C. R. Acad. Sci. Paris, Ser. I 357 (2019) 59–65

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations

Existence of a renormalized solution to the quasilinear Riccati-type equation in Lorentz spaces

Existence d'une solution renormalisée des équations quasi linéaires de type Riccati dans les espaces de Lorentz

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ARTICLE INFO

Article history:
Received 8 November 2018
Accepted 10 December 2018
Available online 14 December 2018

Presented by the Editorial Board

ABSTRACT

In this paper, we prove the existence of a renormalized solution for the quasilinear Riccati-type equation with low integrability-measure data in Lorentz spaces. The result is established in both regular and singular cases. Our proof is based on the gradient estimates for a solution to a class of quasilinear elliptic equations.

RÉSUMÉ

Nous prouvons dans cet article l'existence d'une solution renormalisée des équations quasi linéaires de type Riccati avec des données de mesure d'intégrabilité faibles sur les espaces de Lorentz. Le résultat est établi dans les cas réguliers et singuliers. La preuve est basée sur les estimations du gradient pour une solution d'une classe d'équations quasi linéaires elliptiques.

1. Introduction

The main goal of this paper is to prove the existence of a renormalized solution to the following quasilinear Riccati-type equation in Lorentz spaces

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) &= |\nabla u|^q + \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

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https://doi.org/10.1016/j.crma.2018.12.001
1631-073X/

where Ω is a bounded open subset of $\mathbb{R}^n$ with $n \geq 2$ and μ is a finite Radon measure in Ω. We moreover consider the nonlinearity $A: \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ as a Carathédory vector-valued function that satisfies growth and monotonicity conditions, i.e. there exist positive constants α, β such that for some $p \in (1, n]$ there holds

$$|A(x,y)| \leq \beta |y|^{p-1},$$
$$\langle A(x,y) - A(x,z), y - z\rangle \geq \alpha \left(|y|^2 + |z|^2\right)^{\frac{p-2}{2}} |y-z|^2,$$

for every $y, z \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega$ almost everywhere.

The existence result for Eq. (1) has been considered by several authors, such as T. Mengesha et al. [11], O. Martio [10] and N. C. Phuc [18], [20], [21]. This type of equation arises from physical theory of surface growth [6], [7], also known as the Kardar–Parisi–Zhang (KPZ) equation which can be viewed as a quasilinear stationary version of a time-dependent viscous Hamilton–Jacobi equation. A simple case of (1) is a type of the p-Laplace equation

$$-\Delta_p u = |\nabla u|^q + \mu,$$

when $A(x,y) = |y|^{p-2}y$. Motivated by these works, we continue to study the solvability of Eq. (1) in Lorentz spaces for the supercritical case $q > \frac{n(p-1)}{n-1}$.

Our work is related to the gradient estimate results for the resolution of the following quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) &= \mu \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

The global gradient estimates of renormalized solution to Eq. (2) were firstly given by G. Mingione in [13], by using the 1-fractional maximal function. Later, many gradient estimate results for Eq. (2) have been studied under various assumptions on the domain Ω and different cases of p in Lorentz and Morrey spaces. For instance, in [19], N. C. Phuc gave the Lorentz global bounds to this equation under the p-capacity uniform thickness condition imposed on the complement of Ω for the regular case $p \in \left(2 - \frac{1}{n}, n\right]$. In [21], the authors presented the gradient estimate of a solution to (2) in Lorentz spaces on the Reifenberg flat domain Ω for the singular case $p \in \left(\frac{3n-2}{2n-1}, 2 - \frac{1}{n}\right]$. And then, in [22], we established the gradient estimate of a solution to (2) for the singular case $p \in \left(\frac{3n-2}{2n-1}, 2 - \frac{1}{n}\right]$ under the uniform thickness condition of the domain. For more results in this problem, we refer the reader to several articles by G. Mingione et al. (in [3], [4], [8], [9], [12], [13]), N. C. Phuc (in [1], [19], [20], [21]), Q. H. Nguyen (in [14], [15], [16], [17]), and references therein.

