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Characterization of Kummer hypergeometric Bernoulli polynomials and applications

Sur une caractérisation des polynômes hypergéométriques de Bernoulli–Kummer et applications

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ABSTRACT

In this paper, we present two characterizations of the sequences of Kummer hypergeometric polynomials $B_{a,b,n}(x)$ and Kummer hypergeometric polynomials of the second kind $K_{a,b,n}(x)$, which are respectively defined by the exponential generating functions:

$$\frac{e^{xt}}{M(a,a+b;t)} = \sum_{n=0}^{\infty} B_{a,b,n}(x)\frac{t^n}{n!} \text{ and } \frac{e^{xt}}{U(a,a+b;t)} = \sum_{n=0}^{\infty} K_{a,b,n}(x)\frac{t^n}{n!}$$

$$\text{with } M(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{t^n}{n!},$$

where $U(a,a+b;t)$ is the Kummer hypergeometric function of the second kind.
First we construct Gauss–Weierstrass-type convolution operators $T_{w_{a,b}}$ with a well-chosen kernel (density) function for each sequence of Kummer hypergeometric polynomials and for Kummer hypergeometric polynomials of the second kind. Then we characterize Kummer hypergeometric polynomials as the only Appell polynomials having a weighted-integral mean equal to zero. Our approach is inspired by the Gauss–Weierstrass convolution transform for Hermite polynomials and the Kummer integral representation for confluent hypergeometric functions.

RÉSUMÉ

Dans cet article, nous présentons deux caractérisations des suites $B_{a,b,n}(x)$ et $K_{a,b,n}(x)$ de polynômes hypergéométriques de type Kummer définies par leurs fonctions génératrices :

$$\frac{e^{xt}}{M(a,a+b;t)} = \sum_{n=0}^{\infty} B_{a,b,n}(x)\frac{t^n}{n!} \text{ et } \frac{e^{xt}}{U(a,a+b;t)} = \sum_{n=0}^{\infty} K_{a,b,n}(x)\frac{t^n}{n!}$$

$$\text{avec } M(a,b;t) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{t^n}{n!},$$

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où $U(a, a+b; t)$ est la fonction hypergéométrique de Kummer de seconde espèce. Premièrement, nous construisons des opérateurs de convolution $T_{w_{a,b}}$ du type Gauss–Weierstrass pour chacune des suites de polynômes de Kummer de première et de seconde espèces. Deuxièmement, nous caractérisons les polynômes hypergéométriques de Kummer $B_{a,b,n}(x)$ comme étant les seuls polynômes ayant une moyenne intégrale pondérée égale à zero. Cette approche nous a été inspirée par la transformation de Gauss–Weierstrass pour les polynômes de Hermite et par la représentation intégrale de type Euler–Kummer pour les fonctions hypergéométriques.

1. Introduction

The series representation of the Kummer-confluent hypergeometric function $M(a,b;z)$ is given by

$$M(a,b;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \quad \text{for} \quad b\neq 0,-1,-2,\dots, \tag{1.1}$$

where $(a)_n=a(a+1)(a+2)\dots(a+n-1)$ is the rising factorial of a (also called the Pochhammer symbol of a). $M(a,b;x)$ was introduced as a solution to the Kummer hypergeometric differential equation:

$$xy''(x)+(b-x)y'(x)-ay(x)=0.$$

The general solution is given by

$$y=AM(a,b:x)+Bx^{1-b}M(a-b+1,2-b;x),$$

where A and B are arbitrary constants.

By using $\mathrm{e}^{zt}=\sum_{n=0}^{\infty}\frac{z^n t^n}{n!}$ and the classical beta function $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-a-1}\,\mathrm{d}t=\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}$, we can obtain another interesting representation of $M(a,b;x)$ that will be used in this paper. It is the Kummer integral representation, valid for $\Re b>\Re a>0$ [1, Chapter 13]:

$$M(a,b;z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 \mathrm{e}^{zt}t^{a-1}(1-t)^{b-a-1}\,\mathrm{d}t. \tag{1.2}$$

By replacing t with $1-t$ in the integral representation (1.2), we obtain the Kummer transformation formula [1, Chapter 13]:

$$M(a,b;z)=\mathrm{e}^{z}M(b-a,b;-z). \tag{1.3}$$

Notice that (1.3) also follows from Pfaff's transformation formula for the Gauss hypergeometric function.

