0}\}$ is not totally geodesic, then let $X_0 = \sqrt{1 - |Z_0|^2}\xi_m$ for a unit vector $\xi_m \in W_m(Z)$. Let σ be the unique geodesic with $\sigma(0) = e$ and $\sigma'(0) = X_0 + Z_0$. Then $\operatorname{span}_{\mathbb{R}}\{X_0, Z_0, j(Z_0)X_0\}$ is totally geodesic, so σ is contained in a three-dimensional totally geodesic subgroup. Using (2.3), the Heisenberg-like assumption, and the fact that $|X_0| = |\widehat{X_0}|$, we know $K(X_0 + Z_0) = K(\widehat{X_0} + Z_0)$, and since $\widehat{X_0} + Z_0$ satisfies (i) to (iv) of Theorem 2.8, so does $X_0 + Z_0$. Hence σ is translated by γ with period ω.

For the converse direction, assume that for every $\gamma \in Z(N)$ and every period ω of γ there exists a γ-periodic geodesic σ of period ω that is contained in a three-dimensional totally geodesic subgroup of N. Note that by Theorem 3.7 and continuity of the eigenvalues, it is enough to prove Heisenberg-like on the simple subdomain $\mathcal{U}$ of $\mathfrak{z}$, i.e., we may assume that locally, the number of distinct eigenvalues does not change, and the eigenvalues ϑ_m, their invariant subspaces W_m, and their gradients $\nabla\vartheta_m$ are well-defined and smooth as functions of Z.

Assume that N is not Heisenberg-like, i.e., there exist $Z_0 \in \mathcal{U}$ and $X_m \in W_m(Z_0)$ such that $[j(Z_0)X_m, X_m] \notin \operatorname{span}_{\mathbb{R}}\{Z_0\}$. This implies $\vartheta_m(Z_0) \neq 0$, so $[j(Z_0)^{-1}X_m, X_m] \notin \operatorname{span}_{\mathbb{R}}\{Z_0\}$. We may assume Z_0 and X_m are unit vectors. Recall that

$$\left[j(Z_0)^{-1}\xi_m, \xi_m\right] = \frac{\nabla\vartheta_m(Z_0)}{\vartheta_m(Z_0)} \quad \text{for all } \xi_m \in W_m(Z_0).$$

So $[j(Z_0)^{-1}X_m, X_m] \notin \operatorname{span}_{\mathbb{R}}\{Z_0\}$ if and only if $\nabla\vartheta_m(Z_0) \notin \operatorname{span}_{\mathbb{R}}\{Z_0\}$.

Let $Z_r = 2\pi K(rX_m + Z_0)/\vartheta_m(Z_0)$. Note that

$$|Z_r| = \frac{2\pi|K(rX_m + Z_0)|}{\vartheta_m(Z_0)} = \frac{2\pi|K((rX_m + Z_0)/\sqrt{1 + r^2})|}{\vartheta_m(Z_0/\sqrt{1 + r^2})}.$$

Using the definition of K, it follows that $\lim_{r \to 0}|K(rX_m + Z_0)| = 1$. Let σ_r be the unit speed geodesic such that $\sigma_r(0) = e$ and having initial velocity $(rX_m + Z_0)/\sqrt{1 + r^2}$. By Theorem 2.4, σ_r is an $\exp(Z_r)$-periodic geodesic with period $\omega_r = (2\pi\sqrt{1 + r^2})/\vartheta_m(Z_0)$.

By hypothesis, there exists $X^r_{m'_r} \in W_{m'_r}(Z_r)$ such that $\mathfrak{n}^* = \mathrm{span}_{\mathbb{R}}\{X^r_{m'_r}, j(Z_r)X^r_{m'_r}, Z_r\}$ is totally geodesic, and the corresponding subgroup contains a geodesic that is translated by $\exp(Z_r)$ with period ω_r. To see that $\mathfrak{n}^*$ must be of this form, note that since $\omega_r \neq |Z_r|$, $\mathfrak{n}^*$ must contain a vector of the form $X + Z$ as in Theorem 2.8. Using the covariant derivative, as given in the proof of Lemma 3.3, one easily computes that $\mathfrak{n}^*$ must contain $j(Z)^n X$ for $n \in \mathbb{N}$. If $\{j(Z)^n X, X + Z\colon\ n \in \mathbb{N}\}$ spans a 3-dimensional vector space, then $j(Z)^2 X \in \mathrm{span}_{\mathbb{R}}\{X\}$, which implies X lies in a single invariant subspace $W_m(Z)$. From this we conclude that $K(X+Z) \in \mathrm{span}_{\mathbb{R}}\{Z\}$, which implies $Z \in \mathrm{span}_{\mathbb{R}}\{Z_r\}$.