In the present work, we study the existence of a renormalized solution to Eq. (1) in Lorentz spaces with low integrability measure data under the p-capacity uniform thickness condition on the domain for both singular and regular cases, i.e. $p \in \left(\frac{3n-2}{2n-1}, n\right]$. The basis idea is to apply the gradient estimate results in [19], [22] and Schauder Fixed Point Theorem. Let us now state our main result in the following theorem. We note that the Lorentz space $L^{s,\infty}(\Omega)$ and the norm $|||\cdot|||_{L^{s,\infty}(\Omega)}$ will be defined in the next section.

Theorem 1.1. *Let $n \geq 3$, $\frac{1}{2} + \sqrt{\frac{5}{4}} < p \leq n$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose complement satisfies a p-capacity uniform thickness condition. Assume that*

$$\max\left\{\frac{1}{n} + p - 1;\ \frac{n(p-1)}{n-1}\right\} < q \leq p - 1 + \frac{p(p-1)}{n}. \tag{3}$$

There exists $\delta_0 > 0$ such that if the finite Radon measure μ satisfies

$$|||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)} \leq \delta_0,$$

then the equation (1) admits a renormalized solution u satisfying

$$|||\nabla u|||^q_{L^{n(q-p+1),\infty}(\Omega)} \leq \frac{q(n+1)}{n(q-p+1)}\delta_0 - |||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)}. \tag{4}$$

The rest of the paper is organized as follows. In the next section, we recall the definition of the Lorentz space and then discuss the equivalent between quasi-norm and norm in Lorentz spaces. For the convenience of the reader, we also recall some gradient estimate results for the renormalized solution to Eq. (2) in this section. Finally, we give the proof of Theorem 1.1, which is divided into several lemmas in the last section.

2. Preliminaries

The definition of the p-capacity uniform thickness condition of a domain and of the renormalized solution to Eq. (2) can be found in [19] or [22]. We now recall the definition of Lorentz spaces (see [5]). For some $0 < s < \infty$ and $0 < t \le \infty$, the Lorentz space $L^{s,t}(\Omega)$ is the set of all Lebesgue measurable functions f on Ω such that $\|f\|_{L^{s,t}(\Omega)} < \infty$, where

$$\|f\|_{L^{s,t}(\Omega)} := \begin{cases} \left(s \int_0^\infty \lambda^s \, |\{x \in \Omega : |f(x)| > \lambda\}|^{\frac{t}{s}} \frac{\mathrm{d}\lambda}{\lambda} \right)^{\frac{1}{t}}, & t < \infty, \\ \sup_{\lambda > 0} \lambda \, |\{x \in \Omega : |f(x)| > \lambda\}|^{\frac{1}{s}}, & t = \infty, \end{cases}$$

and $|\mathcal{O}|$ denotes the n-dimensional Lebesgue measure of a set $\mathcal{O} \subset \mathbb{R}^n$. In the case of $t = s$, the Lorentz space $L^{s,s}(\Omega)$ coincides with the Lebesgue space $L^s(\Omega)$. Moreover, for $1 < r < s < \infty$, we have

$$L^s(\Omega) \subset L^{s,\infty}(\Omega) \subset L^r(\Omega).$$

Lemma 2.1. *Let Ω be a subset of $\mathbb{R}^n$ and $E \subset \Omega$ such that $|E| > 0$. For any $0 < r < s < \infty$ and $f \in L^{s,\infty}(\Omega)$, there holds*

$$\int_E |f(x)|^r \mathrm{d}x \le \frac{s}{s-r} |E|^{1-\frac{r}{s}} \|f\|^r_{L^{s,\infty}(\Omega)}. \tag{5}$$

Proof. We first recall that for any $\alpha > 0$, we have

$$|\{x \in \Omega : \ |f(x)| > \alpha\}| \le \frac{1}{\alpha^s} \int_{\{x \in \Omega: \ |f(x)| > \alpha\}} |f(x)|^s \mathrm{d}x.$$

It deduces that

$$|\{x \in E : \ |f(x)| > \alpha\}| \le \min\{|E|, \alpha^{-s} \|f\|^s_{L^{s,\infty}(\Omega)}\}.$$

