



Dynamical systems/Probability theory

On the CLT for rotations and BV functions

Sur le TCL pour les rotations et les fonctions BV

Jean-Pierre Conze, Stéphane Le Borgne

Université de Rennes, CNRS, IRMAR, UMR 6625, 35000 Rennes, France

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ABSTRACT

Let $x \mapsto x + \alpha$ be a rotation on the circle and let φ be a step function. Denote by $\varphi_n(x)$ the ergodic sums $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. For α in a class containing the rotations with bounded partial quotients and under a Diophantine condition on the discontinuities of φ , we show that $\varphi_n / \|\varphi_n\|_2$ is asymptotically Gaussian for n in a set of density 1.

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R É S U M É

Soient $x \mapsto x + \alpha$ une rotation sur le cercle, φ une fonction en escalier et $\varphi_n(x)$ les sommes ergodiques $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. Pour α dans une classe contenant les rotations à quotients partiels bornés et sous une condition diophantienne sur les discontinuités de φ , nous montrons que $\varphi_n / \|\varphi_n\|_2$ est asymptotiquement gaussien pour n dans un ensemble de densité 1.

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1. Introduction

Let α be an irrational number in $]0, 1[$, $[0; a_1, a_2, \dots, a_n, \dots]$ its continued fraction expansion, p_n and q_n its numerators and denominators defined as usual by: $p_0 = 0$, $p_1 = 1$ and $p_{n+1} = a_{n+1}p_n + p_{n-1}$, $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $n \geq 1$. For the rotation $x \rightarrow x + \alpha \pmod 1$ on $X = \mathbb{R}/\mathbb{Z}$ endowed with the Lebesgue measure μ , denote by $\varphi_L(x) = \sum_{j=0}^{L-1} \varphi(x + j\alpha)$ the ergodic sums of a function φ .

Contrasting with the case of expanding maps like $x \rightarrow 2x \pmod 1$, the behavior of the sequence $(\varphi_L)_{L \geq 1}$ depends strongly on the regularity of φ . Under a Diophantine condition on α , too much regularity for φ can imply that φ is a coboundary and that the sums remain bounded. Therefore, it is natural to consider BV (i.e. with bounded variation) functions on the circle, in particular step functions. But still, by Denjoy–Koksma inequality, along the sequence (q_n) of denominators of α , the ergodic sums of a BV function φ are uniformly bounded: $\|\varphi_{q_n}\|_\infty \leq V(\varphi)$, where $V(\varphi)$ denotes the variation. Nevertheless, one can ask if along other sequences of time (L_n) there is a more stochastic behavior.

E-mail addresses: jean-pierre.conze@univ-rennes1.fr (J.-P. Conze), stephane.leborgne@univ-rennes1.fr (S. Le Borgne).

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The study of a Gaussian behavior in distribution in the context of Fourier series and of rotations has a long history, starting with Salem and Zygmund in the 1940s. M. Denker and R. Burton in 1987, then M. Lacey (1993), D. Volný and P. Liardet (1997), M. Weber (2000, 2006) proved the existence of functions, necessarily not BV, whose ergodic sums over rotations satisfy a Central Limit Theorem after self-normalization. For the functions $\psi := 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 0[}$, F. Huveneers [8] proved that, for every irrational α , there is a sequence $(L_n)_{n \in \mathbb{N}}$ such that ψ_{L_n}/\sqrt{n} is asymptotically normally distributed. Let us also mention the recent “temporal” limit theorems for rotations obtained by J. Beck [1], D. Dolgopyat, and O. Sarig [6], M. Bromberg and C. Ulcigrai [2].

An irrational number α is said to be of bounded type (or “bpq”) if it has bounded partial quotients, i.e. if $\sup_k a_k < \infty$. In [4], an almost sure invariance principle for subsequences of ergodic sums of BV functions was shown when α is not bpq. In this note, for a class of rotations containing the bounded type case, we show (Theorem 3.1) a “spatial” asymptotic Gaussian behavior of the ergodic sums φ_n of a BV function, for n in a set W of integers of density 1. We also consider the particular case when (a_k) is ultimately periodic (equivalently, by a theorem of Lagrange, when α is a quadratic irrational) and improve the size estimation of W in this case. The method differs from [4] and relies on a decorrelation property like in [8]. Detailed proofs of the results of this note are given in [5].

