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Brézis–Gallouet–Wainger-type inequality with critical fractional Sobolev space and BMO

Inégalité de type Brézis–Gallouet–Wainger pour un espace de Sobolev fractionnaire critique et BMO

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we prove the Brézis–Gallouet–Wainger-type inequality involving the BMO norm, the fractional Sobolev norm, and the logarithmic norm of $\dot{\mathcal{C}}^{\eta}$, for $\eta \in (0,1)$.

R É S U M É

Dans cette Note, nous montrons l'inégalité de type Brézis–Gallouet–Wainger faisant intervenir la norme BMO, la norme fractionnaire de Sobolev et la norme logarithmique de $\dot{\mathcal{C}}^{\eta}$, pour $\eta \in (0,1)$.

1. Introduction and main results

The main purpose of this paper is to establish the L^∞-bound by means of the BMO norm, or the critical fractional Sobolev norm with the logarithm of $\dot{\mathcal{C}}^{\eta}$ norm. Such a L^∞-estimate of this type is known as the Brézis–Gallouet–Wainger (BGW)-type inequality. Let us remind that Brézis–Gallouet [2], and Brézis–Wainger [3] considered the relation between L^∞, $W^{k,r}$, and $W^{s,p}$, and proved that there holds

$$\|f\|_{L^\infty} \le C\left(1 + \log^{\frac{r-1}{r}}(1 + \|f\|_{W^{s,p}})\right), \quad sp > n \tag{1.1}$$

provided that $\|f\|_{W^{k,r}} \le 1$, for $kr = n$. Its application is to prove the existence of solutions to the nonlinear Schrödinger equations, see details in [2]. We also note that an alternative proof of (1.1) was given by H. Engler [4] for any bounded set in $\mathbb{R}^n$ with the cone condition. Similar embedding for vector functions u with $\operatorname{div} u = 0$ was investigated by Beale–Kato–Majda:

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$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|\mathrm{rot} u\|_{L^\infty}\left(1 + \log(1 + \|u\|_{W^{s+1,p}})\right) + \|\mathrm{rot} u\|_{L^2}\right), \tag{1.2}$$

for $sp > n$, see [1] (see also [9] for an improvement of (1.2) in a bounded domain). An application of (1.2) is to prove the breakdown of smooth solutions to the 3-D Euler equations. After that, estimate (1.2) was enhanced by Kozono and Taniuchi [5] in that $\|\mathrm{rot} u\|_{L^\infty}$ can be relaxed to $\|\mathrm{rot} u\|_{\mathrm{BMO}}$:

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|\mathrm{rot} u\|_{\mathrm{BMO}}\left(1 + \log(1 + \|u\|_{W^{s+1,p}})\right)\right). \tag{1.3}$$

To obtain (1.3), Kozono–Taniuchi [5] proved a logarithmic Sobolev inequality in terms of BMO norm and Sobolev norm, in which, for any $1 < p < \infty$, and for $s > n/p$, there is a constant $C = C(n, p, s)$ such that the estimate

$$\|f\|_{L^\infty} \le C\left(1 + \|f\|_{\mathrm{BMO}}\left(1 + \log^+(\|f\|_{W^{s,p}})\right)\right) \tag{1.4}$$

holds for all $f \in W^{s,p}$. Obviously, (1.4) is a generalization of (1.1).

Besides, it is interesting to note that a Gagliardo–Nirenberg-type inequality with critical Sobolev space directly yields a BGW-type inequality. For example, H. Kozono and H. Wadade [6] proved the Gagliardo–Nirenberg-type inequalities for the critical case and the limiting case of a Sobolev space as follows:

$$\|u\|_{L^q} \le C_n r' q^{\frac{1}{r'}} \|u\|_{L^p}^{\frac{p}{q}} \|(-\Delta)^{\frac{n}{2r}} u\|_{L^r}^{1-\frac{p}{q}} \tag{1.5}$$

holds for all $u \in L^p \cap \dot{H}^{\frac{n}{r},r}$ with $1 \le p < \infty$, $1 < r < \infty$, and for all q with $p \le q < \infty$ (see also Ozawa [10]).

Also,

$$\|u\|_{L^q} \le C_n q \|u\|_{L^p}^{\frac{p}{q}} \|u\|_{\mathrm{BMO}}^{1-\frac{p}{q}} \tag{1.6}$$

holds for all $u \in L^p \cap BMO$ with $1 \le p < \infty$, and for all q with $p \le q < \infty$.