Thus we can estimate as follows

$$\begin{aligned} \int_E |f(x)|^r \mathrm{d}x &= r \int_0^\infty \alpha^{r-1} |\{x \in E : \ |f(x)| > \alpha\}| \mathrm{d}\alpha \\ &\le r \int_0^{\alpha_0} \alpha^{r-1} |\{x \in E : \ |f(x)| > \alpha\}| \mathrm{d}\alpha \\ &\quad + r \int_{\alpha_0}^\infty \alpha^{r-1} |\{x \in E : \ |f(x)| > \alpha\}| \mathrm{d}\alpha \\ &\le r \int_0^{\alpha_0} \alpha^{r-1} |E| \mathrm{d}\alpha + r \int_{\alpha_0}^\infty \alpha^{r-s-1} \|f\|^s_{L^{s,\infty}(\Omega)} \mathrm{d}\alpha \\ &= \alpha_0^r |E| + \frac{r}{s-r} \alpha_0^{r-s} \|f\|^s_{L^{s,\infty}(\Omega)}. \end{aligned}$$

We finally obtain (5) by choosing $\alpha_0 = |E|^{-\frac{1}{s}} \|f\|_{L^{s,\infty}(\Omega)}$. $\square$

In [5], it is known that $\|\cdot\|_{L^{s,\infty}(\Omega)}$ is just a quasi-norm in $L^{s,\infty}(\Omega)$. Let us introduce a norm in $L^{s,\infty}(\Omega)$. For any $s \in (1, \infty)$ and $f \in L^{s,\infty}(\Omega)$, we define

$$|||f|||_{L^{s,\infty}(\Omega)} := \sup_{0 < |E|, \ E \subset \Omega} \left(|E|^{-1+\frac{1}{s}} \int_E |f(x)| \mathrm{d}x \right).$$

The nice feature is that the quasi-norm $\|\cdot\|_{L^{s,\infty}(\Omega)}$ and $|||\cdot|||_{L^{s,\infty}(\Omega)}$ are equivalent in the Lorentz space $L^{s,\infty}(\Omega)$.

Lemma 2.2. *Let Ω be a subset of $\mathbb{R}^n$, for any $s \in (1,\infty)$ and $f \in L^{s,\infty}(\Omega)$, there holds*

$$\|f\|_{L^{s,\infty}(\Omega)} \le |||f|||_{L^{s,\infty}(\Omega)} \le \frac{s}{s-1}\|f\|_{L^{s,\infty}(\Omega)}. \tag{6}$$

Proof. By definition, for any $E \subset \Omega$ such that $|E| > 0$, we have

$$|||f|||_{L^{s,\infty}(\Omega)} \ge |E|^{-1+\frac{1}{s}} \int_E |f(x)|\mathrm{d}x.$$

For every $\alpha > 0$, let us take $E = \{x \in \Omega : \ |f(x)| > \alpha\}$. Then we obtain

$$|||f|||_{L^{s,\infty}(\Omega)} \ge \alpha |\{x \in \Omega : \ |f(x)| > \alpha\}|^{\frac{1}{s}},$$

which yields $\|f\|_{L^{s,\infty}(\Omega)} \le |||f|||_{L^{s,\infty}(\Omega)}$. The remain part can be deduced from Lemma 2.1. $\square$

We now recall the gradient estimate results for both singular and regular cases in [22, Theorem 1.2] and [19, Theorem 1.1].

Theorem 2.3. *Let $n \ge 2$, $p \in \left(\frac{3n-2}{2n-1}, n\right]$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain whose complement satisfies a p-capacity uniform thickness condition. There exists $C > 0$ such that, for any renormalized solution u to (2) with finite Radon measure data μ, $s \in (0, p]$ and $t \in (0,\infty]$, there holds*

$$\|\nabla u\|_{L^{s,t}(\Omega)} \le C\|[\mathbf{M}_1(\mu)]^{\frac{1}{p-1}}\|_{L^{s,t}(\Omega)}.$$

In Theorem 2.3, the fractional maximal function $\mathbf{M}_1$ of finite measure μ is defined by:

$$\mathbf{M}_1(\mu)(x) = \sup_{R>0} \frac{|\mu|(B_R(x))}{R^{n-1}}, \quad \forall x \in \mathbb{R}^n, \tag{7}$$

where $B_R(x)$ denotes the ball of radius R and center x. The boundedness property of the fractional maximal function $\mathbf{M}_1$ is given by the next lemma.