2. Preliminaries

For $u \in \mathbb{R}$, set $\|u\| := \inf_{n \in \mathbb{Z}} |u - n|$. The arguments of the functions are taken modulo 1. Let \mathcal{BV}_0 be the class of centered BV functions. It contains in particular the step functions with a finite number of discontinuities. If φ is in \mathcal{BV}_0 , its Fourier coefficients $c_r(\varphi)$ satisfy: $c_r(\varphi) = \frac{\gamma_r(\varphi)}{r}$, $r \neq 0$, with $K(\varphi) := \sup_{r \neq 0} |\gamma_r(\varphi)| < +\infty$.

The Ostrowski expansion is the key to the analysis of the ergodic sums over the rotation by α . Let us recall its definition. We use the notation $m = m(n) := \ell$, if $n \in [q_\ell, q_{\ell+1}[$, for $n \geq 1$. We can write $n = b_m q_m + r$, with $1 \leq b_m \leq a_{m+1}$, $0 \leq r < q_m$. By iteration, we get for n the following representation: $n = \sum_{k=0}^m b_k q_k$, with $0 \leq b_k \leq a_{k+1}$ for $1 \leq k < m$, and $0 \leq b_0 \leq a_1 - 1$, $1 \leq b_m \leq a_{m+1}$. Therefore, the ergodic sum $\varphi_n(x) = \sum_{j=0}^{n-1} \varphi(x + j\alpha)$ of a function φ can be written:

$$\varphi_n(x) = \sum_{\ell=0}^m \sum_{j=N_{\ell-1}}^{N_\ell-1} \varphi(x + j\alpha) = \sum_{\ell=0}^m \varphi_{b_\ell q_\ell}(x + N_{\ell-1}\alpha), \text{ with } N_{-1} = 0, N_\ell = \sum_{k=0}^{\ell} b_k q_k, \text{ for } \ell \leq m. \tag{1}$$

3. CLT with rate along large subsets of integers

Using (1), we will obtain a Gaussian behavior of φ_n for n in a large set of integers based on the following decorrelation property between the components $\varphi_{b_n q_n}$. The proof (given in [5]) completes and extends the proof of decorrelation in [8]. A historical reference for analogous computations is [7].

Proposition 3.1. *Let ψ and φ be BV centered functions on the circle. If there are constants $A \geq 1$, $p \geq 0$ such that $a_n \leq A n^p$, $\forall n \geq 1$, then we have for constants $C, \theta_1, \theta_2, \theta_3$, for every $1 \leq n \leq m \leq \ell$:*

$$\begin{aligned} \left| \int_X \psi \varphi_{b_n q_n} d\mu \right| &\leq C V(\psi) V(\varphi) \frac{n^{\theta_1}}{q_n} b_n, \quad \left| \int_X \psi \varphi_{b_n q_n} \varphi_{b_m q_m} d\mu \right| \leq C V(\psi) V(\varphi)^2 \frac{m^{\theta_2}}{q_n} b_n b_m, \\ \left| \int_X \psi \varphi_{b_n q_n} \varphi_{b_m q_m} \varphi_{b_\ell q_\ell} d\mu \right| &\leq C V(\psi) V(\varphi)^3 \frac{\ell^{\theta_3}}{q_n} b_n b_m b_\ell. \end{aligned}$$

Let X and Y be two real random variables defined respectively on (Ω, \mathbb{P}) and (Ω_1, \mathbb{P}_1) . Their distance (in distribution) is defined by $d(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}_1(Y \leq x)|$. Below the ergodic sum φ_n is viewed as a r.v. on the circle endowed with the uniform measure. For n such that $\|\varphi_n\|_2$ is big enough, the decorrelation proved in the preceding proposition permits to bound the distance of $\varphi_n/\|\varphi_n\|_2$ to a r.v. Y_1 with distribution $\mathcal{N}(0, 1)$. With the notation of the preliminaries, we have the following proposition.

Proposition 3.2. *For every $\delta > 0$, there is a constant $C(\delta) > 0$ such that*

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq C(\delta) \left(\frac{\max_{j=1}^{m(n)} b_j}{\|\varphi_n\|_2} \right)^{\frac{2}{3}} m(n)^{\frac{1}{4} + \delta}. \tag{2}$$

The proof of the proposition uses a classical method of expansion and truncation of the characteristic function $\int \exp(i\zeta \varphi_n) d\mu$ where ζ is a real parameter. After replacing φ_n by its representation given in (1), one uses the decorrelation inequalities to estimate recursively the integral.