As a result, (1.5) implies

$$\|u\|_{L^\infty} \le C\left(1 + (\|u\|_{L^p} + \|(-\Delta)^{\frac{n}{2r}} u\|_{L^r})\left(\log(1 + \|(-\Delta)^{\frac{s}{2}} u\|_{L^q})\right)^{\frac{1}{r'}}\right) \tag{1.7}$$

for every $1 \le p < \infty$, $1 < r < \infty$, $1 < q < \infty$ and $n/q < s < \infty$.

Furthermore, (1.6) yields

$$\|u\|_{L^\infty} \le C\left(1 + (\|u\|_{L^p} + \|u\|_{\mathrm{BMO}}) \log(1 + \|(-\Delta)^{\frac{s}{2}} u\|_{L^q})\right) \tag{1.8}$$

for every $1 \le p < \infty$, $1 < q < \infty$, and $n/q < s < \infty$.

Thus, (1.7) and (1.8) may be regarded as generalizations of the BGW inequality. Note that, in (1.7) and (1.8), the logarithm term only contains the semi-norm $\|u\|_{\dot{W}^{s,p}}$.

Furthermore, Kozono, Ogawa, Taniuchi [7] proved the logarithmic Sobolev inequalities in Besov space, generalizing the BGW inequality and the Beale–Kato–Majda inequality.

Motivated by the above results, we study in this paper the BGW-type inequality by means of the BMO norm, the fractional Sobolev norm, and the $\dot{C}^\eta$ norm, for $\eta \in (0,1)$. Then, our first result is as follows.

Theorem 1.1. *Let $\eta \in (0,1)$, and $\alpha \in (0,n)$. Then, there exists a constant $C = C(n,\eta) > 0$ such that the estimate*

$$\|f\|_{L^\infty} \le C + C\|f\|_{\mathrm{BMO}}\left(1 + \log^+\left[\sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} \mathrm{d}y + \|f\|_{\dot{C}^\eta}\right]\right) \tag{1.9}$$

holds for all $f \in \dot{C}^\eta \cap BMO$. We accept the notation $\log^+ s = \log s$ if $s \ge 1$, and $\log^+ s = 0$ if $s \in (0,1)$.

Remark 1.2. It is clear that $\left(\sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} \mathrm{d}y\right)$ is finite if $f \in L^1$. On the other hand, if $f \in L^r$, $r > 1$, then for any $\alpha \in (\frac{n}{r}, n)$, we have

$$\sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} \mathrm{d}y \le C\|f\|_{L^r},$$

where the constant C is independent of f.

Remark 1.3. If supp $f \subset B_R$, then (1.9) implies

$$\|f\|_{L^\infty} \le C + C\|f\|_{\mathrm{BMO}}\left(1 + \log^+\left[R^{n-\alpha+\eta} + \|f\|_{\dot{\mathcal{C}}^\eta}\right]\right). \tag{1.10}$$

Remark 1.4. Note that if $f \in W^{s,p}$ with $sp > n$, then (1.9) is stronger than (1.4) since $W^{s,p} \subset \mathcal{C}^{0,\eta} \subset \dot{\mathcal{C}}^\eta$, with $\eta = \frac{sp-n}{p}$.

Concerning the BGW-type inequality involving the fractional Sobolev space, we have the following result.

Theorem 1.5. *Let $s > 0$, $p \ge 1$ be such that $sp = n$. Let $\alpha > 0$, $\eta \in (0,1)$. Then, there exists a constant $C = C(n,s,p,\eta,\alpha) > 0$ such that the estimate*

$$\|f\|_{L^\infty} \le C + C\|f\|_{\dot{W}^{s,p}}\left(1 + \left(\log^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)|}{(|z-y|+1)^\alpha}\mathrm{d}y + \|f\|_{\dot{\mathcal{C}}^\eta}\right)\right)^{\frac{p-1}{p}}\right) \tag{1.11}$$

holds for all $f \in \dot{\mathcal{C}}^\eta \cap \dot{W}^{s,p}$, where $\dot{W}^{s,p}$ is the homogeneous fractional Sobolev space, see its definition below.

