



Partial differential equations

Exact controllability to trajectories for entropy solutions to scalar conservation laws in several space dimensions



Contrôlabilité exacte aux trajectoires pour des lois de conservation scalaires multidimensionnelles

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ABSTRACT

We describe a new method that allows us to obtain a result of exact controllability to trajectories of multidimensional conservation laws in the context of entropy solutions and under a mere non-degeneracy assumption on the flux and a natural geometric condition.

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R É S U M É

On décrit dans cet article une nouvelle méthode permettant d'obtenir un résultat de contrôlabilité exacte aux trajectoires pour des lois de conservation scalaires en plusieurs dimensions d'espace dans le cadre des solutions entropiques et sous une simple hypothèse de non-dégénérescence du flux et une hypothèse géométrique naturelle.

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1. Introduction

In this paper, we consider a scalar conservation law in several space dimensions, i.e. a partial differential equation of the form

$$\partial_t u + \operatorname{div}_x (f(u)) = 0, \quad t \in \mathbb{R}^+, \quad x \in \Omega \subset \mathbb{R}^d, \quad d \geq 1, \quad (1)$$

where Ω is an open set with smooth boundary (C^2 is sufficient), and f , the flux function, is in $C^1(\mathbb{R}, \mathbb{R}^d)$.

We are interested in the following controllability problem. Given an initial datum $u_0 \in L^\infty(\Omega)$, a suitable target profile u_1 , and a positive time T , we aim at constructing an entropy weak solution $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{R})$ of

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$$\begin{cases} \partial_t u + \operatorname{div}_x (f(u)) = 0, & \text{in } (0, T) \times \Omega, \\ u(0, x) = u_0(x), & \text{on } \Omega, \\ u(T, x) = u_1(x), & \text{on } \Omega, \end{cases} \quad (2)$$

by using the boundary data on $(0, T) \times \partial\Omega$ as controls.

Given any extensive quantity u defined on a domain Ω , such as mass or energy, a conservation law for u translates into a partial differential equation the physical observation that the total amount of u in Ω changes at a rate that corresponds to the net flux of u , $f(u)$, through the boundary $\partial\Omega$. This kind of equations is widely used in modeling phenomena such as gas dynamics (Euler equations), electromagnetism, magneto-hydrodynamics, shallow water, combustion, road traffic, population dynamics, and petroleum engineering.

It is well known that even starting from initial data in $C_c^\infty(\mathbb{R}^d)$, the classical solutions to (1) can develop singularities (jump discontinuities) in finite time, see [20] for a very complete introduction to the field.

The most general wellposedness result for classical solutions to the Cauchy problem states that, for any initial datum u_0 in H^s , with $s > 1 + \frac{d}{2}$, there exists a solution to (1) in $C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$, whose life span T can be estimated depending on f and u_0 .

However, most of the literature devoted to conservation laws focuses on a class of weak (distributional) solutions that satisfies an additional selection criterium, necessary to ensure uniqueness, called entropy condition. In the case of a scalar conservation law in several space dimensions, a complete wellposedness theory for entropy solutions to the Cauchy problem is due to Kruzhkov [25].

The problem we aim at solving, see (2), can be addressed both in the framework of classical or of entropy solutions. In the first case, controls, besides driving the state to the target, are also responsible for preventing the formation of singularities. Several results exist in this framework, see [13], [28], and [16] for a survey. Unfortunately, this approach does not allow one to attain many physically relevant states involving jump discontinuities and leads to control strategies that are in general not very robust. Indeed very small perturbations of the control might lead to blow up of the derivatives of the solution before time T .

In the present paper, we are interested in the controllability of entropy solutions. The literature in this framework is less abundant also due to specific technical difficulties, even if we can notice a growing interest of researchers in this field. The classical methodology for exact controllability relies heavily on linearization, which is not possible (or at least horribly technical) anymore around discontinuous solutions. Moreover, Bressan and Coclite showed in [12] that nonlinear conservation laws may fail the linear test. Indeed, they provided a system for which the linearized approximation around a constant state is controllable, while the original nonlinear system cannot reach that same constant state in finite time.