To apply the proposition, we need an information about the quotient $\frac{\max_{j=1}^m b_j}{\|\varphi_n\|_2}$. For it, we will assume that φ satisfies the condition:

$$\exists N_0, \eta, \theta_0 > 0 \text{ such that } \frac{1}{N} \text{Card}\{j \leq N : |\gamma_{q_j}(\varphi)| \geq \eta\} \geq \theta_0, \forall N \geq N_0. \tag{3}$$

Remarks on the validity of (3) for step functions are given later. Let φ in \mathcal{BV}_0 satisfying (3).

Theorem 3.1. 1) Suppose that α is such that $a_n \leq Cn^p, \forall n \geq 1$, for a constant C . For a positive constant B , let $W_B := \{n \in \mathbb{N} : B^{-1} \sqrt{m(n)}/\sqrt{\ln m(n)} \leq \|\varphi_n\|_2 \leq B\sqrt{m(n)}\}$. Then if B is big enough, the asymptotic density of W_B is 1 and, for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that, for $p < \frac{1}{8}$,

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq K(\delta_0) m(n)^{-\frac{1}{12} + \frac{2}{3}p + \delta_0}, \forall n \in W_B. \tag{4}$$

If α has bounded partial quotients, the statement holds with $p = 0$ and $m(n)$ replaced by $\ln n$.

2) Let α be a quadratic irrational. For a positive constant B , let $V_B := \{n \geq 1 : B^{-1} \sqrt{\log n} \leq \|\varphi_n\|_2 \leq B\sqrt{\log n}\}$. Then, there are B, N_0 and two constants $R, \zeta_0 > 0$ such that the density of V_B satisfies:

$$\frac{1}{N} \text{Card}(V_B \cap [1, N]) \geq 1 - RN^{-\zeta_0}, \text{ for } N \geq N_0; \tag{5}$$

and for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that for $n \in V_B$,

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq K(\delta_0) (\log n)^{-\frac{1}{12} + \delta_0}. \tag{6}$$

Sketch of the proof. Statements (4) and (6) follow from Proposition 3.2. It remains to show that W_B has density 1 and that (5) holds. This will show that the variance $\|\varphi_n\|_2^2$ is rather big for n in large sets of integers. Let n be in $[q_{\ell-1}, q_\ell[$. Keeping only the indices q_j in the Fourier series of φ , the variance at time n is bounded from below as follows, with $c = \frac{8}{\pi^2}$, for every $\delta \in]0, \frac{1}{2}[$,

$$\|\varphi_n\|_2^2 \geq c \sum_{j=1}^{\ell} |c_{q_j}(\varphi)|^2 \frac{\|nq_j\alpha\|^2}{\|q_j\alpha\|^2} \geq c\delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2 \mathbf{1}_{\|nq_j\alpha\| \geq \delta}. \tag{7}$$

Modulo 1 we have $q_j\alpha = \theta_j$, with $\theta_j = (-1)^j \|q_j\alpha\|$. We count how many n in an interval of integers $I = [N_1, N_2[$ of length L are such that $\|n\theta_j\| < \delta$. The numbers $n\theta_j$ are separated by steps of length θ_j , these steps encounter integers at most $L(\theta_j^{-1} - 1)^{-1} + 2$ times, and each time it occurs, we get at most $2(1 + \delta\theta_j^{-1})$ times $\|n\theta_j\| < \delta$. Thus, as $|\theta_j| \leq q_{j+1}^{-1}$, the number of n in I such that $\|n\theta_j\| < \delta$ is less than $C(\delta + q_{j+1}^{-1})L$ with a universal constant $C > 0$ if $q_{j+1} \leq 2L$. By summation on the array $(j, n) \in [1, \ell] \times I$, using (7) and (3), we get two positive constants c_1, c_2 (not depending on δ) such that, if $q_\ell \leq 2L$, for every $\delta \in]0, \frac{1}{2}[$, the number of n in I such that $\|\varphi_n\|_2 < c_1 \delta \sqrt{\ell}$ is less than $c_2 (\delta + \ell^{-1})L$. Choosing $N_2 = n, N_1 := q_{m(N)-u_N}$ with $u_N = \lfloor \frac{1}{2}m(N) \rfloor$ and $\delta = (\ln m(N_1))^{-\frac{1}{2}}$, we obtain that W_B has density 1.