Another independent solution to the Kummer differential equation is the Kummer hypergeometric function of the second kind $U(a,b;z)$ having the integral representation:

$$U(a,b;x)=\frac{1}{\Gamma(a)}\int_0^{\infty}\mathrm{e}^{-xt}t^{a-1}(1+t)^{b-a-1}\,\mathrm{d}t,\ \text{for } a>0, x>0. \tag{1.4}$$

A sequence of polynomials $\{P_n(x)\}$ is said to be an Appell polynomial [2] if

$$P_0(x)=1 \qquad \text{and} \quad P_n'(x)=nP_{n-1}(x).$$

This is equivalent to the existence of a continuous function $R(t)$ with $R(0)\neq 0$, such that

$$\frac{\mathrm{e}^{xt}}{R(t)}=\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{n!}.$$

Let us introduce the generating functions:

$$\frac{1}{M(a,a+b;t)}=\sum_{n=0}^{\infty}B_{a,b,n}\frac{t^n}{n!}, \tag{1.5}$$

$$\frac{e^{xt}}{M(a,a+b;t)} = \sum_{n=0}^{\infty} B_{a,b,n}(x)\frac{t^n}{n!}, \tag{1.6}$$

and

$$\frac{e^{xt}}{U(a,a+b;t)} = \sum_{n=0}^{\infty} K_{a,b,n}(x)\frac{t^n}{n!}, \tag{1.7}$$

where $B_{a,b,n}$, $B_{a,b,n}(x)$, and $K_{a,b,n}(x)$ are respectively the Kummer hypergeometric numbers, the Kummer hypergeometric polynomials, and the Kummer hypergeometric polynomials of the second kind. Note that $B_{1,N,n}$ are the hypergeometric Bernoulli numbers $B_{N,n}$ that were introduced by Carlitz [7] and Howard [17,18]. Finally, $B_{1,1,n}$ are the Bernoulli numbers B_n.

A key importance of the Kummer hypergeometric function is its relationship with other classical functions, such as

$$M(a,a;z) = e^z,$$
$$M(a,a+1;-x) = \frac{a}{x^a}\gamma(a,x),$$
$$M(1/2,3/2;-x^2) = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x),$$

and

$$M(p+1/2,2p+1;2ix) = J_p(x)\frac{\Gamma(p+1)}{(\frac{x}{2})^p}e^{ix}.$$

The Kummer hypergeometric functions have a large number of applications in different areas of research such as quantum mechanics (waves) [13], atomic and molecular physics [5], elasticity theory, acoustics, strings, hydrodynamics, random walk theory, optics, fiber optics, binary stars, finances (pricing & options) [6], and in the theory of probability and mechanical statistics. For more applications, see [14] and the references therein.

Recently, various generalizations of Bernoulli polynomials have been extensively investigated (see, e.g., [9], [10], [11], [12], [15], [18], and [20]). In this paper, using the Weierstrass convolution integral, we present characterizations of Kummer hypergeometric polynomials $B_{a,b,n}(x)$ and of Kummer hypergeometric polynomials of the second kind $K_{a,b,n}(x)$. Using the reciprocal theorem, we obtain an explicit determinant form for the Kummer hypergeometric numbers. A nice application of combinatorics Riordan version of Faá Di Bruno's formula for derivatives of composite functions gives us a closed form for the Kummer hypergeometric numbers as well as for the Nörlund–Bernoulli numbers.

2. Analytic characterizations

By using the Cauchy product for power series, it is easy to see that

$$B_{a,b,n}(x+y) = \sum_{k=0}^{n}\binom{n}{k}B_{a,b,k}(x)y^{n-k} = \sum_{k=0}^{n}\binom{n}{k}B_{a,b,k}(y)x^{n-k} \tag{2.8}$$

and

$$x^n = \frac{\Gamma(a+b)}{\Gamma(b)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(a+n-k)}{\Gamma(a+b+n-k)}B_{a,b,k}(x). \tag{2.9}$$

By taking $n=0$, $n=1$ and $n=2$ in Formula (2.9), we respectively obtain:

$$B_{a,b,0}(x) = 1, \quad B_{a,b,1}(x) = x - \frac{a}{a+b}, \quad B_{a,b,2}(x) = x^2 - \frac{2a}{a+b}x + \frac{2a^2}{(a+b)^2} - \frac{a(a+1)}{(a+b)(a+b+1)}.$$