By Theorem 2.8,

$$\omega_r = \sqrt{\frac{4\pi k_{m'_r}(\vartheta_{m'_r}(\overline{Z}_r)|Z_r| - \pi k_{m'_r})}{\vartheta_{m'_r}(\overline{Z}_r)^2}},$$

where $\overline{Z}_r = Z_r/|Z_r|$. Thus

$$\frac{4\pi^2(1+r^2)}{\vartheta_m(Z_0)^2} = \frac{4\pi k_{m'_r}(\vartheta_{m'_r}(\overline{Z}_r)|Z_r| - \pi k_{m'_r})}{\vartheta_{m'_r}(\overline{Z}_r)^2}.$$

Define

$$\lambda_r = \frac{\vartheta_{m'_r}(\overline{Z}_r)}{k_{m'_r}\vartheta_m(Z_0)}.$$

Substituting, we obtain the quadratic equation

$$(1+r^2)\lambda_r^2 - 2\lambda_r\big|K(rX_m + Z_0)\big| + 1 = 0.$$

Solving for λ_r and letting $r \to 0$, we obtain $\lim_{r\to 0}\lambda_r = 1$, i.e.,

$$\lim_{r\to 0}\frac{\vartheta_{m'_r}(\overline{Z}_r)}{k_{m'_r}} = \vartheta_m(Z_0).$$

As $\lim_{r\to 0}\overline{Z}_r = Z_0$, and the finitely many $\vartheta_{m'_r}$ are continuous in Z, we have

$$\lim_{r\to 0}\frac{\vartheta_{m'_r}(\overline{Z}_r)}{k_{m'_r}} = \lim_{r\to 0}\frac{\vartheta_{m'_r}(Z_0)}{k_{m'_r}} = \vartheta_m(Z_0).$$

By passing to a subsequence, if necessary, we may assume m'_r and $k_{m'_r}$ are independent of r. Letting $m' = m'_r$ and $k_{m'} = k_{m'_r}$, we obtain $\vartheta_{m'}(Z_0) = k_{m'}\vartheta_m(Z_0)$.

Assume for the moment that Z_0 satisfies the property that $\vartheta_n(Z_0)$ is an integer multiple of $\vartheta_m(Z_0)$ if and only if $n = m$. Then we are done, since this implies $m' = m$ and $k_{m'} = 1$. So $\mathrm{span}_{\mathbb{R}}\{X^r_m, j(Z_r)X^r_m, Z_r\}$ totally geodesic implies $\nabla\vartheta_m(Z_r) \in \mathrm{span}_{\mathbb{R}}\{Z_r\}$. Since these are all continuous in r, letting $r \to 0$ we obtain $\nabla\vartheta_m(Z_0) \in \mathrm{span}_{\mathbb{R}}\{Z_0\}$, a contradiction.

Now assume $\vartheta_{m'}(Z_0) = k_{m'}\vartheta_m(Z_0)$, where $m' \neq m$. Since $\nabla\vartheta_m(Z_0) \notin \mathrm{span}_{\mathbb{R}}\{Z_0\}$, by continuity there exists $\varepsilon > 0$ such that $\nabla\vartheta_m(Z) \notin \mathrm{span}_{\mathbb{R}}\{Z\}$ for all Z within ε of Z_0. Also, there exists Z_1 within ε of Z_0 and $\delta > 0$ such that the δ-neighborhood of Z_1 is contained in the ε-neighborhood of Z_0, and $\vartheta_{m'}(Z) \neq k_{m'}\vartheta_m(Z)$ for all Z within δ of Z_1. To see this last assertion, note that if this were not the case, then $\vartheta_{m'}(Z) = k_{m'}\vartheta_m(Z)$ would be a dense condition on the ε-neighborhood of Z_0, which would imply $\vartheta_{m'}(Z) = k_{m'}\vartheta_m(Z)$ on a neighborhood of Z_0. Thus $\nabla\vartheta_m(Z_0)$ and $\nabla\vartheta_{m'}(Z_0)$ lie in the same one-dimensional subspace, a contradiction.