If α is a quadratic irrational, the corresponding Ostrowski expansion is associated with a subshift of finite type and we use a result of large deviations to bound the size of the complementary of V_B . \square

Remark 1. There are also examples of rotations for which there is a non-normal non-degenerate limit law for the normalized ergodic sums along the subsequence giving the biggest variance (see a counter-example in [5]).

4. Application to step functions

To be able to apply the results to a centered BV function $\varphi = \sum_{r \neq 0} \frac{\gamma_r(\varphi)}{r} e^{2\pi i r x}$, we have to check the condition (3) on the coefficients $\gamma_{q_j}(\varphi)$. The functions $\{x\} - \frac{1}{2} = \frac{-1}{2\pi i} \sum_{r \neq 0} \frac{1}{r} e^{2\pi i r x}$ and $\mathbf{1}_{]0, \frac{1}{2}[} - \mathbf{1}_{[\frac{1}{2}, 1[} = \sum_r \frac{2}{\pi i(2r+1)} e^{2\pi i(2r+1)x}$ are immediate examples where (3) is satisfied. In the second case, this is because $\gamma_{q_k} = 0$ or $\frac{2}{\pi i}$, depending on whether q_j is even or odd, and two consecutive q_j 's cannot be both even.

In general, for a step function φ , (3) (and therefore by Theorem 3.1 a lower bound for $\|\varphi_n\|_2$ for many n 's) is related to the Diophantine properties of its discontinuities with respect to α . A generic result follows from the following lemma.

Lemma 4.1. If $\varphi = \sum_{j=0}^s v_j \mathbf{1}_{[u_j, u_{j+1}[} - c$ is a centered step function φ on $[0, 1[$ taking a constant value $v_j \in \mathbb{R}$ on the interval $[u_j, u_{j+1}[$, with $u_0 = 0 < u_1 < \dots < u_s < u_{s+1} = 1$ and c a constant such that φ is centered, there is a function $H_\varphi(u_1, \dots, u_s) \geq 0$ such that $|\gamma_r(\varphi)|^2 = \pi^{-2} H_\varphi(ru_1, \dots, ru_s)$.

Since (q_k) is a strictly increasing sequence of integers, for almost every (u_1, \dots, u_s) in \mathbb{T}^s , the sequence $(q_k u_1, \dots, q_k u_s)_{k \geq 1}$ is uniformly distributed in \mathbb{T}^s . Hence, condition (3) is satisfied for a.e. value of (u_1, \dots, u_s) in \mathbb{T}^s , since for a.e. $(u_1, \dots, u_s) \in \mathbb{T}^s$:

$$\lim_n \frac{1}{n} \sum_{k=1}^n H_\varphi(q_k u_1, \dots, q_k u_s) = \int_{\mathbb{T}^s} H_\varphi(u_1, \dots, u_s) du_1 \dots du_s > 0.$$

Remark 2. For example, if $\varphi = \varphi(u, \cdot) = 1_{[0, u[} - u$, $H_\varphi(u) = \sin^2(\pi u)$, if $\varphi = \varphi(w, u, \cdot) = 1_{[0, u]} - 1_{[w, u+w]}$, $H(\varphi) = 4 \sin^2(\pi u) \sin^2(\pi w)$. Observe that, if α is not bpq, in the first example there are many u 's that do not satisfy the previous equidistribution property. Indeed, let $u = \sum_{n \geq 0} b_n q_n \alpha$ mod 1, $b_n \in \mathbb{Z}$, be the so-called Ostrowski expansion of u , where q_n are the denominators of α . It can be shown that, if $\sum_{n \geq 0} \frac{|b_n|}{a_{n+1}} < \infty$, then $\lim_k \|q_k u\| = 0$. There is an uncountable set of u 's satisfying this condition if α is not bpq. The variance degenerates for these values of u , although the cocycle generated by φ is ergodic (therefore not a coboundary) under the only condition $u \notin \mathbb{Z}\alpha + \mathbb{Z}$.

We can also address the case of vectorial functions. For simplicity, consider a centered vectorial function $\Phi = (\varphi_1, \varphi_2)$ with two components $\varphi_i = \sum_{j=0}^{s_i} v_j^i 1_{[u_j^i, u_{j+1}^i]} - c_i$, for $i = 1, 2$. Now we have to control the covariance matrix. We use the following lemma.