In general, we can compute explicitly the polynomials $B_{a,b,n}(x)$ by solving the equation $KB = X$, where K is the lower-triangular matrix:

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{a}{a+b} & 1 & 0 & 0 & \cdots & \vdots \\ \frac{(a)_2}{(a+b)_2} & \binom{2}{1}\frac{a}{a+b} & 1 & 0 & \cdots & \vdots \\ \frac{(a)_3}{(a+b)_3} & \binom{3}{1}\frac{(a)_2}{(a+b)_2} & \binom{3}{2}\frac{a}{a+b} & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{(a)_n}{(a+b)_n} & \binom{n}{1}\frac{(a)_{n-1}}{(a+b)_{n-1}} & \cdots & \cdots & \binom{n}{n-1}\frac{a}{a+b} & 1 \end{bmatrix}, \tag{2.10}$$

$B = (B_{a,b,0}(x), B_{a,b,1}(x), \dots, B_{a,b,n}(x))$ and $X = (1, x, x^2, \dots, x^n)$ are two uni-column matrices. In many situations, the matrix representation of these sequences of numbers provides a powerful computational tool for deriving a number of useful identities for a given sequence, for example the Pascal numbers, etc.

In 1885, Karl Weierstrass [22] introduced the following integral operator:

$$W(f)(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f(x+u) \mathrm{e}^{-\frac{u^2}{4}} \, \mathrm{d}u, \tag{2.11}$$

where f is a continuous function on the real line. Weierstrass used this transformation in his original proof of the well-known approximation theorem of continuous functions by polynomials on a closed interval. The transformation W is also known for its relation to the heat equation, which can be seen in the works by P. Appell in [3] and by E. Hille in [16].

Notice that W is a linear one-to-one operator that transforms any polynomial into another polynomial of the same degree. Furthermore, if $H_n(x)$ denotes a Hermite polynomial of degree n, then $W(H_n(x/2)) = x^n$. A natural question that arises is: can the relation $W(H_n(x/2)) = x^n$ characterize Hermite polynomials?

Let us introduce the following Gauss–Weierstrass type integral operator:

$$T_{w_{a,b}} f(x) = \int_0^1 f(x+u) w_{a,b}(u) \, \mathrm{d}u, \tag{2.12}$$

where $w_{a,b}(u) = \frac{u^{a-1}(1-u)^{b-1}}{B(a,b)}$ is such that $\int_0^1 w_{a,b}(u)\,\mathrm{d}u = 1$. The operator $T_{w_{a,b}}$ is linear and one-to-one.

The following two results present a characterization of the sequence of Kummer hypergeometric polynomials $B_{a,b,n}(x)$.

Lemma 2.1. *$B_{a,b,n}(x)$ is a sequence of Appell polynomials satisfying*

$$T_{w_{a,b}}(B_{a,b,n}(x) = \begin{cases} x^n & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases} \tag{2.13}$$

Proof. It follows from (2.8) that

$$\begin{aligned} \int_0^1 x^{a-1}(1-x)^{b-1} B_{a,b,n}(x)\,\mathrm{d}x &= \sum_{k=0}^{n} \binom{n}{k} B_{a,b,k} \int_0^1 x^{a+n-k-1}(1-x)^{b-1}\,\mathrm{d}x \\ &= \sum_{k=0}^{n} \binom{n}{k} B_{a,b,k} \frac{\Gamma(a+n-k)\Gamma(b)}{\Gamma(a+b+n-k)}. \end{aligned}$$

By applying identity (2.9) for $x = 0$, we obtain the result. □

The reverse implication is as follows.

Theorem 2.2. *Let $P_n(x)$ be a sequence of Appell polynomials satisfying*

$$\int_0^1 x^{a-1}(1-x)^{b-1} P_n(x)\,\mathrm{d}x = \begin{cases} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases} \tag{2.14}$$

Then $P_n(x) = B_{a,b,n}(x)$.