Remark 1.6. As Remark 1.4, we can see that (1.11) is stronger than (1.1). Furthermore, if supp $f \subset B_R$, then (1.9) implies

$$\|f\|_{L^\infty} \le C + C\|f\|_{\dot{W}^{s,p}}\left(1 + \left(\log^+\left[R^{n-\alpha+\eta} + \|f\|_{\dot{\mathcal{C}}^\eta}\right]\right)^{\frac{p-1}{p}}\right). \tag{1.12}$$

Remark 1.7. We consider $f_\delta(x) = -\log(|x|+\delta)\psi(|x|)$, where $\psi \in \mathcal{C}_c^1([0,\infty))$, $0 \le \psi \le 1$, $\psi(|x|) = 1$ if $|x| \le \frac{1}{4}$, and $\delta > 0$ is small enough. It is not hard to see that, for any $\delta > 0$ small enough,

$$||f_\delta||_{L^\infty(\mathbb{R}^d)} \sim |\log(\delta)|, \quad ||f_\delta||_{\mathrm{BMO}(\mathbb{R}^d)} \sim 1, \quad \|f_\delta\|_{\dot{W}^{\frac{n}{p},p}} \sim |\log(\delta)|^{\frac{1}{p}},$$

and

$$\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_\delta(y)|}{(|z-y|+1)^\alpha}\mathrm{d}y \sim 1, \quad \|f_\delta\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n)} \lesssim \delta^{-1}.$$

Therefore, the powers 1 and $\frac{p-1}{p}$ of the term $\log_2^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)|}{(|z-y|+1)^\alpha}\mathrm{d}y + \|f\|_{\dot{\mathcal{C}}^\eta}\right)$ in (1.9) and (1.11), respectively, are sharp, so there are no such estimates of the form:

$$\|f_1\|_\infty \le C + C\|f_1\|_{\mathrm{BMO}}\left(1 + \left(\log_2^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1(y)|}{(|z-y|+1)^\alpha}\mathrm{d}y + \|f_1\|_{\dot{\mathcal{C}}^\eta}\right)\right)^\gamma\right),$$

and

$$\|f_2\|_{L^\infty} \le C + C\|f_2\|_{\dot{W}^{\frac{n}{p},p}}\left(1 + \left(\log^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_2(y)|}{(|z-y|+1)^\alpha}\mathrm{d}y + \|f_2\|_{\dot{\mathcal{C}}^\eta}\right)\right)^{\gamma\frac{p-1}{p}}\right),$$

hold for all $f_1 \in BMO \cap \dot{\mathcal{C}}^\eta$, $f_2 \in \dot{\mathcal{C}}^\eta \cap \dot{W}^{s,p}$, for some $\gamma \in (0,1)$.

Before closing this section, let us introduce some functional spaces that we use throughout this paper. First of all, we recall $\dot{\mathcal{C}}^\eta$, $\eta \in (0,1)$, as the homogeneous Hölder continuous of order η, endowed with the semi-norm:

$$\|f\|_{\dot{\mathcal{C}}^\eta} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\eta}.$$

Next, if $s \in (0,1)$, then we recall $\dot{W}^{s,p}$ the homogeneous fractional Sobolev space, endowed with the semi-norm:

$$\|f\|_{\dot{W}^{s,p}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}.$$

When $s>1$, and s is not an integer, we denote $\dot{W}^{s,p}$ as the homogeneous fractional Sobolev space endowed with the semi-norm:

$$\|f\|_{\dot{W}^{s,p}}=\sum_{|\sigma|=[s]}\|D^{\sigma}f\|_{\dot{W}^{s-[s],p}}.$$

If s is an integer, then

$$\|f\|_{\dot{W}^{s,p}}=\sum_{|\sigma|=[s]}\|D^{\sigma}f\|_{L^{p}}.$$

We refer to [8] for details on the fractional Sobolev space.

After that, we accept the notation $(f)_{\Omega}:=\fint_{\Omega}f=\frac{1}{|\Omega|}\int_{\Omega}f(x)\mathrm{d}x$ for any Borel set Ω. Finally, C is always denoted as a constant that can change from line to line. And $C(k,n,l)$ means that this constant merely depends on k,n,l.

2. Proof of the theorems

We first prove Theorem 1.1.