A separate issue is related to the irreversibility of entropy solutions: the set of admissible target states in time T is reduced and its description, often involving a number of highly technical conditions, is in itself an open problem in most cases, see [3,4,6,7,14].

In the existing literature, we can distinguish essentially three approaches toward the study of exact controllability for conservation laws in one space dimension (consider equation (1) with $d = 1$).

Starting from the pioneering work by Ancona and Marson [4], several results have been obtained using the theory of generalized characteristics introduced by Dafermos in [19], as [4,7,14,24,32] or the explicit Lax–Oleinik representation formula, as [1,6]. The latter technique is applicable only when the flux function f is strictly convex/concave, while the theory of generalized characteristics covers also the (slightly) more general case of a flux function f with one inflection point.

The return method introduced by Coron [16] is the basis of the approach developed by Horsin in [24] and, combined with the classical vanishing viscosity method, plays a key role in [23] and in the only paper to our knowledge in which the flux function f is allowed to have a finite number of inflection points [26].

The asymptotic stabilization of entropy solutions to scalar conservation laws is the topic of [10,33,34].

The only available tool for investigating the exact controllability of systems of conservation laws in one space dimension is the wave front tracking algorithm [11], which has been successfully applied in [3,12,21,22,29].

The asymptotic stabilization of entropy solutions to systems has been studied in [5,9,18].

It seems difficult to investigate the exact controllability of entropy solutions of scalar conservation laws in several space dimensions using the techniques designed for the one-dimensional case. In the present paper, we propose a new approach, inspired by the work on stabilization by Coron [15] and by Coron, Bastin, and d'Andéa Novel [17]; see also the monography [9] for a comprehensive presentation of the method. The conditions we impose on the flux function are technical and will be detailed in the next Section, but we stress that in the special case $d = 1$ they are not related to convexity (or concavity). This means that, even in the one-dimensional case, our result is new, although for this case stronger results are available under more restrictive hypotheses.

The first of our conditions, called later *nondegeneracy condition*, says that the range of u does not contain any interval on which f is affine. This condition is necessary to ensure the existence of traces at the boundary of Ω , see [35].

The second condition, called later *replacement condition*, involves f together with T and Ω . Roughly speaking, once we reduce to the one-dimensional case, it says that all generalized characteristics issued from points (t, x) in $\{0\} \times \Omega$ leave the cylinder $(0, T) \times \Omega$ before time T , so that the dynamics in the domain only depends on the boundary data and not on the initial data for t large enough.

2. Preliminary definitions and notations

In the following, $u \mapsto \text{sign}(u)$ is the function given by

$$\forall u \in \mathbb{R}, \quad \text{sign}(u) := \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0, \end{cases}$$

$\langle \cdot | \cdot \rangle$ denotes the scalar product between two vectors and χ_E is the indicator function of the set E .

Definition 2.1. Given $f \in C^1(\mathbb{R}; \mathbb{R}^d)$ and $u_0 \in L^\infty(\Omega)$, we consider the equation

$$\partial_t u + \text{div}(f(u)) = 0, \quad \text{for } (t, x) \in (0, +\infty) \times \Omega, \tag{3}$$

supplemented with the initial condition

$$u(0, x) = u_0(x), \quad \text{for } x \in \Omega. \tag{4}$$

A function $u \in L^\infty([0, +\infty) \times \Omega)$ is an entropy solution to (3)–(4) in $[0, T] \times \Omega$ if, for any real number k and any non-negative function ϕ in $C_c^1([0, +\infty) \times \Omega)$, we have

$$\begin{aligned} \int_{(0, T) \times \Omega} |u(t, x) - k| \partial_t \phi(t, x) + \text{sign}(u(t, x) - k) \langle f(u(t, x)) - f(k) | \nabla \phi(t, x) \rangle dt dx \\ + \int_{\Omega} \text{sign}(u_0(x) - k) \phi(0, x) dx \geq 0. \end{aligned} \tag{5}$$

We will also say that a function u is an entropy solution (without referring to any initial data) in $(0, +\infty) \times \Omega$ when it satisfies (5) for any non-negative $\phi \in C_c^1((0, +\infty) \times \Omega)$.

