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On the relationship between distinction and irreducibility of parabolic induction

Sur la relation entre distinction et irréductibilité de l'induction parabolique

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ABSTRACT

Let U_{2n} denote the quasi-split unitary group over $2n$ variables with respect to a quadratic extension E/F of p-adic fields. In this short note, we relate $\mathrm{GL}_n(F)$-distinction of ladder representations of $\mathrm{GL}_n(E)$ with irreducibility of its Siegel parabolic induction in U_{2n}.

RÉSUMÉ

Soit U_{2n} le groupe unitaire quasi déployé à $2n$ variables associé à une extension E/F de corps p-adiques. Dans cette courte note, nous établissons un lien entre la propriété de distinction par $\mathrm{GL}_n(F)$ d'une représentation en échelle de $\mathrm{GL}_n(E)$ et l'irréductibilité de son induite parabolique.

1. Introduction

Let G be a topological group, H a closed subgroup of G and χ a complex valued character of the group H. A smooth complex valued representation (π, V) of G is said to be (H, χ)-distinguished (or simply H-distinguished when χ is the trivial character) if there exists a non-zero linear functional ℓ on V such that $\ell(\pi(h)v) = \chi(h)\ell(v)$ for all $h \in H$ and $v \in V$. Suppose now that E/F is a quadratic extension of non-Archimedean local fields of characteristic zero. Let $\omega = \omega_{E/F}$ denote the quadratic character of $F^\times$, whose kernel is the image of the norm map from $E^\times$ to $F^\times$. By composing it with the determinant map, we get a complex valued character of $\mathrm{GL}_n(F)$ of order two, which we again denote by ω. In what follows, $G = \mathrm{GL}_n(E)$, $H = \mathrm{GL}_n(F)$, and χ will either be the trivial character of $\mathrm{GL}_n(F)$ or the character ω defined above.

The study of distinguished representations in general has been central to representation theory of p-adic groups. In particular, the question of classifying and characterizing $(\mathrm{GL}_n(F), \chi)$-distinguished representations of $\mathrm{GL}_n(E)$ has been dealt with in several references in the past (see, for instance, [15], [8], [12], [6]).

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Let $a \mapsto \overline{a}$ denote the action of the non-trivial involution in $\mathrm{Gal}(E/F)$. It induces a natural involution on $\mathrm{GL}_n(E)$, which we also denote by $g \mapsto \overline{g}$. For an irreducible representation π of $\mathrm{GL}_n(E)$, denote by $\overline{\pi}$ the representation of $\mathrm{GL}_n(E)$ given by $\overline{\pi}(g) = \pi(\overline{g})$ for $g \in \mathrm{GL}_n(E)$. It was shown in [3, Proposition 12] that any irreducible representation of $\mathrm{GL}_n(E)$ that is $(\mathrm{GL}_n(F), \chi)$-distinguished satisfies $\overline{\pi} \cong \pi^\vee$, where $\pi^\vee$ denotes the contragredient of π. The converse of this statement, although not true in general (see, for instance, [7, §8]), holds for many interesting irreducible representations of $\mathrm{GL}_n(E)$. As shown in [6], an appropriate form of the converse holds for ladder representations of $\mathrm{GL}_n(E)$, a set of irreducible representations of general linear groups introduced in [9]. We remark that ladder representations are known to have several interesting properties, which make them important objects of study for questions concerning general linear groups. A noteworthy subset of the set of all ladder representations is the set of all Speh representations, which are known to be the 'building blocks' of the unitarizable dual of general linear groups (see [17]).

Another interesting aspect of (H, χ)-distinguished representations of G is their relation to the image of the standard base change map (as well as the twisted base change map), which takes irreducible representations of U_n (the quasi-split unitary group over n variables with respect to E/F) to irreducible representations of $\mathrm{GL}_n(E)$, via the local Langlands correspondence for the two groups. A precise statement in this context was obtained for cuspidal representations of $\mathrm{GL}_3(E)$ in [2], which was later generalized to the class of ladder representations in [7]. In this article, we demonstrate yet another aspect of (H, χ)-distinction for ladder representations of $\mathrm{GL}_n(E)$, by relating it to the question of irreducibility of parabolic induction in the group U_{2n}.

