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Topological and equidistributional refinement of the André–Pink–Zannier conjecture at finitely many places

Raffinements topologiques et équidistributionnels de la conjecture d'André–Pink–Zannier en un nombre fini de places

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A B S T R A C T

We present some applications of recent results in homogeneous dynamics to an unlikely intersections problem in Shimura varieties (the André–Pink–Zannier conjecture) and its refinements.

R É S U M É

On présente quelques applications des résultats récents en dynamique homogène à un problème d'intersections atypiques dans les variétés de Shimura (la conjecture de André–Pink–Zannier) et ses raffinements.

Version française abrégée

On présente ici des applications des résultats de [8] à la conjecture d'André–Pink–Zannier restreinte à un nombre fini de places. On considère une variété de Shimura $Sh_K(G,X)$, un ensemble fini S de nombres premiers et un point $s=[x,t]$ de $Sh_K(G,X)$ défini sur un corps E (à noter qu'on ne fait aucune hypothèse sur les propriétés de E). La S-orbite de Hecke de s est l'ensemble $H_S(s)=\{[x,tg], g\in G(\mathbf{Q}_S)\}$. La S-conjecture d'André–Pink–Zannier est l'énoncé du fait que les composantes irreducibles de l'adhérence de Zariski d'un sous-ensemble de $H_S(s)$ sont des sous-variétés faiblement spéciales. Nous faisons une hypothèse de S-Shafarevich qui affirme que, pour toute extension finie F de E, $H_S(s)$ ne contient qu'un nombre fini de points F-rationnels. Notons que, par le théorème de Faltings, cette propriété vaut pour toutes les variétés de Shimura de type abélien. Dans [9], on définit aussi d'autres propriétés (S-Tate, S-algébricité, S-Mumford–Tate) et on démontre différentes implications entre ces propriétés. On définit également une notion de sous-variété S-réelle faiblement spéciale. Il s'agit d'une sous-variété réelle analytique homogène qui n'admet pas forcément de structure de

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variété complexe. L'adhérence de Zariski d'une telle variété est faiblement spéciale. À toute sous-variété S-réelle faiblement spéciale on peut associer canoniquement une mesure de probabilité sur $Sh_K(G, X)$ de support cette variété.

On considère une suite (s_n) de points de $H_S(s)$ à laquelle on attache une suite de mesures discrètes de probabilité (μ_n) sur $Sh_K(G, X)$. On démontre alors (en utilisant et en adaptant les résultats de [8]) que la suite (μ_n) converge étroitement vers une combinaison linéaire de mesures associées à des sous-variétés S-réelles faiblement spéciales. De plus, pour tout n assez grand, le support de μ_n est contenu dans le support de cette mesure limite. On déduit de ce résultat une version topologique de la S-conjecture d'André–Pink–Zannier affirmant que l'adhérence topologique d'un sous-ensemble de $H_S(s)$ est une union finie de sous-variétés S-réelles faiblement spéciales. De plus, si on fait une hypothèse plus forte – celle de S-Mumford–Tate –, alors on démontre que les sous-variétés S-réelles faiblement spéciales en question sont des sous-variétés faiblement spéciales.

1. Introduction

This note presents the main results of [9]. The results are motivated by the André–Pink–Zannier conjecture, which is a special case of the Zilber–Pink conjectures (see [1] and [11]) on unlikely intersections in mixed Shimura varieties. The case of mixed Shimura varieties was already formulated in [6, Conj. 1.6].

Let (G, X) be a Shimura datum and K be a compact open subgroup of $G(\mathbf{A}_f)$. The following double coset space is the set of complex points of the corresponding Shimura variety (see [3]):

$$Sh_K(G, X) = G(\mathbf{Q})\Big\backslash X \times G(\mathbf{A}_f)\Big/ K.$$

We recall that $Sh_K(G, X)$ admits a canonical model over a certain explicitly described number field $E(G, X)$, called the reflex field of the Shimura datum (G, X). The notion of a weakly special subvariety of $Sh_K(G, X)$ is explained and studied in [10]. In [4], is it proven that weakly special subvarieties of $Sh_K(G, X)$ are exactly the totally geodesic ones.

Let $x \in X$ and $t \in G(\mathbf{A}_f)$. We denote by $s = [x, t]$ the image of $(x, t) \in X \times G(\mathbf{A}_f)$ in $Sh_K(G, X)$. The Hecke orbit of s is the set $H(s) = \{[x, g] | g \in G(\mathbf{A}_f)\}$. The André–Pink–Zannier conjecture is the statement that irreducible components of the Zariski closure of any subset of $H(s)$ are weakly special subvarieties. For Shimura varieties of abelian type and components of dimension one, this statement was proved by Orr in [5] using the theory of o-minimality. Orr's approach is currently not generalisable to higher dimensional subvarieties or Shimura varieties of exceptional type. Our approach in this paper is very different from Orr's — it relies on recent results in homogeneous dynamics due to the first author and Zamojski (see [8] and [7]). When our techniques apply, they yield much finer conclusions than just a characterisation of the Zariski closure.