We thus have $Z_1 \in \mathcal{U}$ such that $\nabla\vartheta_m(Z_1) \notin \mathrm{span}_{\mathbb{R}}\{Z_1\}$, and a δ-neighborhood around Z_1 such that $\vartheta_{m'}(Z_1)$ is not an integer multiple of $\vartheta_m(Z_1)$. (We can shrink ε to avoid $\vartheta_{m'}(Z)$ being any integer multiple of $\vartheta_m(Z)$.) If we repeat this process at most $\mu = \mu(Z_0)$ times, we obtain $Z_\mu \in \mathcal{U}$ such that $\nabla\vartheta_\mu(Z_\mu) \notin \mathrm{span}_{\mathbb{R}}\{Z_\mu\}$, and such that Z_μ satisfies the property that $\vartheta_n(Z_\mu)$ is an integer multiple of $\vartheta_m(Z_\mu)$ if and only if $n = m$, and we are done. □

4. The length spectrum and isospectrality

Two-step nilmanifolds have played a vital role in comparing the Laplace spectrum of Riemannian manifolds. Until recently, all known examples of isospectral manifolds were also locally isometric, i.e., shared a common Riemannian cover. The locally isometric examples of isospectral manifolds are known to have the same length spectrum (ignoring multiplicities). See [23,25] for references and more details.

Then Gordon [18,19], [22] introduced a method for producing pairs of isospectral two-step nilmanifolds that need not be locally isometric, i.e., she does not assume a common cover. We briefly review this construction and prove that all known examples of isospectral two-step nilmanifolds arising from this method must have the same length spectrum. This result does not follow from the generic results of [9], as nilmanifolds do not satisfy the genericity hypotheses of that paper.

The following definitions are used in the Gordon construction.

Definitions 4.1. –
(a) Let $\mathrm{spec}(M,g)$ denote the eigenvalues of the Laplace–Beltrami operator Δ acting on smooth functions on the Riemannian manifold (M,g).
(b) Let $N(j)$ denote the simply connected nilpotent Lie group with Lie algebra determined by the inner product spaces $\mathfrak{V}, \mathfrak{z}$, and the mapping $j:\mathfrak{z} \to \mathrm{so}(\mathfrak{V})$ as described in (1.7). Let $\langle\ ,\ \rangle$ denote the left invariant metric on $N(j)$ corresponding to the inner product on $\mathfrak{V} \oplus \mathfrak{z}$. Let Γ be a cocompact, discrete subgroup of $N(j)$. Then $N(j,\Gamma) = \Gamma\backslash N(j)$ with the metric induced from $\langle\ ,\ \rangle$ on $N(j)$ is a Riemannian two-step nilmanifold.
(3) Define $\pi: N(j) \to N(j)/N^{(1)}$ where $N^{(1)}$ is the derived group of $N(j)$. The Lie group $\pi(N(j))$ is abelian and isomorphic to Euclidean space. The image $\pi(\Gamma)$ is a cocompact discrete subgroup (i.e., a lattice) of $\pi(N(j))$. Let $N_0(j,\Gamma) = \pi(\Gamma)\backslash\pi(N(j))$ together with the unique metric making $N(j,\Gamma) \to N_0(j,\Gamma)$ a Riemannian submersion with totally geodesic fibers. One easily checks that the manifold $N_0(j,\Gamma)$ is a flat torus.

Definition 4.2. – Let $\mathfrak{V}$ and $\mathfrak{z}$ be fixed and let j and j' be two linear transformations from $\mathfrak{z}$ to $\mathrm{so}(\mathfrak{V})$. We say *$j$ is isospectral to j'*, denoted $j \sim j'$, if for all $Z \in \mathfrak{z}$, $j(Z)$ and $j'(Z)$ are similar, i.e., the eigenvalues of $j(Z)$ and $j'(Z)$ coincide.

We examine pairs of isospectral two-step nilmanifolds, constructed as follows.

THEOREM 4.3 ([22, Theorem 3.4]). – *In the notation of* (4.1), *let $N(j,\Gamma)$ and $N(j',\Gamma')$ be compact, strictly nonsingular, two-step Riemannian nilmanifolds associated with the same inner product spaces $\mathfrak{V}$ and $\mathfrak{z}$. Assume*
(i) $\log(\Gamma \cap N^{(1)}) = \log(\Gamma' \cap N'^{(1)})$,
(ii) $\mathrm{spec}(N_0(j,\Gamma)) = \mathrm{spec}(N_0(j',\Gamma'))$, *and*
(iii) $j \sim j'$.
Then $\mathrm{spec}(N(j,\Gamma)) = \mathrm{spec}(N(j',\Gamma'))$.