Proof of Theorem 1.1. It is enough to prove that

$$|f(0)|\le C+C\|f\|_{\mathrm{BMO}}\left(1+\log_2^{+}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y+\|f\|_{\dot{C}^{\eta}}\right)\right). \tag{2.1}$$

Let $m_0\in\mathbb{N}$, set $B_\rho:=B_\rho(0)$, we have

$$\begin{aligned}|f(0)|&=\left|f(0)-\fint_{B_{2^{-m_0}}}f+\sum_{j=-m_0}^{m_0-1}\left(\fint_{B_{2^j}}f-\fint_{B_{2^{j+1}}}f\right)+\fint_{B_{2^{m_0}}}f\right|\\&\le\fint_{B_{2^{-m_0}}}|f-f(0)|+\sum_{j=-m_0}^{m_0-1}\fint_{B_{2^j}}|f-(f)_{B_{2^{j+1}}}|+C2^{-m_0(n-\alpha)}\int_{B_{2^{m_0}}}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y\\&\le\fint_{B_{2^{-m_0}}}|y|^{\eta}\|f\|_{\dot{C}^{\eta}}\mathrm{d}y+2m_0\|f\|_{\mathrm{BMO}}+C2^{-m_0(n-\alpha)}\int_{B_{2^{m_0}}}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y\\&\le C2^{-m_0\min\{n-\alpha,\eta\}}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y+\|f\|_{\dot{C}^{\eta}}\right)+Cm_0\|f\|_{\mathrm{BMO}}.\end{aligned}$$

Choosing

$$m_0=\left[\frac{\log_2^{+}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y+\|f\|_{\dot{C}^{\eta}}\right)}{\min\{n-\alpha,\eta\}}\right]+1,$$

we get (2.1). The proof is complete. □

Next, we prove Theorem 1.5.

Proof of Theorem 1.5. To prove it, we need the following lemmas.

Lemma 2.1. *Let $a_0=1$, and let $(a_0,a_1,\dots,a_{k+1})\in\mathbb{R}^{k+2}$, for any $k\ge1$, be a unique solution to the following system:*

$$\sum_{j=0}^{k+1}a_j2^{jl}=0,\quad\forall\, l=0,\dots,k. \tag{2.2}$$

Then we have:

$$a := \sum_{j=0}^{k}(k-j+1)a_j \neq 0. \tag{2.3}$$

Moreover, for any $m \geq 1$, and for $b, b_l \in \mathbb{R}$, $l = -m, \ldots, m$, we have

$$\sum_{l=-m}^{m-1}\left[\sum_{j=0}^{k+1} a_j b_{j+l}\right] = \sum_{l=m}^{k+m}\left[\sum_{j=l-m+1}^{k+1} a_j\right] b_l + \sum_{l=-m}^{k-m}\left[\sum_{j=0}^{l+m} a_j\right](b_l - b) + ab. \tag{2.4}$$

As a result, we obtain

$$|b| \leq \frac{1}{|a|}\left[\sum_{j=0}^{k+1}|a_j|\right]\sum_{l=-m}^{k-m}|b_l - b| + \frac{1}{|a|}\sum_{l=-m}^{m-1}\left|\sum_{j=0}^{k+1} a_j b_{j+l}\right| + \frac{1}{|a|}\left[\sum_{j=0}^{k+1}|a_j|\right]\sum_{l=m}^{k+m}|b_l|. \tag{2.5}$$

Proof. First of all, we note that $a_j \neq 0$, for $j = 0, \ldots, k+1$. Set

$$Q(x) = \sum_{j=0}^{k+1} a_j x^j.$$

Then,

$$Q'(1) = \sum_{j=1}^{k+1} j a_j.$$

On the other hand, by (2.2), we have $Q(2^l) = 0$, for $l = 0, \ldots, k$. Thus,

$$Q(x) = a_{k+1}\prod_{l=0}^{k}(x - 2^l), \text{ and } Q'(1) = \prod_{l=1}^{k}(1 - 2^l).$$

This implies

$$\sum_{j=1}^{k+1} j a_j = \prod_{j=1}^{k}(1 - 2^l) \neq 0. \tag{2.6}$$

Next, we observe that

$$0 = (k+1)\sum_{j=0}^{k+1} a_j = a + \sum_{j=1}^{k+1} j a_j = 0.$$

The last equation and (2.6) yield $a = -\prod_{j=1}^{k}(1 - 2^l) \neq 0$.