In order to apply results from dynamics, we make a somewhat restrictive assumption, which we now introduce. Let S be a finite set of prime numbers, and let $\mathbf{Q}_S = \prod_{p \in S} \mathbf{Q}_p$. The S-Hecke orbit of $s = [x, t]$ is $H_S(s) = \{[x, tg] | g \in G(\mathbf{Q}_S)\}$. Intuitively, one may think of an S-Hecke orbit as the set of abelian varieties isogeneous to a given abelian variety via an isogeny whose degree is only divisible by primes in S. Our object of study is the following one.

Conjecture 1 *(S-André–Pink–Zannier conjecture). The Zariski closure $\overline{\Sigma}$ of a subset $\Sigma \subseteq H(s)$ is a finite union of weakly special subvarieties.*

An essential assumption we must make is the S-Shafarevich assumption (see next section), which is the natural generalisation of the Shafarevich conjecture proved by Faltings for abelian varieties. In particular, this assumption is true for all Shimura varieties of abelian type.

Under this additional assumption (see section 4 below), we are able to characterise, not only the Zariski closure, but also the closure of subsets of S-Hecke orbits with respect to the Archimedean topology via equidistribution arguments.

2. Galois monodromy properties

Let $s = [x, t]$ be a point of $Sh_K(G, X)$ and $E \subset \mathbf{C}$ be a field of definition of s. Let $M \subset G$ be the Mumford–Tate group of x. We recall the following (see section 3 of [9] for details), possibly passing to a finite extension of E.

(i) There exists a continuous S-adic monodromy representation

$$\rho_{s,S} : Gal(\overline{E}/E) \to M(\mathbf{A}_f) \cap K \cap G(\mathbf{Q}_S)$$

which has the property that for any $g \in G(\mathbf{Q}_S)$ and $\sigma \in Gal(\overline{E}/E)$,

$$\sigma(s) = [x, \rho(\sigma) \cdot tg]. \tag{2.1}$$

(ii) The *S-adic monodromy group* is $U_S = \rho_{s,S}\left(Gal(\overline{E}/E)\right)$. It is a compact S-adic Lie subgroup of $M(\mathbf{A}_f) \cap G(\mathbf{Q}_S) \simeq M(\mathbf{Q}_S)$.

(iii) The *algebraic S-adic monodromy group*, denoted $H_S \subset M_S$, is $H_S = \overline{U_S}^{\mathrm{Zar}}$.

We now introduce the following definitions: we say that the point s over E is

(i) of *S-Mumford–Tate type* if U_S is open in $M(\mathbf{A}_f) \cap G(\mathbf{Q}_S)$;
(ii) of *S-Shafarevich type* if for every finite extension F of E, there are only finitely many F-rational points in $H_S(s)$;
(iii) of *S-Tate type* if M and H_S^0 (neutral component of H_S) have the same centraliser in $G(\mathbf{Q}_S)$.

We say that the point s over E

(iv) satisfies *S-semisimplicity* if H_S is a reductive group;
(v) satisfies *S-algebraicity* if the subgroup U_S of $H_S(\mathbf{Q}_S)$ is open.

These properties are heavily dependent on the choice of the field E, particularly on whether E is of finite type over $\mathbf{Q}$ or not. For a discussion of this, we refer to Section 3.4 of [9]. There we provide examples where all the above properties fail when E is not of finite type, even in the case of an elliptic curve over $\mathbf{Q}$. Note that a point of S-Mumford–Tate type is obviously of S-Tate type. One proves ([9], Proposition 3.6) that S-Tate-type property (iii) implies the algebraicity of U_S if E is of finite type. As a consequence, for such an E, the S-Mumford–Tate-type property (i) is equivalent to its *a priori* weaker variant: $H_S^0 = M_{\mathbf{Q}_S}$.

Our main results are proved under the S-Shafarevich hypothesis (ii), which is implied by (and in fact not far from being equivalent to) the S-Tate hypothesis (iii) together with the S-semisimplicity assumption (iv). More precisely ([9], Proposition 3.7), the S-Shafarevich property is equivalent to S-semisimplicity together with the property that the centraliser of U_S in $G(\mathbf{Q}_S)$ is compact modulo the centraliser of M in $G(\mathbf{Q}_S)$.