THEOREM 4.4. – *If $N(j,\Gamma)$ and $N(j',\Gamma')$ are isospectral by satisfying the hypotheses of Theorem* 4.3, *then $N(j,\Gamma)$ and $N(j',\Gamma')$ have the same length spectrum, ignoring multiplicities.*

Before proving Theorem 4.4, we need the following.

Definition 4.5. – Let $K:\mathfrak{V}\oplus\mathfrak{z}\to\mathfrak{z}$ and $K':\mathfrak{V}\oplus\mathfrak{z}\to\mathfrak{z}$ be the mappings defined in (2.2) determined by $(\mathfrak{V},\mathfrak{z},j)$ and $(\mathfrak{V},\mathfrak{z},j')$, respectively. We say K *is similar to* K', denoted $K\sim K'$, if for all $Z\in\mathfrak{z}$, $\mu(Z)=\mu'(Z)$, and for all $X\in\mathfrak{V}$ there exists $X'\in\mathfrak{V}$ such that $K(X+Z)=K'(X'+Z)$ and such that

(i) $|X_1|=|X_1'|$, and

(ii) if $\vartheta_m(Z)\neq 0$, then $|\xi_m|=|\xi_m'|$, $m=1,\dots,\mu(Z)$;

we also require that for all $Z\in\mathfrak{z}$ and $X'\in\mathfrak{V}$ there exists $X\in\mathfrak{V}$, such that $K(X+Z)=K'(X'+Z)$ and such that (i) and (ii) hold. Here μ,μ' are the counting functions associated to j,j' respectively, as defined in (1.17), and X_1,X_1',ξ_m, and ξ_m' are determined by decomposing $X=X_1+\sum_m\xi_m$, and $X'=X_1'+\sum_m\xi_m'$ as in (2.1).

LEMMA 4.6. – *Let* $\mathfrak{V}$ *and* $\mathfrak{z}$ *be fixed and let* j *and* j' *be linear transformations from* $\mathfrak{z}$ *to* $\mathrm{so}(\mathfrak{V})$. *If* $j\sim j'$ *then* $K\sim K'$.

Proof. – Consider $K(X+Z_0)$. Decompose X as X_1+X_2, $X_2=\sum_m\xi_m$ as in (2.1). Observe that $K(X_1+X_2+Z_0)=K(X_2+Z_0)$. By similarity, $\ker j(Z_0)$ is nonempty if and only if $\ker j'(Z_0)$ is nonempty, so it is enough to prove the theorem in the case $X_1=0$, i.e., we may assume $\vartheta_m(Z_0)$ is nonzero in what follows.

Let $\mathcal{U}$ be the simple subdomain of $\mathfrak{z}$ defined in (1.18). By Proposition 1.19, $\vartheta_m(Z)$ is smooth on $\mathcal{U}$, $m=1,\dots,\mu$. For Z in $\mathcal{U}$, Lee and Park [33, Lemma 3.2] showed that for $\vartheta_m(Z)\neq 0$,

$$\frac{1}{\vartheta_m(Z)}\nabla\vartheta_m(Z)=\left[j(Z)^{-1}\widehat{X}_m(Z),\widehat{X}_m(Z)\right], \tag{4.7}$$

where $\widehat{X}_m(Z)$ is any smooth (in Z) choice of unit vector in $W_m(Z)$.

If Z_0 is in $\mathcal{U}$ then by (4.7), if $\xi_m\neq 0$,

$$\left[j(Z_0)^{-1}\xi_m,\xi_m\right]=|\xi_m|^2\left[j(Z_0)^{-1}\widehat{X}_m(Z_0),\widehat{X}_m(Z_0)\right]=|\xi_m|^2\frac{1}{\vartheta_m(Z_0)}\nabla\vartheta_m(Z_0).$$

Note that this value depends only on the local behavior of the eigenvalues $\vartheta_m(Z_0)$ and the magnitude of the orthogonal projection of X_2 onto $W_m(Z_0)$, $m=1,\dots,\mu(Z_0)$.

If $j\sim j'$, then for each m, let ξ_m' be a vector in $W_m'(Z_0)$, the invariant subspace of j' associated to the eigenvalue $\vartheta_m(Z_0)$, such that $|\xi_m'|=|\xi_m|$. Let $X_2'=\sum_m\xi_m'$. By (4.7), $K(X_2+Z_0)=K'(X_2'+Z_0)$, and we are done.