We now give a more specific idea of the main result of this paper. For an irreducible representation π of $\mathrm{GL}_n(E)$, let $\pi \rtimes \mathbf{1}_0$ denote the parabolic induction of π from the standard Siegel–Levi subgroup of U_{2n}. Recall that a proper ladder representation of $\mathrm{GL}_n(E)$ is a ladder representation that is not parabolically induced from any proper Levi subgroup of $\mathrm{GL}_n(E)$. (For instance, Speh representations are examples of proper ladders.) It was shown in [6] that a proper ladder representation is at most one of H-distinguished or (H, ω)-distinguished. For simplicity, here in the introduction, we state our main result only for the special case of proper ladders; see Theorem 2 for the general case. We refer the reader to §2 for any unexplained notation.

Theorem 1. *Let $\pi = L(\mathfrak{m})$ be a proper ladder representation of $\mathrm{GL}_n(E)$. Suppose that $\overline{\pi} \cong \pi^\vee$ and let t denote the cardinality of the multi-set $\mathfrak{m}$. Then π is $(\mathrm{GL}_n(F), \omega^{t+1})$-distinguished if and only if the representation $\pi \rtimes \mathbf{1}_0$ is irreducible.*

Theorem 1 extends [2, Corollary 1.4], which proves the result for the case of discrete series representations (i.e. when $t = 1$). The result is not true for general irreducible representations (see Remark 1) and we do not expect it to potentially provide us with more examples of (H, χ)-distinguished representations of G than what is already known. However, this connection between distinction and irreducibility of parabolic induction is useful. (For instance, the cuspidal case of the result has been crucially used in the proofs of several results in [14] and [13] which deal with $\mathrm{Sp}_{2n}(F)$-distinguished representations of U_{2n}.) We expect this connection to be of further use in future, in more questions concerning the group U_{2n}, especially in the ones where ladder representations play a role.

Furthermore, we expect such a connection between distinction and irreducibility of the Siegel parabolic induction to exist for any arbitrary classical group defined over a p-adic field, for distinction by an appropriate subgroup of the corresponding general linear group. We defer the investigation of this problem to a later work.

2. Notation and preliminaries

Let E/F be a quadratic extension of a non-Archimedean local field of characteristic zero. Let $|\cdot|_E$ denote the absolute value of E. We denote by $a \mapsto \bar{a}$ the action of the non-trivial element of $\mathrm{Gal}(E/F)$. Let $\omega = \omega_{E/F}$ denote the quadratic character of $F^\times$, whose kernel is the image of the norm map from $E^\times$ to $F^\times$. Set

$$J_n = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}$$

where $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n(E)$ and define

$$U_{2n} = \{g \in \mathrm{GL}_{2n}(E) \mid {}^t\bar{g}\, J_n\, g = J_n\}.$$

By standard Siegel parabolic in U_{2n} we refer to the standard parabolic subgroup $P \subseteq U_{2n}$ whose Levi component is isomorphic to $\mathrm{GL}_n(E)$. As usual, we denote the normalized parabolic induction in U_{2n} (resp., $\mathrm{GL}_n(E)$) by $\rtimes$ (resp., $\times$).

2.1. Definition of distinction

Let π be a representation of $\mathrm{GL}_n(E)$ and $H = \mathrm{GL}_n(F)$. Let χ be a character of H. We say that π is (H, χ)-distinguished if there exists a non-zero linear form ℓ on the space of π such that $\ell(\pi(h)v) = \chi(h)\ell(v)$ for all $h \in H$ and v in the space of π.

Let ω be the character of H given by composing the character $\omega_{E/F}$ of $F^\times$ with the determinant map. In this paper, the character χ will either be the trivial character of H or the character ω. If $\chi = 1$, we will shorten the phrase "$(H, 1)$-distinguished" to "H-distinguished".