Now, we prove (2.4). We denote by *LHS* (resp. *RHS*) the left-hand side (resp. the right-hand side) of (2.4). It is not difficult to verify that

$$\sum_{l=-m}^{k-m}\left[\sum_{j=0}^{l+m} a_j\right] b = ab.$$

Then, a direct computation shows

$$\begin{aligned} RHS &= a_0 b_{-m} + (a_0 + a_1) b_{1-m} + \ldots + (a_0 + \ldots + a_k) b_{k-m} \\ &\quad + (a_1 + \ldots + a_{k+1}) b_m + (a_2 + \ldots + a_{k+1}) b_{m+1} + \ldots + a_{k+1} b_{k+m} = a_0 \sum_{l=-m}^{k-m} b_l \\ &\quad + a_1\left(\sum_{l=1-m}^{k-m} b_l + \sum_{l=m}^{m} b_l\right) + \ldots + a_k\left(\sum_{l=k-m}^{k-m} b_{k-m} + \sum_{l=m}^{m+k-1} b_l\right) + a_{k+1}\left(\sum_{l=m}^{m+k} b_l\right). \end{aligned}$$

Note that $\left(\sum_{j=0}^{k+1} a_j\right) \sum_{l=k+1-m}^{m-1} b_l = 0$. Thus,

$$\begin{aligned}
RHS &= RHS + \left(\sum_{j=0}^{k+1} a_j\right) \sum_{l=k+1-m}^{m-1} b_l \\
&= \sum_{j=0}^{k+1} a_j \left(\sum_{l=j-m}^{j+m-1} b_l\right) \\
&= \sum_{l=m}^{k+m} \left(\sum_{j=l-m+1}^{k+1} a_j\right) b_l + \sum_{l=k+1-m}^{m-1} \left(\sum_{j=0}^{k+1} a_j\right) b_l + \sum_{l=-m}^{k-m} \left(\sum_{j=0}^{l+m} a_j\right) b_l \\
&= \sum_{l=m}^{k+m} \left(\sum_{j=l-m+1}^{k+1} a_j\right) b_l + \sum_{l=-m}^{k-m} \left(\sum_{j=0}^{l+m} a_j\right) b_l \\
&= LHS.
\end{aligned}$$

We get (2.4).

Finally, (2.5) follows from (2.4) by using the triangle inequality. In other words, we get Lemma 2.1. □

Next, we have the following lemma.

Lemma 2.2. *Assume $a_0, a_1, ..., a_{k+1}$ as in Lemma 2.1. Let $\Omega_j = B_{2^{j+1}} \backslash B_{2^j}$, where $B_\rho := B_\rho(0)$ for any $\rho > 0$. Then, there holds:*

$$\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} f\right| \le C \fint_{B_{2^{k+3}} \backslash B_{2^{-1}}} \left|D^k f(y) - \left(D^k f\right)_{B_{2^{k+3}} \backslash B_{2^{-1}}}\right| dy. \tag{2.7}$$

For any $l \in \mathbb{R}$, we set $E_l = B_{2^{k+l+3}} \backslash B_{2^{l-1}}$. As a consequence of (2.7), we obtain:

$$\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f\right| \le C 2^{kl} \fint_{E_l} \fint_{E_l} \left|D^k f(y) - D^k f(y')\right| dy\, dy'. \tag{2.8}$$

Moreover, by the triangle inequality, we get from (2.8):

$$\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f\right| \le C\, 2^{kl} \fint_{E_l} \left|D^k f(y)\right| dy. \tag{2.9}$$

Proof. Assume that (2.7) is not true, which is a contradiction. There exists then a sequence $(f_m)_{m\ge 1} \subset W^{k,1}(B_{2^{k+3}} \backslash B_{2^{-1}})$ such that

$$\int_{B_{2^{k+3}} \backslash B_{2^{-1}}} |D^k f_m(y) - \left(D^k f_m\right)_{B_{2^{k+3}} \backslash B_{2^{-1}}}| dy \le \frac{1}{m}, \tag{2.10}$$

and

$$|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} f_m| = 1, \quad \forall m \ge 1.$$

Let us put

$$\tilde{f}_m(x) = f_m(x) - P_{k,m}(x), \quad \text{with } P_{k,m}(x) = \sum_{l=0}^{k} \sum_{\alpha_1+...+\alpha_n=l} c_{l,k,m}(\alpha_1, ..., \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n},$$

and $c_{l,k,m}(\alpha_1,...,\alpha_n)$ is a constant such that

$$\left(D^l\tilde{f}_m\right)_{B_{2^{k+3}}\setminus B_{2^{-1}}}=0,\ \forall\, l=0,...,k. \tag{2.11}$$