Proof. Let $G(x,t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}$. Since the sequences $P_n(x)$ are Appell, it easily follows that $G(x,t) = \frac{\mathrm{e}^{xt}}{R(t)}$, for a continuous function $R(t)$ with $R(0) \neq 0$. So,

$$\begin{aligned} \int_0^1 x^{a-1}(1-x)^{b-1} G(x,t)\,\mathrm{d}x &= \sum_{n=0}^{\infty} (\int_0^1 x^{a-1}(1-x)^{b-1} P_n(x)\,\mathrm{d}x \frac{t^n}{n!} \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \end{aligned}$$

On the other hand,

$$\int_0^1 x^{a-1}(1-x)^{b-1}G(x,t)\,\mathrm{d}x = \frac{1}{R(t)}\int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{e}^{xt}\,\mathrm{d}x$$
$$= \frac{1}{R(t)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}M(a,a+b;t).$$

Hence, $R(t) = M(a,a+b;t)$. The result follows by comparing coefficients to coefficients. □

Theorem 2.3. *The only sequence* $\{P_n(x)\}$ *of Appell polynomials satisfying* $T_{w_{a,b}}(P_n(x)) = x^n$ *is the sequence of Kummer hypergeometric polynomials* $B_{a,b,n}(x)$.

Proof. Note that, by Kummer integral representation, we have:

$$T_{w_{a,b}}(\mathrm{e}^{xt}) = \frac{\mathrm{e}^{xt}}{B(a,b)}\int_0^1 \mathrm{e}^{xu}u^{a-1}(1-u)^{b-1}\,\mathrm{d}u = \mathrm{e}^{xt}M(a,a+b;t). \tag{2.15}$$

Suppose that $\frac{\mathrm{e}^{xt}}{R(t)} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}$. By applying the operator $T_{w_{a,b}}$ to both sides, we obtain:

$$\frac{1}{R(t)}T_{w_{a,b}}(\mathrm{e}^{xt}) = \sum_{n=0}^{\infty} T_{w_{a,b}}(P_n(x))\frac{t^n}{n!}.$$

By using (2.13), we get

$$\frac{1}{R(t)}\mathrm{e}^{xt}M(a,a+b;t) = \mathrm{e}^{xt}.$$

Hence, $R(t) = M(a,a+b;t)$. □

For the case when $a = 1$, we obtain the following characterizations of the hypergeometric Bernoulli polynomials $B_{N,n}(x)$.

Corollary 2.4. *The only sequence of Appell polynomials satisfying* $T_{w_N}(P_n(x)) = x^n$ *is the sequence of hypergeometric Bernoulli polynomials* $B_{N,n}(x)$*, where*

$$T_{w_N} f(x) = \int_0^1 f(x+u)N(1-u)^{N-1}\,\mathrm{d}u. \tag{2.16}$$

Corollary 2.5. *The only sequence of Appell polynomials satisfying*

$$\int_0^1 (1-x)^{N-1}P_n(x)\,\mathrm{d}x = \begin{cases} \frac{1}{N} & \text{for } n = 0 \\ 0 & \text{for } n > 0 \end{cases} \tag{2.17}$$

is the sequence of hypergeometric Bernoulli polynomials $B_{N,n}(x)$.

In a similar way, we can introduce the following Gauss–Weierstrass integral operator

$$T_{w*_{a,b}} f(x) = \frac{1}{\Gamma(a)}\int_0^{\infty} f(x-u)w*_{a,b}(u)\,\mathrm{d}u, \quad \text{with} \quad w*_{a,b}(u) = u^{a-1}(1+u)^{b-a-1} \tag{2.18}$$

and obtain the following characterization of Kummer hypergeometric polynomials of the second kind $K_{a,b,n}(x)$.

Theorem 2.6. *Suppose that a sequence* $\{P_n(x)\}$ *of Appell polynomials satisfies* $T_{w*_{a,b}}(P_n(x)) = x^n$. *Then* $P_n(x)$ *is a Kummer hypergeometric polynomial of the second kind.*

3. Some closed forms for Kummer hypergeometric numbers

Given a series $\sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$. Then the coefficients b_n of the reciprocal series $\sum_{n=0}^{\infty} b_n x^n = (\sum_{n=0}^{\infty} a_n x^n)^{-1}$ are given by

$$b_n = \sum_{j=1}^{n} \frac{(-1)^j}{(a_0)^{j+1}} a_{k_1} a_{k_2} \dots a_{k_j}, \quad \text{for} \quad k_1 + k_2 + \dots + k_j = n. \tag{3.19}$$

The coefficients b_n can also be given in determinant form as in [19, p. 116]:

$$b_n = \frac{(-1)^n}{(a_0)^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & \vdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & a_3 & \ddots & \ddots & a_0 \\ a_n & \cdots & \cdots & \cdots & a_2 & a_1 \end{vmatrix}. \tag{3.20}$$

By applying (3.19), we obtain the following results.