Now assume that $Z_0\notin\mathcal{U}$. As $\mathcal{U}$ is dense in $\mathfrak{z}$, Z_0 is a limit point of $\mathcal{U}$. By (1.19.4) and continuity of the set of eigenvalues [32, §II.5.7], two (or more) of the eigenvalues must approach each other as Z approaches $Z_0\in\mathfrak{z}-\mathcal{U}$. That is, there exists $Z_t\to Z_0$, $Z_t\in\mathcal{U}$, such that as $t\to 0$,

$$\lim_{t\to 0}\vartheta_m(Z_t)=\vartheta_h(Z_0)\quad\text{and}\quad\lim_{t\to 0}\vartheta_{m'}(Z_t)=\vartheta_h(Z_0).$$

We proceed as though only two eigenvalues approach each other; it will be clear that this does not affect the argument.

By Anselone [1, Theorem 4.16], since $j(Z_t)\to j(Z_0)$ and skew-symmetry holds, we may define

$$V_m(Z_0)=\lim_{t\to 0}W_m(Z_t)\qquad\text{and}\quad V_{m'}(Z_0)=\lim_{t\to 0}W_{m'}(Z_t).$$

Note that since $W_m(Z_t)$ and $W_{m'}(Z_t)$ are orthogonal, invariant subspaces of $j(Z_t)$ for all t, their limit spaces $V_m(Z_0)$, $V_{m'}(Z_0)$ are orthogonal, invariant subspaces of $j(Z_0)$, and

$$W_h(Z_0) = V_m(Z_0) \oplus V_{m'}(Z_0).$$

We refer to $V_m(Z_0), V_{m'}(Z_0)$ as *refined invariant subspaces* of $j(Z_0)$.

Let $X_h \in W_h(Z_0)$. We now show that $K(X_h + Z_0)$ depends only on the norms of the projections of X_h onto the refined invariant subspaces of $W_h(Z_0)$, and on the nearby behavior of ϑ_m and $\vartheta_{m'}$. Once we do this, using the similarity of j and j', we choose $X'_h \in W'_h(Z_0)$ such that the norms of projections of X_h and X'_h onto their corresponding refined invariant subspaces are equal. (Clearly, this implies that the norms of their projections onto corresponding invariant subspaces are equal.) Then $K(X_h + Z_0) = K'(X'_h + Z_0)$ and we are done.

Consider first the case $X_h \in V_m(Z_0)$, i.e., X_h is contained in a single refined invariant subspace of $j(Z_0)$. Then for all t there exists $X^t_m \in W_m(Z_t)$ such that $\lim_{t\to 0} X^t_m = X_h$. Using continuity of the Lie bracket and $Z_t \in \mathcal{U}$,

$$\begin{aligned} K(X_h + Z_0) &= Z_0 + \frac{1}{2}\big[j(Z_0)^{-1}X_h, X_h\big] \\ &= Z_0 + \frac{1}{2}\lim_{t\to 0}\big\{\big[j(Z_t)^{-1}X^t_m, X^t_m\big]\big\} \\ &= Z_0 + \frac{1}{2}|X_h|^2 \lim_{t\to 0}\left\{\frac{\nabla\vartheta_m(Z_t)}{\vartheta_m(Z_t)}\right\} \end{aligned}$$

and we are done in this case.

Now consider the case $X_h = X_m + X_{m'}$ where $X_m \in V_m(Z_0)$ and $X_{m'} \in V_{m'}(Z_0)$. We show that

$$K(X_h + Z_0) = K(X_m + Z_0) + K(X_{m'} + Z_0) - Z_0.$$

From above, the right hand side of this equation depends only on $|X_m|$, $|X_{m'}|$ and the nearby behavior of ϑ_m and $\vartheta_{m'}$. Now

$$\begin{aligned} &K(X_m + Z_0) + K(X_{m'} + Z_0) - Z_0 \\ &\quad = Z_0 + \frac{1}{2}\big[j(Z_0)^{-1}X_m, X_m\big] + \frac{1}{2}\big[j(Z_0)^{-1}X_{m'}, X_{m'}\big]. \end{aligned}$$

On the other hand, since X_h is in the single invariant subspace $W_h(Z_0)$,

$$K(X_m + X_{m'} + Z_0) = Z_0 + \frac{1}{2}\big[j(Z_0)^{-1}(X_m + X_{m'}), X_m + X_{m'}\big].$$

Thus

$$\begin{aligned} &K(X_m + X_{m'} + Z_0) - \big\{K(X_m + Z_0) + K(X_{m'} + Z_0) - Z_0\big\} \\ &\quad = \frac{1}{2}\big[j(Z_0)^{-1}X_m, X_{m'}\big] + \frac{1}{2}\big[j(Z_0)^{-1}X_{m'}, X_m\big] \\ &\quad = \frac{1}{2\vartheta_h(Z_0)^2}\big(\big[X_m, j(Z_0)X_{m'}\big] + \big[X_{m'}, j(Z_0)X_m\big]\big). \end{aligned}$$