2.2. Representation theory of general linear groups

Let $\Pi(\mathrm{GL}_n(E))$ be the category of smooth representations of $\mathrm{GL}_n(E)$ of finite length and let $\mathrm{Irr}(\mathrm{GL}_n(E))$ be the class of irreducible representations in $\Pi(\mathrm{GL}_n(E))$. Let $\mathrm{Cusp}(\mathrm{GL}_n(E))$ be the set of all cuspidal representations in $\mathrm{Irr}(\mathrm{GL}_n(E))$. Set

$$\mathrm{Cusp}_{\mathrm{GL}} = \sqcup_{n=1}^{\infty}\mathrm{Cusp}(\mathrm{GL}_n(E)), \quad \mathrm{Irr}_{\mathrm{GL}} = \sqcup_{n=0}^{\infty}\mathrm{Irr}(\mathrm{GL}_n(E)).$$

Let $\nu(g) = |\det g|_E$ for any $g \in \mathrm{GL}_n(E)$ and $n \in \mathbb{N}$. We will freely use in this paper the notation and terminology introduced in [18]. A Zelevinsky segment is a subset of $\mathrm{Cusp}_{\mathrm{GL}}$ of the form

$$[a,b]_{(\rho)} = \{\nu^i\rho \mid i = a, a+1, \dots, b\}$$

where $a, b \in \mathbb{R}$ such that $b-a+2$ is a positive integer and $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ (so that $[a, a-1]_{(\rho)}$ is the empty segment). Segments will often be denoted by Δ and finite multi-sets of segments by $\mathfrak{m}$. We denote the set of all finite multi-sets of segments by $\mathcal{O}$. For $\mathfrak{m} \in \mathcal{O}$, we write $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ as an unordered t-tuple. When choosing a specific order, by abuse of notation we write $\mathfrak{m} = (\Delta_1, \dots, \Delta_t)$ as an ordered t-tuple. The Langlands classification provides a bijection between the sets $\mathcal{O}$ and $\mathrm{Irr}_{\mathrm{GL}}$. We denote by $\mathfrak{m} \mapsto L(\mathfrak{m})$ the above bijection (see, e.g., [16]).

The action of the non-trivial element of $\mathrm{Gal}(E/F)$ induces a natural involution on $\mathrm{GL}_n(E)$, which we denote by $g \mapsto \overline{g}$. For $\pi \in \mathrm{Irr}(\mathrm{GL}_n(E))$, denote by $\overline{\pi}$ the representation in $\mathrm{Irr}(\mathrm{GL}_n(E))$ given by $\overline{\pi}(g) = \pi(\overline{g})$. Let $\pi^\vee$ denote the contragredient of a representation $\pi \in \mathrm{Irr}(\mathrm{GL}_n(E))$. For a Zelevinsky segment Δ, we set $\Delta^\vee = \{\rho^\vee \mid \rho \in \Delta\}$ and $\overline{\Delta} = \{\overline{\rho} \mid \rho \in \Delta\}$. We also let $\mathfrak{m}^\vee = \{\Delta_1^\vee, \dots, \Delta_t^\vee\}$ for $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ and similarly define the multi-set $\overline{\mathfrak{m}}$.

For $\pi \in \mathrm{Irr}_{\mathrm{GL}}$, let $\mathrm{Supp}\,(\pi)$ denote the cuspidal support of π as defined in [18, §1.10]. For $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ define its *cuspidal line*,

$$\rho^{\mathbb{Z}} = \{\nu^m\rho \mid m \in \mathbb{Z}\}.$$

For a representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ with central character z_π, let $\alpha = \exp(\pi) \in \mathbb{R}$ be the exponent of π. It is the unique real number such that $\nu^{-\alpha}z_\pi$ is a unitary character. For a segment $\Delta = [a,b]_{(\rho)}$, define its exponent $\exp(\Delta)$ to be $\frac{(a+b)}{2}\exp(\rho)$. Thus we have $\exp(\Delta) = \exp(L(\Delta))$. For a multi-set $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ of segments, let $\mathfrak{m}_{>0}$ be the multi-set defined by

$$\mathfrak{m}_{>0} = \{\Delta_i \mid 1 \le i \le t,\ \exp(\Delta_i) > 0\}.$$

2.3. Ladder representations

Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$. Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ where $\Delta_i = [a_i, b_i]_{(\rho)}$ ($a_i, b_i \in \mathbb{Z}$). By renumbering the segments if required, we can assume that $a_1 \ge \dots \ge a_t$. Then $\mathfrak{m}$ is called a *ladder* multi-set if further we have

$$a_1 > \dots > a_t \quad \text{and} \quad b_1 > \dots > b_t.$$

It is called a *proper ladder* if additionally $a_i \le b_{i+1}+1$ for all $i = 1, \dots, t-1$. A representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ is called a ladder (resp., proper ladder) representation if $\pi = L(\mathfrak{m})$ where $\mathfrak{m}$ is a ladder (resp., proper ladder) multi-set. It follows directly from [9, Theorem 16] that any ladder representation π can be written as

$$\pi = \pi_1 \times \dots \times \pi_k \tag{1}$$

where each π_i is a proper ladder. The decomposition is unique up to a reordering of the π_i.