By a sake of brief, we denote $c_{l,m}=c_{l,k,m}(\alpha_1,...,\alpha_n)$. Since $P_{k,m}$ is a polynomial of at most degree k, then $D^kP_{k,m}=const$. This, (2.10), and (2.11) imply

$$\int\limits_{B_{2^{k+3}}\setminus B_{2^{-1}}}|D^k\tilde{f}_m(y)|\,dy=\int\limits_{B_{2^{k+3}}\setminus B_{2^{-1}}}|D^kf_m(y)-\left(D^kf_m\right)_{B_{2^{k+3}}\setminus B_{2^{-1}}}|\,dy\le\frac{1}{m}.$$

It follows from the compact embeddings that there exists a subsequence of $(\tilde{f}_m)_{m\ge1}$ (still denoted as $(\tilde{f}_m)_{m\ge1}$) such that $\tilde{f}_m\to\tilde{f}$ strongly in $L^1(B_{2^{k+3}}\setminus B_{2^{-1}})$, and

$$D^k\tilde{f}=0,\quad\text{in } B_{2^{k+3}}\setminus B_{2^{-1}}.$$

This implies that $\tilde{f}$ is a polynomial of at most degree $(k-1)$, i.e.:

$$\tilde{f}(x)=\sum_{l=0}^{k-1}\sum_{\alpha_1+...+\alpha_n=l}c'_{l,k}(\alpha_1,...,\alpha_n)x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n},\ \ \forall\, x\in B_{2^{k+3}}\setminus B_{2^{-1}}.$$

On the other hand, we observe that, for any $l=0,...,k$,

$$\begin{aligned}&\sum_{j=0}^{k+1}a_j\fint_{\Omega_j}\sum_{\alpha_1+...+\alpha_n=l}c(\alpha_1,...,\alpha_n)x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}dx_1dx_2...dx_n\\&=\sum_{j=0}^{k+1}a_j\fint_{\Omega_1}\sum_{\alpha_1+...+\alpha_n=l}c(\alpha_1,...,\alpha_n)(2^jx_1)^{\alpha_1}(2^jx_2)^{\alpha_2}...(2^jx_n)^{\alpha_n}dx_1dx_2...dx_n\\&=\fint_{\Omega_1}\sum_{\alpha_1+...+\alpha_n=l}c(\alpha_1,...,\alpha_n)\left(\sum_{j=0}^{k+1}a_j2^{jl}\right)x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}\,dx_1dx_2...dx_n=0,\end{aligned}$$

by (2.2). This implies

$$\sum_{j=0}^{k+1}a_j\fint_{\Omega_j}\tilde{f}=0, \tag{2.12}$$

and

$$\left|\sum_{j=0}^{k+1}a_j\fint_{\Omega_j}\tilde{f}_m\right|=\left|\sum_{j=0}^{k+1}a_j\fint_{\Omega_j}f_m\right|=1.$$

Remind that $\tilde{f}_m\to\tilde{f}$ strongly in $L^1(B_{2^{k+3}}\setminus B_{2^{-1}})$; then we have

$$\left|\sum_{j=0}^{k+1}a_j\fint_{\Omega_j}\tilde{f}\right|=1.$$

Now we complete the proof of (2.7).

The proof of (2.8) (resp. (2.9)) is trivial; we leave it to the reader. This puts an end to the proof of Lemma 2.2. □

Now, we are ready to prove Theorem 1.5.

It is enough to show that

$$|f(0)|\le C+C\|f\|_{\dot{W}^{s,p}}\left(1+\log_2^+\left(\int\limits_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^\alpha}dy+\|f\|_{\dot{C}^\eta}\right)\right)^{\frac{p-1}{p}}. \tag{2.13}$$

Set $s_1 = s - k$, $s_1 \in [0, 1)$. Then, we divide our study into the two cases.

i) Case 1: $s_1 \in (0, 1)$**.** We apply Lemma 2.1 with $b = f(0)$, $b_j = \fint_{\Omega_j} f$. Then, for any $m_0 \geq 1$, there is a constant $C = C(k) > 0$ such that

$$|f(0)| \leq C\left(\sum_{l=-m_0}^{k-m_0}\left|\fint_{\Omega_l} f - f(0)\right| + \sum_{l=-m_0}^{m_0-1}\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f\right| + \sum_{l=m_0}^{k+m_0}\left|\fint_{\Omega_l} f\right|\right). \tag{2.14}$$