3. Canonical measures

Let R_S (the 'S-Ratner class') denote the set of subgroups L of G defined over $\mathbf{Q}$ such that no proper subgroup $M < L$, assumed to be defined over $\mathbf{Q}$, is such that $M(\mathbf{R} \times \mathbf{Q}_S)$ contains the subgroup $L^\dagger$ of $L(\mathbf{R} \times \mathbf{Q}_S)$ generated by the unipotent elements of the latter.

Let L be a group in R_S and let $L(\mathbf{R})^+$ be the neutral connected component of $L(\mathbf{R})$. An *S-real weakly special submanifold* is a subset of $Sh_K(G,X)$ of the following form (for some $(x,t) \in X \times G(\mathbf{A}_f)$),

$$Z_{L,(x,t)} = \left\{[l \cdot x, t] \middle| l \in L(\mathbf{R})^+\right\}.$$

We have a natural identification

$$(\Gamma \cap L(\mathbf{R})^+)\backslash L(\mathbf{R})^+/(L(\mathbf{R})^+ \cap C) \simeq Z_{L,(x,t)}$$

where Γ is an arithmetic subgroup of G depending on g and C a maximal compact subgroup of $G^{\mathrm{der}}(\mathbf{R})$ depending on x. As L belongs to R_S, the intersection $\Gamma \cap L(\mathbf{R})^+$ is a lattice in $L(\mathbf{R})^+$. We obtain a probability measure μ_Z on $Z = Z_{L,(x,g)}$ deduced from a Haar measure on $L(\mathbf{R})^+$. The measure μ_Z is canonical (it does not depend on the choice of L and (h,g) (see Lemma 1.12 of [9]).

4. Main theorems

Theorem 2 *(Inner equidistributional S-André–Pink–Zannier). Let s be a point of $Sh_K(G,X)$ defined over a field E that we assume to be of S-Shafarevich type. Let $(s_n)_{n\geq 0}$ be a sequence of points in the S-Hecke orbit $H_S(s)$ of s, and denote*

$$\mu_n = \frac{1}{|Gal(\overline{E}/E)|} \sum_{\zeta \in Gal(\overline{E}/E)\cdot s_n} \delta_\zeta$$

the sequence of discrete probability measures attached to the Galois orbits of the s_n.

After possibly extracting a subsequence and replacing E by a finite extension, there exists a finite set $\mathcal{F}$ of weakly S-special real submanifolds Z with canonical probability measure μ_Z such that

(i) (equidistribution) the sequence $(\mu_n)_{n\geq 0}$ tightly converges to a barycentre μ_∞ of the μ_Z,
(ii) (inner equidistribution) for all $n \geq 0$, we have

$$Supp(\mu_n) \subset Supp(\mu_\infty) = \bigcup_{Z \in \mathcal{F}} Z,$$

(iii) if s is of S-Mumford–Tate type, then every Z in $\mathcal{F}$ is a weakly special subvariety.

Note that conclusion (iii) applies to special points, in which case every Z in $\mathcal{F}$ is actually a *special* subvariety. From Theorem 2, we deduce the following theorem, which is more directly related to the André–Pink–Zannier conjecture. The deduction is elementary, but quite lengthy and tedious. Details can be found in section 2 of [9].

Theorem 3 *(Topological and Zariski S-André–Pink–Zannier). Keep the notations of Theorem 2. Consider a subset $\Sigma \subset H_S(s)$ and denote*

$$\Sigma_E = Gal\left(\overline{E}/E\right) \cdot \Sigma = \left\{\sigma(x) \;\middle|\; \sigma \in Gal(\overline{E}/E), x \in \Sigma\right\}.$$

(i) If s is of S-Shafarevich type, then the topological closure of Σ_E is a finite union of weakly S-special real submanifolds. Furthermore, the Zariski closure of Σ is a finite union of weakly special subvarieties.
(ii) If s is of S-Mumford–Tate type, then the topological closure of Σ_E is a finite union of weakly special subvarieties.

It follows from the hyperbolic Ax–Lindemann–Weierstrass conjecture proved in [2] that the Zariski closure of a weakly S-special real submanifold is a weakly special subvariety. Hence the Conjecture 1 follows from the above theorem, under the S-Shafarevich assumption applied to s and a field of definition of s and $\overline{\Sigma}$.

Finally, one easily proves a converse statement of Theorem 3, which shows that the S-Shafarevich property assumption is essential and optimal.

Theorem 4. *Let s be a point in a Shimura variety $Sh_K(G, X)$ defined over a field E and let $H_S(s)$ be its S-Hecke orbit.*

Assume that, for any sequence $(s_n)_{n\geq 0}$ in $H_S(s)$, for any finite extension F of E, there is an extracted subsequence for which the associated measure

$$\mu_n = \frac{1}{|Gal(\overline{E}/F)|} \sum_{\zeta \in Gal(\overline{E}/F)\cdot s_n} \delta_\zeta$$

converges weakly to a limit μ_∞ in such a way that

$$\forall n \geq 0, \operatorname{Supp}(\mu_n) \subseteq \operatorname{Supp}(\mu_\infty). \tag{4.1}$$

Then s is of S-Shafarevich type.