3. Main result

Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be such that $\overline{\rho} \cong \rho^\vee$. It is known that $\nu^x\rho \rtimes \mathbf{1}_0$ is reducible for precisely one $x \in \mathbb{R}_{\ge 0}$ and the reducibility point lies in the set $\{0, \frac{1}{2}\}$ (by [5, Theorems 3.1 and 3.2]). We begin by noting the cuspidal case of Theorem 1, which is a special case of [2, Corollary 1.4].

Proposition 1. *Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be such that $\overline{\rho} \cong \rho^\vee$. Then the representation ρ is H-distinguished if and only if $\rho \rtimes \mathbf{1}_0$ is irreducible.*

Next we require the following easy consequence of the irreducibility results obtained in [11]. Define

$$\mathcal{S} = \{\rho \in \mathrm{Cusp}_{\mathrm{GL}} \mid \rho \rtimes \mathbf{1}_0 \text{ is reducible}\}.$$

Lemma 1. *Let $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ be a ladder representation such that $\overline{\pi} \cong \pi^\vee$. Then $\pi \rtimes \mathbf{1}_0$ is irreducible if and only if $\mathcal{S} \cap \mathrm{Supp}\,(\pi) = \emptyset$.*

Proof. The "only if" part of the lemma is a direct consequence of [11, Theorem 1.1]. For the "if" part, let $\pi = L(\mathfrak{m})$ where $\mathfrak{m} = (\Delta_1, \ldots, \Delta_t)$ be a ladder multi-set ordered such that $\exp(\Delta_i) > \exp(\Delta_{i+1})$, $i = 1, \ldots, t-1$. Since $\pi^\vee = \overline{\pi}$, we have that $\Delta^\vee_{t-i+1} \cong \overline{\Delta_i}$. Note that in this case $(\overline{\mathfrak{m}}^\vee)_{>0} = \mathfrak{m}_{>0} = \{\Delta_1, \ldots, \Delta_{[\frac{t}{2}]}\}$ and the representation $L(\mathfrak{m}_{>0}) \times L(\mathfrak{m}_{>0})$ is irreducible (see [10, Proposition 4.6(b)]). The result now follows from a direct application of [11, Theorem 1.2]. $\square$

Theorem 2. *Let $\pi = L(\mathfrak{m}) \in \mathrm{Irr}_{\mathrm{GL}}$ be a ladder representation. Suppose that $\overline{\pi} \cong \pi^\vee$. Let t denote the cardinality of the multi-set $\mathfrak{m}$. Write $\pi = \pi_1 \times \cdots \times \pi_k$ (as in eq. (1)) where π_i are proper ladder representations.*

(1) *Suppose that k is an odd integer. Then π is (H, ω^{t+1})-distinguished if and only if $\pi \rtimes \mathbf{1}_0$ is irreducible. In this case, π is not both H-distinguished and (H, ω)-distinguished.*
(2) *Suppose that k is an even integer. Then $\pi \rtimes \mathbf{1}_0$ is irreducible, and π is both H-distinguished and (H, ω)-distinguished.*

Proof. Let $\pi = \pi_1 \times \cdots \times \pi_k$, where each π_i is a proper ladder representation. By rearranging the π_i if required, we assume that $\exp(\pi_i) > \exp(\pi_{i+1})$, $i = 1, \ldots, k-1$. Let $\pi = L(\mathfrak{m})$ where $\mathfrak{m} = (\Delta_1, \ldots, \Delta_t)$ be a ladder multi-set ordered such that $\exp(\Delta_i) > \exp(\Delta_{i+1})$, $i = 1, \ldots, t-1$. Let $\mathfrak{m}_i$ be such that $\pi_i = L(\mathfrak{m}_i)$. Since $\overline{\pi} \cong \pi^\vee$, we have $\overline{\pi_i} \cong \pi^\vee_{k-i+1}$, $i = 1, \ldots, k$ and $\overline{\Delta_i} \cong \Delta^\vee_{t-i+1}$, $i = 1, \ldots, t$. Moreover, there exists $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ such that $\overline{\rho} \cong \rho^\vee$ and $\mathrm{Supp}\,(\pi) \subseteq \rho^{\mathbb{Z}} \sqcup (\nu^{1/2}\rho)^{\mathbb{Z}}$. Set $\rho' = \rho$ if $\mathrm{Supp}\,(\pi) \subseteq \rho^{\mathbb{Z}}$ and $\rho' = \nu^{1/2}\rho$ otherwise (i.e. $\mathrm{Supp}\,(\pi) \subseteq (\rho')^{\mathbb{Z}}$).