Concerning the first term on the right-hand side of (2.14), we have

$$\sum_{l=-m_0}^{k-m_0}\left|\fint_{\Omega_l} f - f(0)\right| \leq \sum_{l=-m}^{k-m}\fint_{\Omega_l}|f - f(0)| \leq \sum_{l=-m}^{k-m}\fint_{\Omega_l}|x|^{\eta}\|f\|_{\dot{C}^{\eta}}\,\mathrm{d}x.$$

Thus,

$$\sum_{l=-m}^{k-m}\left|\fint_{\Omega_l} f - f(0)\right| \leq \sum_{l=-m}^{k-m} 2^{(l+1)\eta}\|f\|_{\dot{C}^{\eta}} \leq C(\eta, k)2^{-m\eta}\|f\|_{\dot{C}^{\eta}}. \tag{2.15}$$

Next, we use (2.8) in Lemma 2.2 to obtain

$$\sum_{l=-m}^{m-1}\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f\right| \leq C\sum_{l=-m}^{m-1} 2^{kl}\fint_{E_l}\fint_{E_l}\left|D^k f(y) - D^k f(z)\right|\mathrm{d}y\,\mathrm{d}z, \tag{2.16}$$

where $E_l = B_{2^{k+l+3}} \backslash B_{2^{l-1}}$. It follows from Hölder's inequality:

$$\sum_{l=-m_0}^{m_0-1} 2^{kl}\fint_{E_l}\fint_{E_l}|D^k f(y) - D^k f(z)|\,\mathrm{d}y\,\mathrm{d}z \leq$$

$$\sum_{l=-m_0}^{m_0-1} 2^{kl}|E_l|^{-2}\left(\int_{E_l}\int_{E_l}\frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\left(\int_{E_l}\int_{E_l}|y-z|^{\frac{n+s_1 p}{p-1}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{p-1}{p}}.$$

Since $y, z \in E_l$, we have $|y - z| \leq |y| + |z| \leq 2^{k+l+4}$. Thus, the right-hand side of the indicated inequality is less than

$$C(n, p, k)2^{kl+\frac{l(n+s_1 p)}{p}}|E_l|^{\frac{-2}{p}}\sum_{l=-m_0}^{m_0-1}\left(\int_{E_l}\int_{E_l}\frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}.$$

Note that $n = sp = (k + s_1)p$, and $|E_l|^{\frac{-2}{p}} \leq C(n, p, k)2^{-2l\frac{n}{p}}$.

Then, there is a constant $C = C(k, s, n) > 0$ such that

$$\sum_{l=-m_0}^{m_0-1} 2^{kl}\fint_{E_l}\fint_{E_l}|D^k f(y) - D^k f(z)|\,\mathrm{d}y\,\mathrm{d}z \leq C\sum_{l=-m_0}^{m_0-1}\left(\int_{E_l}\int_{E_l}\frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}. \tag{2.17}$$

Thanks to the inequality

$$\sum_{j=-m_0}^{m_0-1} c_j^{\frac{1}{p}} \leq (2m_0)^{\frac{p-1}{p}}\left(\sum_{j=-m_0}^{m_0-1} c_j\right)^{\frac{1}{p}}, \tag{2.18}$$

we have

$$
\begin{aligned}
&\sum_{l=-m_0}^{m_0-1}\left(\int_{E_l}\int_{E_l}\frac{|D^k f(y)-D^k f(z)|^p}{|y-z|^{n+s_1p}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}\\
&\quad\le (2m_0)^{\frac{p-1}{p}}\left(\sum_{l=-m_0}^{m_0-1}\int_{E_l}\int_{E_l}\frac{|D^k f(y)-D^k f(z)|^p}{|y-z|^{n+s_1p}}\mathrm{d}y\,\mathrm{d}z\right)^{\frac{1}{p}}.
\end{aligned}
\tag{2.19}
$$