We now introduce a simple geometric condition which is sufficient (though clearly not necessary) to obtain our controllability result.

Definition 2.2. Let Ω be a smooth open set of \mathbb{R}^d , I be a segment of \mathbb{R} and $f : \mathbb{R} \rightarrow \mathbb{R}^d$ a C^1 flux function. We say that the triple (f, Ω, I) satisfies the replacement condition in time $t > 0$ if there exists a vector $w \in \mathbb{R}^d$ and a positive number c such that

$$L := \sup_{x \in \Omega} \langle w | x \rangle - \inf_{x \in \Omega} \langle w | x \rangle < +\infty, \tag{6}$$

$$\forall u \in I, \quad \langle f'(u) | w \rangle \geq c, \quad \text{and } t = \frac{L}{c}. \tag{7}$$

Definition 2.3. We say that the flux f is non-degenerate if, for any couple $(\tau, \zeta) \in \mathbb{R} \times \mathbb{R}^d \setminus \{(0, 0)\}$, we have

$$\mathcal{L}(\{z \in \mathbb{R} : \tau + \langle \zeta | f'(z) \rangle = 0\}) = 0,$$

where \mathcal{L} is the Lebesgue measure.

We can now state our main theorem on exact controllability to trajectories for a conservation law in any space dimension.

Theorem 1. Let $v \in C^0((0, +\infty); L^1(\Omega)) \cap L^\infty((0, +\infty) \times \Omega)$ be an entropy solution to (3) and u_0 be a function in $L^\infty(\Omega)$.

We suppose that both u_0 and v take values in a segment I such that (f, Ω, I) satisfies the replacement condition in time t . We also suppose that the flux f is non-degenerate.

Then, for any times T_1 and T_2 larger than t , there exists an entropy solution u to (3) satisfying

$$u(0, x) = u_0(x), \quad u(T_1, x) = v(T_2, x) \quad \text{for almost every } x \in \Omega.$$

Remark 1. For the sake of simplicity, we omit to write here the exact form of the controls we use. In the next Section, we precise in which sense the boundary conditions on $\partial\Omega$ are taken into account by entropy solutions and in the last Section, in the proof of Theorem 1, we write our controls in a fully explicit way.

Remark 2. A characterization of the set of admissible target profiles at fixed time $T \geq 0$ for a scalar conservation law in several space dimensions is not available in the literature. We stress, however, that in the statement of Theorem 1, we really need to assume that v is an entropy solution on the cylinder $(0, +\infty) \times \Omega$ because the complete knowledge of v is necessary in our proof.

3. Boundary conditions and entropy solutions

We have so far avoided the precise formulation of boundary conditions for (3). In general, given the initial boundary value problem

$$\begin{cases} \partial_t u + \operatorname{div}(f(u)) = 0, & (0, +\infty) \times \Omega, \\ u(t, x) = u_b(t, x), & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases} \tag{8}$$

its entropy solution u does not satisfy the boundary condition in the usual sense, as the trace of u on $\partial\Omega$ does not coincide with the prescribed Dirichlet datum. The situation is easier to visualize in the one-dimensional case, as we can see in the following example.

Example 1. Assume $d = 1$, $\Omega = (0, +\infty)$, $f(u) = \frac{u^2}{2}$ and impose in (8) constant initial and boundary data, $u_0 = -1$ and $u_b = -\frac{1}{2}$. Then the initial condition is transported along characteristic curves with negative slope up to the boundary, while no characteristic can spring out of the boundary itself. The trace of the solution at $x = 0$ can only take the value $u(t, 0^+) = u_0$, and u_b can not be attained.