Theorem 3.1. *The Kummer hypergeometric numbers are given by*

$$B_{a,b,n} = n!(-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & \vdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & a_3 & \ddots & \ddots & a_0 \\ a_n & \cdots & \cdots & \cdots & a_2 & a_1 \end{vmatrix}, \tag{3.21}$$

where $a_j = \frac{(a)_j}{j!(a+b)_j}$.

For the case when $a = 1$, we obtain an explicit determinant form for the hypergeometric Bernoulli numbers as follows.

Corollary 3.2. *The hypergeometric Bernoulli numbers are given by*

$$B_{N,n} = n!(-1)^n \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & \vdots \\ a_3 & a_2 & a_1 & a_0 & \cdots & \vdots \\ a_4 & a_3 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & a_3 & \ddots & \ddots & a_0 \\ a_n & \cdots & \cdots & \cdots & a_2 & a_1 \end{vmatrix}, \tag{3.22}$$

where $a_j = \frac{1}{(N+1)_j}$.

By using the Kummer's transformation formula (1.3), we obtain the following results.

Theorem 3.3.

$$B_{a,b,n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} B_{b,a,k} \tag{3.23}$$

and

$$(-1)^n B_{a,b,n}(x) = B_{b,a,n}(1-x). \tag{3.24}$$

In the case when $a = b = 1$, the identities (3.22) and (3.23) are reduced to the well-known recurrence identity for the Bernoulli numbers and the reflection identity for the Bernoulli polynomials.

Corollary 3.4.

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} B_k \tag{3.25}$$

and

$$B_n(1-x) = (-1)^n B_n(x). \tag{3.26}$$

As an application of the Euler type integral representation of the Kummer hypergeometric functions, we present some closed forms of the Kummer hypergeometric polynomials and numbers in terms of the Stirling numbers of the second kind $S(n,k)$.

Theorem 3.5. *For n non-negative integer, we have:*
(i)

$$B^r_{a,b,n}(x) = \sum_{k=0}^{n} (-1)^k (r)_k B_{n,k}\left(a_1, a_2, \dots a_{n-k+1}\right)$$

and ***(ii)***

$$B^r_{a,b,n} = \sum_{k=0}^{n} (-1)^k (r)_k B_{n,k}\left(\frac{a}{a+b}, \frac{a(a+1)}{(a+b)(a+b+1)}, \dots, \frac{(a)_{n-k+1)}}{(a+b)_{n-k+1)}}\right),$$

where $a_p = \sum_{j=0}^{p} \binom{p}{j} (-x/r)^{p-j} \frac{\Gamma(a+j)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+j)}$.

In order to prove the theorem above, we need to introduce some notions and lemmas.

The partial Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ defined by the series expansion (generating function):

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}. \tag{3.27}$$

It is easy to see that by taking $x_m = 1$, in (3.10), we obtain:

$$B_{n,k}(1, 1, \dots, 1) = S(n,k).$$

Moreover, we have the following technical lemma, for which we provide a direct proof.

Lemma 3.6. *For* $n \geq k \geq 1$, *we have:*

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = n! \sum_{j=0}^{k} \frac{S(n+j, j)}{(n+j)!(k-j)!} (-1)^{k-j}. \tag{3.28}$$

Proof. Applying (3.10), with $x_m = \frac{1}{m+1}$, for $m = 1, 2, \dots, n-k+1$, we obtain:

$$\begin{aligned}\sum_{n\geq k} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{m\geq 1} \frac{1}{m+1} \frac{t^m}{m!}\right)^k \\ &= \frac{1}{k!}\left(\sum_{m\geq 1} \frac{t^m}{(m+1)!}\right)^k \\ &= \frac{1}{k!} \frac{1}{t^k} (\mathrm{e}^t - 1 - t)^k\end{aligned}$$

$$= \frac{1}{k!}\frac{1}{t^k}\sum_{j=0}^{k}\binom{k}{j}(e^t-1)^j(-t)^{k-j}$$

$$= \frac{1}{k!}\frac{1}{t^k}\sum_{j=0}^{k}\binom{k}{j}j!\sum_{n\geq j}S(n,j)\frac{t^n}{n!}(-t)^{k-j}$$

$$= \sum_{n\geq k}\left(n!\sum_{j=0}^{k}\frac{S(n+j,j)}{(n+j)!(k-j)!}(-1)^{k-j}\right)\frac{t^n}{n!}.$$