Lemma 4.2. *Let Λ be a compact space and let $(F_\lambda, \lambda \in \Lambda)$ be a family of nonnegative non-identically null continuous functions on \mathbb{T}^d depending continuously on λ . If a sequence (z_k) is equidistributed in \mathbb{T}^d , then $\exists N_0, \eta, \theta_0 > 0$ such that $\text{Card}\{n \leq N : F_\lambda(z_n) \geq \eta\} \geq \theta_0 N, \forall N \geq N_0, \forall \lambda \in \Lambda$.*

Let us consider a linear combination $\varphi_{a,b} = a\varphi_1 + b\varphi_2$. Denote by \underline{u} the parameter $(u_1^1, \dots, u_{s_1}^1, u_1^2, \dots, u_{s_2}^2)$ in $\mathbb{T}^{s_1+s_2}$ and apply the lemma to $F_\lambda(\underline{u}) = H_{a\varphi_1+b\varphi_2}(\underline{u})$, for $\lambda = (a, b)$ in the unit sphere. The set of points \underline{u} for which $(q_k \underline{u})_{k \geq 1}$ is equidistributed in $\mathbb{T}^{s_1+s_2}$ has full measure in $\mathbb{T}^{s_1+s_2}$.

Applying Lemma 4.2 with $z_k = q_k \underline{u}$ for such a point \underline{u} , we obtain that, generically with respect to the discontinuities of (φ_1, φ_2) , condition (3) is satisfied by $a\varphi_1 + b\varphi_2$ uniformly with respect to (a, b) in the set of unit vectors. Therefore, generically, a bi-dimensional analogue of Theorem 3.1 holds for Φ .

There are also special values of the parameter for which the result holds: let us consider the vectorial function appearing in the model of rectangular periodic billiard in the plane studied in [3] (see also [4]): $\Phi = (\varphi_1, \varphi_2)$ with $\varphi_1 = 1_{[0, \frac{\alpha}{2}] - 1_{[\frac{1}{2}, \frac{1}{2} + \frac{\alpha}{2}]}$, $\varphi_2 = 1_{[0, \frac{1}{2} - \frac{\alpha}{2}] - 1_{[\frac{1}{2}, 1 - \frac{\alpha}{2}]}$. The Fourier coefficients of φ_1 and φ_2 of order r are null for r even.

If q_j is even, then $\gamma_{q_j}(\varphi_{a,b})$ is null. If q_j is odd, we have $\gamma_{q_j}(\varphi_{a,b}) = a + O(\frac{1}{q_{j+1}})$, if p_j is odd, $= b + O(\frac{1}{q_{j+1}})$, if p_j is even. It follows that, if α is such that, in average, there is a positive proportion of pairs (p_j, q_j) that are (even, odd) and the same for (odd, odd), then we have for $\Phi = (\varphi_1, \varphi_2)$ a bi-dimensional analogue of Theorem 3.1.

References

[1] J. Beck, Randomness of the square root of 2 and the giant leap, part 1, 2, *Period. Math. Hung.* 60 (2) (2010) 137–242; 62 (2) (2011) 127–246.
 [2] M. Bromberg, C. Ulcigrai, A temporal central limit theorem for real-valued cocycles over rotations, preprint, arXiv:1705.06484.
 [3] J.-P. Conze, E. Gutkin, On recurrence and ergodicity for geodesic flows on non-compact periodic polygonal surfaces, *Ergod. Theory Dyn. Syst.* 32 (2) (2012) 491–515.
 [4] J.-P. Conze, S. Isola, S. Le Borgne, Diffusive behaviour of ergodic sums over rotations, arXiv:1705.10550, *Stoch. Dyn.* (2019), <https://doi.org/10.1142/S0219493719500163>, in press.
 [5] J.-P. Conze, S. Le Borgne, On the CLT for rotations and BV functions, arXiv:1804.09929v2.
 [6] D. Dolgopyat, O. Sarig, Temporal distributional limit theorems for dynamical systems, *J. Stat. Phys.* 166 (2017) 680–713.
 [7] G.H. Hardy, J.E. Littlewood, Some problems of Diophantine approximation: a series of cosecants, *Bull. Calcutta Math. Soc.* 20 (1930) 251–266.
 [8] F. Huveneers, Subdiffusive behavior generated by irrational rotations, *Ergod. Theory Dyn. Syst.* 29 (4) (2009) 1217–1233.