For this reason, the boundary conditions should be interpreted in a broader sense, made precise by Leroux [27], and Bardos, Leroux, and Nédélec [8]. In the setting of the example above, we say that the boundary condition is fulfilled in the sense of Bardos–Leroux–Nédélec as soon as the solution to the Riemann problem with data $u_L = u_b$ and $u_R = u(t, 0^+)$ only contains waves of non-positive speed (i.e. waves which do not enter the domain). In the general multidimensional case, this takes the following form.

Definition 3.1. Let $I(a, b)$ denote the interval of extrema a and b , and let $\eta(x)$ be the outer unit normal of $\partial\Omega$ at $(t, x) \in (0, T) \times \partial\Omega$. Then we say that the boundary condition u_b in the IBVP (8) is fulfilled at $(t, x) \in (0, T) \times \partial\Omega$ if for any $k \in I(u_b(t, x), u(t, x))$

$$\operatorname{sign}(u(t, x) - u_b(t, x)) (f(u(t, x)) \cdot \eta(x) - f(k) \cdot \eta(x)) \geq 0.$$

We have to precise, however, that the above definition is not exactly the one we adopt in the present work as existence of traces is not guaranteed for the solution to (8) in the L^∞ setting. The first results dealing with this problem were by Otto [31], see also [30]. We use more recent results by Ammar, Carillo, and Wittbold [2], which build upon those ideas. We also recall a (simplified version of a) result by Vasseur [35], showing that if the flux satisfies the non-degeneracy condition, then any entropy solution $u \in L^\infty$ admits a trace at the boundary.

Let us recall some definitions and results in [2]. We use the following notations. For any real numbers α and k , and any point $x \in \partial\Omega$, we call $\eta(x)$ the outer unit normal at x and introduce

$$\begin{aligned} \omega^+(x, k, \alpha) &:= \max_{k \leq r, s \leq \max(\alpha, k)} |(f(r) - f(s))\eta(x)|, \\ \omega^-(x, k, \alpha) &:= \max_{\min(\alpha, k) \leq r, s \leq k} |(f(r) - f(s))\eta(x)|. \end{aligned}$$

For integrals on the boundary, we denote the surface measure at $x \in \partial\Omega$ by $d\sigma(x)$.

Definition 3.2. Given a boundary condition $u_b \in L^\infty((0, +\infty) \times \partial\Omega)$ and an initial data $u_0 \in L^\infty(\Omega)$ we say that u is an entropy solution to (8) when the following hold for any $k \in \mathbb{R}$ and any non-negative function $\zeta \in C_c^\infty([0, +\infty) \times \mathbb{R}^d)$

$$\begin{aligned} &\int_0^T \int_\Omega (u(t, x) - k)^+ \partial_t \zeta(t, x) + \chi_{\{u(t, x) > k\}} (f(u(t, x)) - f(k)) |\nabla \zeta(t, x)| \, dx \, dt \\ &+ \int_{\partial\Omega} \int_0^T \zeta(t, x) \omega^+(x, k, u_b(t, x)) \, dt \, d\sigma(x) + \int_\Omega (u_0(x) - k)^+ \zeta(0, x) \, dx \geq 0, \end{aligned} \tag{9}$$

$$\int_0^T \int_{\Omega} (k - u(t, x))^+ \partial_t \zeta(t, x) + \chi_{\{u(t, x) < k\}} \langle f(u(t, x)) - f(k) | \nabla \zeta(t, x) \rangle dx dt + \int_{\partial\Omega} \int_0^T \zeta(t, x) \omega^-(x, k, u_b(t, x)) dt d\sigma(x) + \int_{\Omega} (k - u_0(x))^+ \zeta(0, x) dx \geq 0. \quad (10)$$

The following two theorems were proven in [2] (see [2], Theorem 2.3 and 2.4).