We show that this value is zero by showing that for any vector Z in $\mathcal{U}$,

$$\big\langle Z, \big[X_m, j(Z_0)X_{m'}\big] + \big[X_{m'}, j(Z_0)X_m\big]\big\rangle = 0.$$

Note that $\langle Z, [X_m, j(Z_0)X_{m'}]\rangle$ equals $-\langle j(Z)j(Z_0)X_{m'}, X_m\rangle$ by the definition of $j(Z)$. Also, $\langle Z, [X_h, j(Z_0)X_h']\rangle$ equals $-\langle j(Z_0)j(Z)X_m, X_{m'}\rangle$, using the definitions of $j(Z)$ and $j(Z_0)$ three times. So that

$$\langle Z, [X_m, j(Z_0)X_{m'}] + [X_{m'}, j(Z_0)X_m]\rangle = \langle (j(Z)j(Z_0) + j(Z_0)j(Z))X_{m'}, X_m\rangle.$$

Let $\widehat{X}_m(Z)$ and $\widehat{X}_{m'}(Z)$ be any smooth (in Z) choice of unit vectors in $W_m(Z)$ such that $\widehat{X}_m(Z_0) = X_m/|X_m|$, and $\widehat{X}_{m'}(Z_0) = X_{m'}/|X_{m'}|$. Let $\widehat{X}_h(Z) = \widehat{X}_{m'}(Z)$. Note that $\langle \widehat{X}_h(Z_0), \widehat{X}_m(Z_0)\rangle = 0$. By Lee and Park [33, Lemma 3.2]

$$\big(j(Z)j(Z_0) + j(Z_0)j(Z)\big)\widehat{X}_h(Z_0) = -\big(j(Z_0)^2 + \vartheta_h(Z_0)^2 Id\big)\frac{\partial \widehat{X}_h}{\partial z} - 2\vartheta_h(Z_0)\frac{\partial \vartheta_h}{\partial z}\widehat{X}_h(Z_0),$$

provided $\partial\widehat{X}_h/\partial z$ and $\partial\vartheta_h/\partial z$ make sense.

Here $\partial/\partial z$ is the derivative of the expression taken in the direction Z. Let $A(s) = j(Z_0 + sZ)$. Then $A(s)^2$ is self-adjoint for every $s \in \mathbb{R}$ and is analytic in the real variable s. By [16, Theorem S6.3], the eigenvalues and normalized eigenvectors of $A(s)^2$ can be chosen to depend analytically on s. A curve $-\vartheta_k(Z_0 + sZ)^2$ corresponds to one of the analytic eigenvalue curves, and $\widehat{X}_k(Z_0 + sZ)$ corresponds to one of the analytic eigenvector curves of $A(s)^2$. We choose analytic curves such that $\vartheta_k(Z_0) = \vartheta_h(Z_0)$ and $\widehat{X}_k(Z_0) = \widehat{X}_h(Z_0)$. Then define

$$\frac{\partial \vartheta_h(Z_0)}{\partial z} = \frac{d\vartheta_k(Z_0 + sZ)}{ds}\bigg|_{s=0}, \quad \text{and} \quad \frac{\partial \widehat{X}_h(Z_0)}{\partial z} = \frac{d\widehat{X}_k(Z_0 + sZ)}{ds}\bigg|_{s=0},$$

which exist and are bounded.

While $\partial\widehat{X}_h/\partial z$ and $\partial\vartheta_h/\partial z$ may not be well defined, they are defined up to the choice of the curves $\vartheta_k(Z_0 + sZ)$ and $\widehat{X}_k(Z_0 + sZ)$, which is all we need, since

$$\begin{aligned}
&\big\langle \big(j(Z)j(Z_0) + j(Z_0)j(Z)\big)\widehat{X}_{m'}(Z_0), \widehat{X}_m\big\rangle \\
&= \left\langle \big(j(Z_0)^2 + \vartheta_h(Z_0)^2\big)\frac{\partial \widehat{X}_h(Z_0)}{\partial z}, \widehat{X}_m \right\rangle \\
&= \big(\vartheta_h(Z_0)^2 - \vartheta_h(Z_0)^2\big)\left\langle \frac{\partial \widehat{X}_h(Z_0)}{\partial z}, \widehat{X}_m \right\rangle = 0,
\end{aligned}$$

and we are done. □

Proof of Theorem 4.4. – This follows from the formulation of the length spectrum given in Theorem 2.4.