The result follows by comparing coefficients to coefficients. □

The well-known Faà di Bruno's formula states that, if two functions f and g have sufficient numbers of derivatives, then

$$\frac{d^n}{dt^n}f(g(t)) = \sum \frac{n!}{k_1!k_2!\ldots k_n!}f^{(k)}(g(t))\left(\frac{g'(t)}{1!}\right)^{k_1}\left(\frac{g''(t)}{2!}\right)^{k_2}\ldots\left(\frac{g^{(n)}(t)}{n!}\right)^{k_n}. \tag{3.29}$$

This formula is first given by Louis François Antoine Arbogast in 1800 [4, p. 60]. A restatement of Arbogast–Faà di Bruno's formula in terms of set partitions, also called the Bell polynomials version, was given by Riordan [21]:

$$\frac{d^n}{dt^n}f(g(t)) = \sum_{k=0}^{n}f^{(k)}(g(t))B_{n,k}\left(g'(t),g''(t),\ldots g^{(n-k+1)}(t)\right). \tag{3.30}$$

We have the following well-known identities (for more details see [8, p.136 formula (3n) and formula (3f)]):

$$B_{n,k}(x_1+y_1,x_2+y_2,\ldots) = \sum_{l\leq k,\nu\leq n}\binom{n}{\nu}B_{\nu,l}(x_1,x_2,\ldots)B_{n-\nu,k-l}(y_1,y_2,\ldots) \tag{3.31}$$

and

$$B_{n,k}(abx_1,ab^2x_2,ab^3x_3,\ldots) = a^kb^nB_{n,k}(x_1,x_2,x_3,\ldots). \tag{3.32}$$

Proof of Theorem 3.5. Using the GW-type integral operator, we can re-write the exponential generating function associated with the generalized Kummer hypergeometric polynomials as follows:

$$G_r(x,t) = \left(\frac{1}{M(a,a+b;t)}\right)^r e^{xt} = \left(\frac{1}{\frac{1}{B(a,b)}\int_0^1 e^{(u-x/r)t}u^{a-1}(1-u)^{b-1}\,du}\right)^r = f(g(t)),$$

with $f(t)=\frac{1}{t^r}$ and $g(t)=\frac{1}{B(a,b)}\int_0^1 e^{(u-x/r)t}u^{a-1}(1-u)^{b-1}\,du$. Then,

$$f^{(k)}(t) = \frac{(-1)^k(r)_k}{t^{k+r}}$$

and

$$g^{(k)}(0) = \frac{1}{B(a,b)}\int_0^1(u-x/r)^ku^{a-1}(1-u)^{b-1}\,du$$

$$= \sum_{j=0}^{k}\binom{k}{j}(-x/r)^{k-j}\frac{1}{B(a,b)}\int_0^1u^{k+a-1}(1-u)^{b-1}\,du$$

$$= \sum_{j=0}^{k}\binom{k}{j}(-x/r)^{k-j}\frac{\Gamma(j+a)\Gamma(a+b)}{\Gamma(j+a+b)\Gamma(a)}.$$

Hence, the result of (i) using Riordan's version of the Arbogast–Faà di Bruno formula.

Statement (ii) comes from (i) by taking $x=0$. □

By taking $a=b=1$, we obtain the following closed formulas for Nörlund–Bernoulli polynomials and numbers.

Corollary 3.7. *For n a non-negative integer, we have:*

(i)

$$B_n^r(x) = \sum_{k=0}^{n}(-1)^k (r)_k \sum_{r+s=l}\sum_{l+m=n}\binom{n}{l}(1-x)^{l+r} B_{l,r}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{l-r+2}\right)$$
$$x^{s+m}(-1)^m B_{m,s}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m-s+2}\right)$$

*and **(ii)***

$$B_n^r = n!\sum_{k=0}^{n}(-1)^k (r)_k \sum_{j=0}^{k}\frac{(-1)^{k-j}S(n+j,j)}{(n+j)!(k-j)!},$$

where

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{n+k}{k-j}S(n+j,j).$$

$S(n,k)$ are the Stirling numbers of the second kind defined by

$$S(n,k) = \frac{1}{k!}\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}i^n \qquad \textit{with} \qquad \frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty}S(n,k)\frac{t^n}{n!}.$$

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