Suppose that k is an even integer. Observe that, in this case, t is also an even integer, $\Delta_{t/2} \in \mathfrak{m}_{k/2}$ and $\Delta_{t/2+1} \in \mathfrak{m}_{k/2+1}$. Write $\Delta_{t/2} = [a, b]_{(\rho)}$ with $a, b \in \frac{1}{2}\mathbb{Z}$ such that $b - a \in \mathbb{Z}$. Then $\Delta_{t/2+1} = [-b, -a]_{(\rho)}$. Since these two segments belong to different proper ladder multi-sets of a ladder multi-set, we have that $a \geq 1$. Thus neither ρ nor $\nu^{1/2}\rho$ lies in $\mathrm{Supp}\,(\pi)$. By Lemma 1, we get that, in this case, $\pi \rtimes \mathbf{1}_0$ is irreducible. The fact that π is both H-distinguished and (H, ω)-distinguished is demonstrated in [6, Theorem 4.3]. This finishes the proof of part (2).

We now prove part (1). Suppose that k is an odd integer. Let ρ and ρ' be as defined above. By [8, Theorem 7] or [1] (see also [6, Proposition 3.8]), there exists a unique element $\gamma(\rho) \in \{0, 1\}$ such that ρ is $(H, \omega^{\gamma(\rho)})$-distinguished. Define

$$\gamma(\rho') = \begin{cases} \gamma(\rho) & \text{if } \rho' = \rho \\ 1 - \gamma(\rho) & \text{if } \rho' = \nu^{1/2}\rho. \end{cases}$$

We will deal with the two cases, namely whether or not $\rho \cong \rho'$, separately. Assume first that $\rho' \cong \nu^{1/2}\rho$. By [6, Theorem 4.6], we have that in this case π cannot be both H-distinguished or (H, ω)-distinguished. Furthermore, by [6, Theorem 4.3], π is (H, ω^{t+1})-distinguished if and only if $\gamma(\rho') = 0$, which in this case is equivalent to the statement that ρ is (H, ω)-distinguished. By Proposition 1, this is equivalent to the statement that the representation $\rho \rtimes \mathbf{1}_0$ is reducible. Note that, since k is odd, $\pi^\vee_{(k+1)/2} \cong \overline{\pi_{(k+1)/2}}$, and that $\rho' \in \mathrm{Supp}\,(\pi_{(k+1)/2})$. Thus $\mathrm{Supp}\,(\pi) \cap \mathcal{S} = \emptyset$ if and only if $\rho' \rtimes \mathbf{1}_0$ is irreducible. This implies, by [5, Theorems 3.1 and 3.2], that $\mathrm{Supp}\,(\pi) \cap \mathcal{S} = \emptyset$ if and only if $\rho \rtimes \mathbf{1}_0$ is reducible. Thus π is (H, ω^{t+1})-distinguished if and only if $\mathrm{Supp}\,(\pi) \cap \mathcal{S} = \emptyset$. Appealing to Lemma 1 now proves part (1) when $\rho' \cong \nu^{1/2}\rho$. The case when $\rho' \cong \rho$ is dealt with similarly and thus we omit the proof. $\square$

Remark 1. Evidently, the statement of Theorem 2 holds for Speh representations, which are the 'building blocks' of the unitarizable dual of general linear groups. However, the statement does not hold for an arbitrary unitarizable representation. To see this, consider a representation $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ that is (H, ω)-distinguished. By [12, Theorem 4.2] and [4, Proposition 26], the representation $\rho \times \rho$ is H-distinguished and (H, ω)-distinguished, respectively. By Proposition 1, we get that $(\rho \times \rho) \rtimes \mathbf{1}_0$ is reducible.

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