Lemma 2.4. *Let $1 < s < n$ and μ be a finite Radon measure on $\mathbb{R}^n$. There exists a constant $C = C(n,s) > 0$ such that*

$$\|\mathbf{M}_1[\mu]\|_{L^{\frac{sn}{n-s},\infty}(\mathbb{R}^n)} \le C\|\mu\|_{L^{s,\infty}(\mathbb{R}^n)}.$$

Applying Theorem 2.3 and Lemma 2.4, we obtain the following corollary, which is useful for the proof of our main theorem.

Corollary 2.5. *Under the hypotheses of Theorem 2.3 and Lemma 2.4, assume moreover that $\frac{sn}{n-s} \le p$. Then there exists a positive constant C such that, for any renormalized solution u to (2) with given measure data μ and for some $q > 0$, there holds*

$$|||\nabla u|^q\|_{L^{\frac{s(p-1)n}{q(n-s)},\infty}(\Omega)} \le C\|\mu\|^{\frac{q}{p-1}}_{L^{s,\infty}(\Omega)}.$$

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. The main idea of the proof comes from Schauder Fixed Point Theorem for a continuous map $T : \ V \to V$, where V is closed, convex and $T(V)$ is precompact under the strong topology of $W_0^{1,1}(\Omega)$. We have divided the proof into a sequence of lemmas under all hypotheses of Theorem 1.1.

We here emphasize that if q satisfies (3) then we obtain $n(q-p+1) > 1$ and $\frac{n(q-p+1)}{q} > 1$. For any $\lambda > 0$, let us first introduce the set V_λ as follows

$$V_\lambda = \left\{u \in W_0^{1,1}(\Omega) : \ |||\nabla u|||_{L^{n(q-p+1),\infty}(\Omega)} \le \lambda\right\}.$$

Lemma 3.1. *For any $\lambda > 0$, the set V_λ is convex and closed under the strong topology of $W_0^{1,1}(\Omega)$.*

Proof. We first show that V_λ is closed under the strong topology of $W_0^{1,1}(\Omega)$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence in V_λ such that u_k converges strongly in $W_0^{1,1}(\Omega)$ to a function u. We need to show that $u \in V_\lambda$. Let E be a subset of Ω such that $|E| > 0$, we have

$$|E|^{-1+\frac{1}{n(q-p+1)}} \int_E |\nabla u_k| dx \le |||\nabla u_k|||_{L^{n(q-p+1),\infty}(\Omega)} \le \lambda.$$

We note that ∇u_k converges to ∇u almost everywhere. By the Fatou lemma it follows that

$$|E|^{-1+\frac{1}{n(q-p+1)}} \int_E |\nabla u| dx \le \lambda.$$

We thus get

$$|||\nabla u|||_{L^{n(q-p+1),\infty}(\Omega)} = \sup_{0<|E|,\, E\subset\Omega} \left(|E|^{-1+\frac{1}{n(q-p+1)}} \int_E |\nabla u| dx \right) \le \lambda,$$

which leads to $u \in V_\lambda$.

We next prove that V_λ is convex. For any $u, v \in V_\lambda$ and $t \in [0, 1]$, we need to show that $w = tu + (1-t)v \in V_\lambda$. Let E be a subset of Ω such that $|E| > 0$, let us set $s = n(q-p+1)$ for simplicity; we have

$$\begin{aligned} |E|^{-1+\frac{1}{s}} \int_E |\nabla w| dx &\le |E|^{-1+\frac{1}{s}} \left(t \int_E |\nabla u| dx + (1-t) \int_E |\nabla v| dx \right) \\ &\le t|||\nabla u|||_{L^{s,\infty}(\Omega)} + (1-t)|||\nabla v|||_{L^{s,\infty}(\Omega)} \\ &\le t\lambda + (1-t)\lambda = \lambda. \end{aligned}$$

We obtain that $|||\nabla w|||_{L^{s,\infty}(\Omega)} \le \lambda$, which gives $w \in V_\lambda$. $\square$

Next, we introduce a mapping T that has a fixed point by Schauder Fixed Point Theorem. For any $v \in V_\lambda$, let u be a unique renormalized solution to the equation

$$\begin{cases} -\mathrm{div}(A(x, \nabla u)) &= |\nabla v|^q + \mu \quad \text{in } \Omega, \\ \qquad\qquad u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{8}$$

We define the map $T: V_\lambda \to V_\lambda$ by $T(v) = u$.