We briefly sketch the argument: assume that the S-Shafarevich property fails. Then there exists a finite extension F of E and a sequence of pairwise distinct F-points s_n. The associated probability measures μ_n are Dirac masses at s_n. We can always replace s_n by a suitable subsequence. If the sequence (s_n) diverges, then (4.1) can not hold. If the sequence (s_n) converges, then (μ_n) converges to a Dirac mass, which means that (s_n) is stationary, which is again a contradiction.

5. Strategy of proofs

In this section, we explain the strategy of the proof of the main Theorem 2. For full details, we refer the reader to sections 4 and 5 of [9]. Let s be a point of $Sh_K(G, X)$ and E its field of definition. Fix a finite set S of primes and assume that the S-Shafarevich property holds for s and E. Assume for simplicity (this causes no loss of generality) that we can write $s = [x, 1]$. Let $(s_n = [x, g_n])$, $g_n \in G(\mathbf{Q}_S)$ be a sequence of points of $H_S(s)$ and let (μ_n) be the associated sequence of probability measures as in the statement of 2. Our aim is to prove inner equidistribution of the measures μ_n. The main difficulty is to reduce ourselves to a situation in which we can apply the main results of [8] (summarised in [7]).

We reduce ourselves to the case where the group G is semisimple (see section 4.2 of [9]). This is one of the technical difficulties of the proof and it is necessary in order to apply the results of [8]. Let U_S be the S-adic monodromy group associated with S. Let $\Gamma = G(\mathbf{Q}) \cap (G(\mathbf{Q}_S)K)$. There is a natural map

$$\pi_G : \Gamma\backslash G(\mathbf{R} \times \mathbf{Q}_S) \to Sh_K(G, X).$$

We let μ_{U_S} be the measure on $\Gamma\backslash G(\mathbf{R} \times \mathbf{Q}_S)$ obtained as the direct image of the Haar probability measure on U_S. We define $\widetilde{\mu_n} = \mu_{U_S} \cdot g_n$. We have the lifting property

$$\pi_{G\star}(\widetilde{\mu_n}) = \mu_n.$$

We now apply the theorem of Richard–Zamojski (Theorem 3 of [8]), which shows that there exists a group L in R_S, elements n_∞ and g_∞ of $G(\mathbf{R} \times \mathbf{Q}_S)$ and an explicitly describes open subgroup L^{++} of $L(\mathbf{R} \times \mathbf{Q}_S)$ such that (up to restricting to a subsequence),

$$\lim_{n\to\infty} \widetilde{\mu_n} = \mu_{L^{++}} \cdot n_\infty \star \nu \cdot g_\infty \tag{5.1}$$

where $\mu_{L^{++}}$ is the probability induced by the Haar measure on L^{++} and ν is the Haar probability of U_S (which acts on the right on $\Gamma\backslash G(\mathbf{R} \times \mathbf{Q}_S)$). Here the S-Shafarevich assumption is essential, indeed it guarantees that the g_ns are not "too close to the centraliser of U_S", which is one of the main technical assumptions of [8]. Using the fact that L^{++} contains $L(\mathbf{R})^+$ (as it is open in $L(\mathbf{R} \times \mathbf{Q}_S)$), the finiteness of the class number of L and the fact that right action of U_S contributes only finitely many components, one shows that the pushforward of the limit measure by π_G is a finite linear combination of canonical measures associated with S-real weakly special subvarieties, as claimed. Details can be found in section 4.5 of [9].

To prove that the equidistribution is inner, we use a more refined conclusion in the theorem 0 of [8], called the "focusing criterion"', as coined by Eskin, Mozes, and Shah. It gives a precise description of the sequences g_n such that we obtain a given limiting measure (5.1). We use again the S-Shafarevich assumption, as well as some properties of the centraliser of U_S and the fact that the g_ns have trivial Archimedean factor, and we are able to choose the elements g_n in such a way that the inclusion of supports for n large enough holds (after extracting a subsequence). Technical details can be found in section 5.2 of [9].

Assuming the S-Mumford–Tate conjecture, we are able to show that our limit S-real weakly special submanifolds are actually weakly special subvarieties: $L(\mathbf{R})^+$ is normalised by the image of x seen as a cocharacter of the Deligne torus. This implies that the real symmetric space $L(\mathbf{R})^+ \cdot h$ has a complex structure and hence the associated S-real weakly special submanifold is a weakly special subvariety.

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