Theorem 2. *Given initial and boundary data $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty((0, +\infty) \times \partial\Omega)$, there exists a unique entropy solution to (8).*

Theorem 3. *Given initial data u_0, v_0 in $L^\infty(\Omega)$ and boundary data u_b, v_b in $L^\infty((0, +\infty) \times \partial\Omega)$, the corresponding entropy solutions u and v satisfy*

$$\int_0^T \int_{\Omega} (u(t, x) - v(t, x))^+ \partial_t \zeta(t, x) + \chi_{\{u(t, x) > v(t, x)\}} \langle f(u(t, x)) - f(v(t, x)) | \nabla \zeta(t, x) \rangle dx dt + \int_{\partial\Omega} \int_0^T \omega^-(x, u_b(t, x), v_b(t, x)) \zeta(t, x) dt d\sigma(x) + \int_{\Omega} (u_0(x) - v_0(x))^+ \zeta(0, x) dx \geq 0, \quad (11)$$

for any non-negative function $\zeta \in C_c^\infty([0, +\infty) \times \mathbb{R}^d)$.

Let us finally recall a simplified version of the result obtained by Vasseur in [35], which is sufficient for our use.

Theorem 4. *Assume that the flux f is non-degenerate and that the domain Ω is C^2 . Then if $u \in L^\infty((0, +\infty) \times \Omega)$ is an entropy solution of (3) in the sense of Definition 2.1, i.e. (5) is satisfied for any k and any non-negative function ϕ in $C_c^1((0, +\infty) \times \Omega)$, then there exists boundary data $u_b \in L^\infty((0, T) \times \partial\Omega)$ and initial data $u_0 \in L^\infty(\Omega)$ such that u is the unique entropy solution to the mixed problem (8) in the sense of Definition 3.2.*

4. Proof of the main result

Lemma 4.1. *Consider $J := [A, B] \subset \mathbb{R}$ and suppose that*

$$u_0(x) \in J, \quad \text{for a.e. } x \in \Omega, \\ u_b(t) \in J, \quad \text{for a.e. } t \geq 0.$$

Then the unique entropy solution to the IBVP (8) with initial and boundary data u_0 and u_b , u satisfies

$$u(t, x) \in J \quad \text{for a.e. } (t, x) \text{ in } (0, +\infty) \times \Omega.$$

Proof. We prove here in full details that $u(t, x) \leq B$ for a.e. (t, x) in $(0, +\infty) \times \Omega$. The inequality $A \leq u(t, x)$ can be obtained analogously.

By hypothesis, we have for almost every x in Ω and (t, y) in $(0, +\infty) \times \partial\Omega$

$$u_0(x) \leq B, \quad u_b(t, y) \leq B,$$

then for any fixed time $\tilde{t} \geq 0$, taking a sequence $\zeta_n \in C_c^\infty(\mathbb{R}) \rightarrow \chi_{(-\infty, \tilde{t}]}$ and using $k = B$, from (9) we obtain

$$\int_{\partial\Omega} \int_0^{\tilde{t}} \omega^+(y, B, u_b(t, y)) dt d\sigma(y) + \int_{\Omega} (u_0(x) - B)^+ - (u(\tilde{t}, x) - B)^+ dx \geq 0. \quad (12)$$

It is clear that for a.e. x in Ω and (t, y) in $(0, \tilde{t}) \times \partial\Omega$ we have

$$\omega^+(y, B, u_b(t, y)) = 0, \quad (u_0(x) - B)^+ = 0,$$

then (12) implies

$$(u(\bar{t}, x) - B)^+ = 0, \quad \text{for a.e. } x \text{ in } \Omega$$

which is indeed

$$u(\bar{t}, x) \leq B, \quad \text{for a.e. } x \text{ in } \Omega. \quad \square$$

Proposition 4.2. *Let u and v be entropy solutions to (8) with respective initial data u_0 and v_0 and the same boundary datum u_b . Let us also suppose that all data take values in an interval I that satisfies the replacement condition in time $t = \frac{1}{c}$ (with a direction w). Then we can conclude that*

$$\forall t \geq \bar{t}, \quad u(t, x) = v(t, x), \quad \text{for almost every } x \text{ in } \Omega.$$