Lemma 3.2. *There exists $\delta_0 > 0$ and $\lambda_0 > 0$ such that if*

$$|||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)} \le \delta_0, \tag{9}$$

then the map $T: V_{\lambda_0} \to V_{\lambda_0}$ is well defined.

Proof. For simplicity, we denote $s = \frac{n(q-p+1)}{q}$ and note that $s > 1$. By Corollary 2.5 and Lemma 2.2, there exists a positive constant C such that, for any renormalized solution u to (2), we have

$$|||\nabla u|||^{p-1}_{L^{qs,\infty}(\Omega)} \le C|||\mu|||_{L^{s,\infty}(\Omega)}. \tag{10}$$

Applying Lemma 3.3 below with $r = \dfrac{q}{p-1} > 1$, $b = \dfrac{s}{s-1}C$ and $c = C$, there exists $\delta_0 > 0$ such that, if

$$|||\mu|||_{L^{s,\infty}(\Omega)} \le \delta_0,$$

then the function f defined by (11) admits one root $t_0 > 0$. This means that

$$\frac{s}{s-1}Ct_0 + C|||\mu|||_{L^{s,\infty}(\Omega)} = t_0^{\frac{p-1}{q}}.$$

Let us set $\lambda_0 = t_0^{\frac{1}{q}}$. By the definition of T, for any $v \in V_{\lambda_0}$, $u = T(v) \in W_0^{1,1}(\Omega)$ is the unique renormalized solution to (8). Applying (10) and Lemma 2.2, we obtain

$$
\begin{aligned}
|||\nabla u|||_{L^{qs,\infty}(\Omega)}^{p-1} &\le C|||\,|\nabla v|^q+\mu|||_{L^{s,\infty}(\Omega)}\\
&\le C\left[\frac{s}{s-1}\|\nabla v\|^q_{L^{qs,\infty}(\Omega)}+|||\mu|||_{L^{s,\infty}(\Omega)}\right]\\
&\le C\left[\frac{s}{s-1}\lambda_0^q+|||\mu|||_{L^{s,\infty}(\Omega)}\right]\\
&= C\left[\frac{s}{s-1}t_0+|||\mu|||_{L^{s,\infty}(\Omega)}\right]\\
&= t_0^{\frac{p-1}{q}}=\lambda_0^{p-1},
\end{aligned}
$$

which yields $T(v)=u\in V_{\lambda_0}$. We conclude that the map T is well defined. $\square$

Lemma 3.3. *Given $r>1$ and $b\ge c>0$, there exists a positive constant δ_0 such that for any $a\in(0,\delta_0]$, the function $f:[0,\infty)\to\mathbb{R}$ defined by*

$$
f(t)=(bt+ca)^r-t, \tag{11}
$$

has at least one root $t_0=t_0(r,a,b,c)>0$.

Proof. Let us choose $\delta_0=(r-1)(br)^{-\frac{r}{r-1}}>0$. Then, for any $a\in(0,\delta_0]$, the function f given by (11) satisfies $f(0)>0$ and $\lim_{t\to\infty}f(t)=\infty$. Moreover, $f'(t)=br(bt+ca)^{r-1}-1$, thus $f'(t)=0$ if and only if $t=t^*$, where

$$
t^*=\frac{1}{b}(br)^{-\frac{1}{r-1}}-\frac{ca}{b}=\frac{r}{r-1}\delta_0-\frac{ca}{b}.
$$

It follows that the minimum value of f on $[0,\infty)$ is

$$
f(t^*)=(bt^*+ca)\frac{1}{br}-t^*=\frac{ca}{b}-\delta_0\le a-\delta_0\le 0.
$$

For this reason, we conclude that f has exactly one root $t_0\in(0,t^*]$, which completes the proof. $\square$

Lemma 3.4. *$T:V_{\lambda_0}\to V_{\lambda_0}$ is continuous, and $\overline{T(V_{\lambda_0})}$ is a compact set under the strong topology of $W_0^{1,1}(\Omega)$.*