Moreover, we observe that $\sum_{l=-\infty}^{+\infty}\chi_{E_l\times E_l}(y_1,y_2)\le k+4$, for all $(y_1,y_2)\in\mathbb{R}^n\times\mathbb{R}^n$. Thus,

$$
\sum_{l=-m_0}^{m_0-1}\int_{E_l}\int_{E_l}\frac{|D^k f(y)-D^k f(z)|^p}{|y-z|^{n+s_1p}}\mathrm{d}y\,\mathrm{d}z\le (k+4)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|D^k f(y)-D^k f(z)|^p}{|y-z|^{n+s_1p}}\mathrm{d}y\,\mathrm{d}z. \tag{2.20}
$$

Combining (2.17), (2.19) and (2.20) yields

$$
\sum_{l=-m_0}^{m_0-1}2^{kl}\fint_{E_l}\fint_{E_l}|D^k f(y)-D^k f(z)|\,\mathrm{d}y\,\mathrm{d}z\le C(k,s,n)m_0^{\frac{p-1}{p}}\|f\|_{\dot{W}^{s,p}}. \tag{2.21}
$$

It remains to treat the last term. Then, it is not difficult to see that, for any $\alpha>0$,

$$
\begin{aligned}
\sum_{l=m_0}^{k+m_0}\left|\fint_{\Omega_l}f\right| &\le C(k,n)2^{-m_0n}\int_{B_{2^{k+m_0}}}|f|\\
&\le C(k,n,\alpha)2^{-m_0(n-\alpha)}\int_{B_{2^{k+m_0}}}\frac{|f(x)|\,\mathrm{d}x}{(|x|+1)^{\alpha}}.
\end{aligned}
\tag{2.22}
$$

Inserting (2.15), (2.21), and (2.22) into (2.14) yields

$$
|f(0)|\le C2^{-m_0\min\{n-\alpha,\eta\}}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y+\|f\|_{\dot{C}^{\eta}}\right)+Cm_0^{\frac{p-1}{p}}\|f\|_{\dot{W}^{s,p}}. \tag{2.23}
$$

By choosing

$$
m_0=\left[\frac{\log_2^+\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^{\alpha}}\mathrm{d}y+\|f\|_{\dot{C}^{\eta}}\right)}{\min\{n-\alpha,\eta\}}\right]+1,
$$

we obtain (2.13).

ii) Case 2: $s_1=0$ **(**$s=k$**).** The proof is similar to the one of the case $s_1\in(0,1)$. There is just a difference of estimating the second term on the right-hand side of (2.14) as follows.

Using (2.9), we get:

$$
\sum_{l=-m_0}^{m_0-1}\left|\sum_{j=0}^{k+1}a_j\fint_{\Omega_{j+l}}f\right|\le C\sum_{l=-m_0}^{m_0-1}2^{kl}\fint_{E_l}|D^k f|. \tag{2.24}
$$

Applying Hölder's inequality, we have

$$\sum_{l=-m_0}^{m_0-1} 2^{kl} \fint_{E_l} |D^k f| \le \sum_{l=-m_0}^{m_0-1} 2^{kl} |E_l|^{-1/p} \left(\int_{E_l} |D^k f|^p \right)^{1/p}$$
$$\le C(n,k) \sum_{l=-m_0}^{m_0-1} \left(\int_{E_l} |D^k f|^p \right)^{1/p}$$
$$\le C m_0^{\frac{p-1}{p}} \left(\sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p} . \tag{2.25}$$

We utilize the fact $\sum_{l=-\infty}^{\infty} \chi_{E_l}(y) \le k+4$, $\forall\, y \in \mathbb{R}^n$ again in order to get

$$\left(\sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p} \le (k+4) \left(\int_{\mathbb{R}^n} |D^k f|^p \right)^{1/p} . \tag{2.26}$$

From (2.26), (2.25), and (2.24), we get

$$\sum_{l=-m_0}^{m_0-1} \left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| \le C(k,n) \|f\|_{\dot{W}^{s,p}} . \tag{2.27}$$

Thus, we obtain another version of (2.23) as follows:

$$|f(0)| \le C 2^{-m_0 \min\{n-\alpha,\eta\}} \left(\int_{\mathbb{R}^n} \frac{|f(y)|}{(|y|+1)^{\alpha}} \mathrm{d}y + \|f\|_{\dot{C}^{\eta}} \right) + C m_0^{\frac{p-1}{p}} \|f\|_{\dot{W}^{s,p}} . \tag{2.28}$$

By the same argument as above (after (2.23)), we get the proof of the case $s_1 = 0$. This completes the proof of Theorem 1.5. □

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