Proof. Let us define for $\theta > 0$ the functional J_θ by

$$\forall t \geq 0, \quad J_\theta(t) := \int_{\Omega} |u(t, x) - v(t, x)| e^{-\theta(w|x)} dx.$$

Given $\bar{t} \geq 0$, we apply (11) to the ordered couples (u, v) and (v, u) , then adding the inequalities we get

$$\begin{aligned} & \int_0^{\bar{t}} \int_{\Omega} ((v(t, x) - u(t, x))^+ + (u(t, x) - v(t, x))^+) \partial_t \zeta(t, x) \\ & + \langle (\chi_{\{v(t,x) > u(t,x)\}} f(v(t, x)) - f(u(t, x))) + (\chi_{\{u(t,x) > v(t,x)\}} f(u(t, x)) - f(v(t, x))) | \nabla \zeta(t, x) \rangle dx dt \\ & + \int_{\partial\Omega} \int_0^{\bar{t}} 2\omega^-(x, u_b(t, x), u_b(t, x)) \zeta(t, x) dt d\sigma(x) + \int_{\Omega} ((v_0(x) - u_0(x))^+ + (u_0(x) - v_0(x))^+) \zeta(0, x) dx \geq 0, \end{aligned}$$

which is actually

$$\begin{aligned} 0 \leq & \int_0^{\bar{t}} \int_{\Omega} |u(t, x) - v(t, x)| \partial_t \zeta(t, x) + \text{sign}(u(t, x) - v(t, x)) \langle f(u(t, x)) - f(v(t, x)) | \nabla \zeta(t, x) \rangle dx dt \\ & + \int_{\Omega} |u_0(x) - v_0(x)| \zeta(0, x) dx. \end{aligned}$$

We consider a sequence $(\zeta_n)_n \subset C_c^\infty(\mathbb{R})$ converging in L^1 to $\chi_{(-\infty, \bar{t}]} e^{-\theta(w|x)}$, so that in the limit $n \rightarrow \infty$, we get

$$J_\theta(\bar{t}) \leq J_\theta(0) + \int_0^{\bar{t}} \int_{\Omega} \text{sign}(u(t, x) - v(t, x)) \langle f(u(t, x)) - f(v(t, x)) | -w \theta e^{-\theta(w|x)} \rangle dx dt. \tag{13}$$

But since

$$\begin{aligned} \forall (a, b) \in I^2, \quad \text{sign}(a - b) \langle f(a) - f(b) | w \rangle &= \text{sign}(a - b) \left\langle \int_0^1 f'(b + s(a - b)) ds (a - b) | w \right\rangle \\ &= \text{sign}(a - b) (a - b) \int_0^1 \langle f'(b + s(a - b)) | w \rangle ds \\ &\geq |a - b| \int_0^1 c ds \\ &\geq c |a - b|, \end{aligned}$$

from (13), Lemma 4.1 and the replacement condition, we obtain

$$J_\theta(\bar{t}) \leq J_\theta(0) - \theta \int_0^{\bar{t}} J_\theta(s) \, ds.$$

Thanks to the classical Gronwall's lemma, we end up with

$$J_\theta(\bar{t}) \leq e^{-c\theta\bar{t}} J_\theta(0).$$

As \bar{t} was arbitrarily chosen, if $M := \sup_{x \in \Omega} \langle w|x \rangle$ and $m = \inf_{x \in \Omega} \langle w|x \rangle$, we can write that, for all $t \geq 0$,

$$\|u(t) - v(t)\|_{L^1(\Omega)} e^{-\theta M} \leq J_\theta(t) \leq \|u(t) - v(t)\|_{L^1(\Omega)} e^{-\theta m}.$$