Proof. We first prove that T is continuous under the strong topology of $W_0^{1,1}(\Omega)$. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence in V_{λ_0} such that v_k converges strongly in $W_0^{1,1}(\Omega)$ to a function $v\in V_{\lambda_0}$. For every $k\in\mathbb{N}$, $u_k=T(v_k)$ is the renormalized solution to the equation

$$
\begin{cases}
-\operatorname{div}(A(x,\nabla u_k)) &= |\nabla v_k|^q+\mu \quad \text{in } \Omega,\\
\qquad\qquad u_k &= 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{12}
$$

with

$$
|||\nabla v_k|||_{L^{n(q-p+1),\infty}(\Omega)}\le\lambda_0. \tag{13}
$$

According to Lemma 2.1, Lemma 2.2 and (13), we obtain that

$$
\|\nabla v_k\|_{L^r(\Omega)}\le\lambda_0, \tag{14}
$$

for any $q<r<n(q-p+1)$. Hence, there exists a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}}$ of $\{v_k\}$ such that ∇v_{k_j} converges to ∇v almost everywhere in Ω. By (14) and Vitali's Convergence Theorem, we have that ∇v_{k_j} converges to ∇v strongly in $L^q(\Omega)$. It follows that ∇v_k converges to ∇v strongly in $L^q(\Omega)$.

By the stability result of the renormalized solution in [2, Theorem 3.4], there exists a subsequence $\{u_{k_j}\}$ such that $\{u_{k_j}\}$ converges to u almost everywhere in Ω, where u is the unique renormalized solution to the following equation:

$$
\begin{cases}
-\operatorname{div}(A(x,\nabla u)) &= |\nabla v|^q+\mu \quad \text{in } \Omega,\\
\qquad\qquad u &= 0 \quad \text{on } \partial\Omega.
\end{cases}
$$

Moreover, ∇u_{k_j} also converges to ∇u almost everywhere in Ω. It is similar to the above; using again Vitali's Convergence Theorem with the facts that $n(q-p+1)>1$ and

$$|||\nabla u_{k_j}|||_{L^{n(q-p+1),\infty}(\Omega)} \leq \lambda_0,$$

we deduce that u_k converges strongly to u in $W_0^{1,1}(\Omega)$. It follows that T is continuous.

The compactness of the set $\overline{T(V_{\lambda_0})}$ under the strong topology of $W_0^{1,1}(\Omega)$ can be proved by the same method as in the above. Indeed, let $\{u_k\} = \{T(v_k)\}$ be a sequence in $T(V_{\lambda_0})$ where $\{v_k\} \subset V_{\lambda_0}$, then we have (12), (13). Using the proof of Theorem 3.4 in [2], there exist a subsequence $\{u_{k_j}\}$ and a function $u \in W_0^{1,1}(\Omega)$ such that $\nabla u_{k_j} \to \nabla u$ almost everywhere in Ω. Finally, by Vitali's Convergence Theorem, we obtain that $\{u_{k_j}\}$ strongly converges to u in $W_0^{1,1}(\Omega)$. $\square$

Proof of Theorem 1.1. By Lemma 3.1, Lemma 3.2 and Lemma 3.4, there exist positive constants δ_0 and λ_0 such that, if

$$|||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)} \leq \delta_0,$$

then the map $T: V_{\lambda_0} \to V_{\lambda_0}$ is continuous and $\overline{T(V_{\lambda_0})}$ is compact under the strong topology of $W_0^{1,1}(\Omega)$, where V_{λ_0} is closed and convex. Using Schauder's Fixed Point Theorem, T has a fixed point in V_{λ_0}. This gives a solution u to Eq. (1). Moreover, in the proof of Lemma 3.2, we obtain

$$\begin{aligned} |||\nabla u|||^q_{L^{n(q-p+1),\infty}(\Omega)} &\leq \frac{q}{q-p+1}\delta_0 - \frac{s-1}{s}|||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)} \\ &\leq \frac{q(n+1)}{n(q-p+1)}\delta_0 - |||\mu|||_{L^{\frac{n(q-p+1)}{q},\infty}(\Omega)}, \end{aligned}$$

where $s = \frac{n(q-p+1)}{q}$. This finishes the proof. $\square$

Acknowledgements

The second author was supported by Ho Chi Minh City University of Education under grant CS.2018.19.02TD.

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