Consider elements of the form $\gamma = \exp(Z)$ with $Z \neq 0$. In this case, $V = 0$, $Z_V = 0$, $Z_V^{\perp} = Z$, and $K_V^{\perp}(X_2 + Z_0) = K(X_2 + Z_0)$ for all $X_2 \in \mathfrak{V}$, $Z_0 \in \mathfrak{z} - \{0\}$. Hence all the periods of γ are of the form $|Z|/|K(X_2 + Z_0)|$ where $X_2 + Z_0$ satisfies (i)–(iv) of Theorem 2.4. Note that (ii) is automatically satisfied, as $\ker j(Z_0)$ is empty by the nonsingularity assumption. The fact that these periods must correspond now follows from Lemma 4.6 and the first and third hypotheses of Theorem 4.3.

Consider elements of the form $\gamma = \exp(V + Z_V + Z_V^{\perp})$ with $V \neq 0$. By the nonsingularity assumption, $[V, \mathfrak{n}] = \mathfrak{z}$, so $Z_V^{\perp} = 0$ and $K_V^{\perp}(X + Z_0) \equiv 0$. By Theorem 2.4, the unique period of γ is $|V|$. Geometrically, the closed geodesics corresponding to these periods are horizontal in the Riemannian submersion $N(j,\Gamma) \to N_0(j,\Gamma)$. Thus, the set of periods of the form $|V|$ is precisely the length spectrum of the quotient torus $N_0(j,\Gamma)$. (See [25] for more details.)

It is known that pairs of flat tori are isospectral if and only if they have the same length spectrum, including multiplicities. The result now follows from the second hypothesis of Theorem 4.3. □

Remark 4.8. – The following pair of isospectral two-step nilmanifolds was constructed using a generalization of Theorem 4.3 [22, Proposition 3.7]. This is the only example of a pair of isospectral nilmanifolds whose length spectrum could not previously be compared and that is not subsumed by Theorem 4.4. We prove by a direct computation in Theorem 4.11 below that even this example has the same length spectrum. In summary, all known examples of isospectral two-step Riemannian nilmanifolds also have the same length spectrum, ignoring multiplicities. The authors have recently shown that all examples constructed using the generalized method must have the same length spectrum; a manuscript is in preparation.

Example 4.9 ([22, Items 2.3, 3.9, 3.10, 3.11]). – Let a_1, a_2, a_3 be integers satisfying $0 < a_1 < a_2 < a_3$. Define j_a by

$$j_a = \begin{pmatrix} 0 & -a_1 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix}.$$

Let $b_1, b_2, b_3, b'_1, b'_2, b'_3$, be integers with g.c.d.$(b_1, b_2, b_3) =$ g.c.d.(b'_1, b'_2, b'_3) such that there exists u in the interval

$$\left[\max\left\{\frac{-{b_1}^2}{{a_2}^2 - {a_1}^2}, \frac{-{b_3}^2}{{a_3}^2 - {a_2}^2}\right\}, \frac{{b_2}^2}{{a_3}^2 - {a_1}^2}\right]$$

satisfying

$$(*) \qquad \begin{aligned} {b'_1}^2 &= {b_1}^2 + u\left({a_2}^2 - {a_1}^2\right), \\ {b'_2}^2 &= {b_2}^2 + u\left({a_1}^2 - {a_3}^2\right), \\ {b'_3}^2 &= {b_3}^2 + u\left({a_3}^2 - {a_2}^2\right). \end{aligned}$$

Define j_b by

$$j_b = \begin{pmatrix} 0 & 0 & b_1 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_1 & 0 & 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -b_2 & 0 & -b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define $j_{b'}$ analogously.

Let $\mathfrak{V}$ be six-dimensional Euclidean space with the standard inner product, and standard basis denoted $\{e_1, e_2, \ldots, e_6\}$. Let $\mathfrak{z}$ be two-dimensional Euclidean space with standard inner product, and standard basis denoted $\{\zeta_a, \zeta_b\}$. Define $j : \mathfrak{z} \to \mathrm{so}(\mathfrak{V})$ by $j(z_1\zeta_1 + z_2\zeta_2) = z_1 j_a + z_2 j_b$. Likewise, define $j' : \mathfrak{z} \to \mathrm{so}(\mathfrak{V})$ by $j'(z_1\zeta_1 + z_2\zeta_2) = z_1 j_a + z_2 j_{b'}$. By comparing their characteristic polynomials, one computes that $j \sim j'$.