So, we can compute

$$\begin{aligned} \|u(t) - v(t)\|_{L^1(\Omega)} &\leq e^{\theta M} J_\theta(t) \\ &\leq e^{\theta M - \theta ct} J_\theta(0) \\ &\leq e^{-\theta c(\frac{M-m}{c} - t)} \|u_0 - v_0\|_{L^1(\Omega)} \\ &\leq e^{-\theta c(\frac{L}{c} - t)} \|u_0 - v_0\|_{L^1(\Omega)}. \end{aligned}$$

So for any $t \geq \frac{L}{c}$, letting $\theta \rightarrow +\infty$, we obtain

$$u(t, x) = v(t, x) \quad \text{for almost every } x \text{ in } \Omega. \quad \square$$

We are ready to prove Theorem 1.

Proof. We aim at proving that there exists an entropy solution to the problem

$$\begin{cases} \partial_t u + \operatorname{div}_x (f(u)) = 0, & \text{in } (0, T_1) \times \Omega, \\ u(0, x) = u_0(x), & \text{on } \Omega, \\ u(T_1, x) = v(T_2, x), & \text{on } \Omega. \end{cases}$$

In view of the well-posedness result stated in Theorem 2, our goal is achieved once we construct suitable boundary conditions, which can be interpreted as controls in our setting.

Case $T_2 > T_1$.

Thanks to Theorem 4, it makes sense to consider

$$\begin{aligned} w_0(x) &= v(T_2 - T_1, x), && \text{for a.e. } x \in \Omega, \\ w_b(s, x) &= v(T_2 - T_1 + s, x), && \text{for a.e. } x \in \partial\Omega \text{ and } s \geq 0. \end{aligned}$$

We call w the unique entropy solution to the IBVP (8) with data w_0, w_b on $(0, T_1) \times \Omega$. The form of the equation implies that, for almost every (s, x) in $(0, T_1) \times \Omega$, $w(s, x) = v(T_2 - T_1 + s, x)$.

By hypothesis, $T_1 \geq t$, so, as a direct application of Proposition 4.2, we can conclude that the entropy solutions to the mixed problems of the form (8) with initial data u_0 and w_0 , respectively, and common boundary datum w_b satisfy

$$u(T_1, x) = w(T_1, x) \quad \text{for a.e. } x \in \Omega,$$

which means

$$u(T_1, x) = v(T_2, x) \quad \text{for a.e. } x \in \Omega.$$

Case $T_1 > T_2$.

We define

$$\begin{aligned} w_0(x) &= v(T_2 - t, x), && \text{for a.e. } x \in \Omega, \\ w_b(s, x) &= v(T_2 - t + s, x), && \text{for a.e. } x \in \partial\Omega \text{ and } s \geq 0, \end{aligned}$$

where t is the time given by the replacement condition.

We call w the unique entropy solution to the IBVP (8) with data w_0, w_b on $(0, t) \times \Omega$. The form of the equation implies that, for almost every (s, x) in $(0, t) \times \Omega$, $w(s, x) = v(T_2 - t + s, x)$.

We consider now a boundary condition of the following form

$$u_b(t, x) = \begin{cases} b, & \text{for } t \in (0, T_1 - t), \quad x \in \partial\Omega, \\ w_b(t - (T_1 - t), x) & \text{for } t \in (T_1 - t, +\infty), \quad x \in \partial\Omega, \end{cases}$$

where b is any constant state in the interval I .

The IBVP (8) with data u_0, u_b admits a unique entropy solution u in $(0, +\infty) \times \Omega$.

We call \tilde{u}_0 the profile of u at time $t = T_1 - t$.

Now it is clear that if we apply Proposition 4.2 to the entropy solutions \tilde{u} et w to (8) with respective initial data \tilde{u}_0 and w_0 and common boundary data w_b we obtain

$$\tilde{u}(t, x) = w(t, x), \quad \text{for a.e. } x \in \Omega,$$

which means

$$u(T_1, x) = v(T_2, x), \quad \text{for a.e. } x \in \Omega. \quad \square$$

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