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ and $(\mathfrak{n}', \langle\ ,\ \rangle)$ denote the resulting metric Lie algebras, and $N(j)$, $N(j')$ the corresponding simply connected Lie groups. Let Γ denote the image under the exponential mapping of the integer lattice of $\mathfrak{n}$ (i.e., the integer span of $e_1,\dots,e_6,\zeta_1,\zeta_2$). Define Γ' analogously. One easily checks that Γ (respectively, Γ') is a cocompact, discrete subgroup of $N(j)$ (respectively, $N(j')$).

THEOREM 4.10 ([22]). – *The manifolds $N(j,\Gamma_{a,b})$ and $N(j',\Gamma'_{a,b'})$ are isospectral.*

THEOREM 4.11. – *The manifolds $N(j,\Gamma_{a,b})$ and $N(j',\Gamma'_{a,b'})$ have the same length spectrum.*

Proof. – The central lengths are equal by Theorem 4.6 above, so we need only consider the noncentral lengths.

Let $\gamma=\exp(V+Z)\in\Gamma,\ V\neq 0$. A straightforward calculation shows that if

$$V\in\operatorname{span}_{\mathbb{R}}\{e_2,\ e_4,\ e_6,\ b_3e_1-b_2e_3+b_1e_5\},$$

then $[V,\mathfrak{n}]=\operatorname{span}\{Z_2\}$; otherwise $[V,\mathfrak{n}]=\mathfrak{z}$. Likewise for $\gamma'=\exp(V'+Z')\in\Gamma',\ V'\neq 0$, if

$$V'\in\operatorname{span}_{\mathbb{R}}\left\{e_2',\ e_4',\ e_6',\ b_3'e_1'-b_2'e_3'+b_1'e_5'\right\},$$

then $[V',\mathfrak{n}']=\operatorname{span}\{Z_2\}$; otherwise $[V',\mathfrak{n}']=\mathfrak{z}$.

Assume γ satisfies

$$V=n_1e_2+n_2e_4+n_3e_6+n_4(b_3e_1-b_2e_3+b_1e_5)/\text{g.c.d.}(b_1,b_2,b_3),$$

i.e., $[V,\mathfrak{n}]=\operatorname{span}\{Z_2\}$. Now let

$$\omega=\sqrt{|V|^2+\frac{|Z_V^{\perp}|^2}{|K_V^{\perp}(X_2+Z_0)|^2}}$$

be a period of γ. The requirement that $V\in\ker j(Z_0)$ of Theorem 2.4 implies that $Z_0\in\operatorname{span}\{Z_1\}$. Let

$$V'=n_1e_2'+n_2e_4'+n_3e_6'+n_4\big(b_3'e_1'-b_2'e_3'+b_1'e_5'\big)/\text{g.c.d.}\big(b_1',b_2',b_3'\big).$$

Note that $|V|=|V'|$, by $(*)$ and by the assumption on the greatest common divisors. Also, $V'\in\ker j'(Z_0)$. Now define $X_2{}'$ as a vector in $\mathfrak{V}'$, such that $\langle X_2',W_m'(Z_0)\rangle=\langle X_2,W_m(Z_0)\rangle$ for all $m=1,\dots,\mu(Z_0)$. Then as in the proof of 4.5 above, $K(X_2+Z_0)=K'(X_2'+Z_0)$. Since $V\in\ker j(Z_0)$, $V'\in\ker j'(Z_0)$ and $[V,\mathfrak{n}]=[V',\mathfrak{n}']$, we have $Z_V^{\perp}=Z_{V'}^{\perp}$, and $K_V^{\perp}(X_2+Z_0)=K'^{\perp}_{V'}(X_2'+Z_0)$.

Note that we have constructed a length-preserving correspondence between vectors $\overline{V}$ in $\pi(\Gamma)$ and $\pi(\Gamma')$ that have the property $\dim[\pi^{-1}(\overline{V}),\mathfrak{n}]<2$. Since $\pi(\Gamma)$ and $\pi(\Gamma')$ produce isospectral tori, this correspondence must extend to a length-preserving correspondence between the entire lattices $\pi(\Gamma)$ and $\pi(\Gamma')$. In particular, if $[V,\mathfrak{n}]=\mathfrak{z}$, there exists $V'\in\mathfrak{V}'$ with $|V'|=|V|$ and $[V',\mathfrak{n}']=\mathfrak{z}$, which implies $Z_V^{\perp}=0$, and we are done. $\square$

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(Manuscript received September 17, 1998;
accepted, after revision, May 6, 1999.)

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