C. R. Acad. Sci. Paris, Ser. I 356 (2018) 503–508

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

A note on a global strong solution to the 2D Cauchy problem of density-dependent nematic liquid crystal flows with vacuum ☆

Sur la solution forte globale du problème de Cauchy pour l'écoulement d'un cristal liquide nématique bidimensionnel, dépendant de la densité et avec vide

Xin Zhong

School of Mathematics and Statistics, Southwest University, Chongqing 400715, People's Republic of China

A R T I C L E   I N F O

Article history:
Received 10 January 2018
Accepted 3 April 2018
Available online 16 April 2018

Presented by the Editorial Board

A B S T R A C T

In Li–Liu–Zhong (*Nonlinearity* 30 (2017) 4062–4088), the authors proved the existence of a unique global strong solution to the Cauchy problem of 2D nonhomogeneous incompressible nematic liquid crystal flows with vacuum as far-field density provided the initial data density and the gradient of orientation decay not too slow at infinity, and the basic energy $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla \mathbf{d}_0\|_{L^2}^2$ is small. In this note, we aim at precisely describing this smallness condition.

R É S U M É

Dans Li–Liu–Zhong (*Nonlinearity* 30 (2017) 4062–4088), les auteurs démontrent l'existence d'une unique solution forte globale au problème de Cauchy pour l'écoulement d'un cristal liquide nématique, incompressible, non homogène, bidimensionnel, avec vide. Ce résultat est valide dans la mesure où la densité initiale donnée et le gradient de dérive d'orientation ne sont pas trop lents à l'infini et l'énergie de base $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla \mathbf{d}_0\|_{L^2}^2$ est petite. Le but de la présente Note est d'expliciter précisément cette dernière condition de petitesse.

☆ Supported by the Postdoctoral Science Foundation of Chongqing (No. xm2017015), China Postdoctoral Science Foundation (No. 2017M610579), Fundamental Research Funds for the Central Universities (No. XDJK2017C050), and the Doctoral Fund of Southwest University (No. SWU116033).

E-mail address: xzhong1014@amss.ac.cn.

https://doi.org/10.1016/j.crma.2018.04.011
1631-073X/

1. Introduction

In this note, we are concerned with the global existence of solutions to the following two-dimensional (2D) simplified version of nematic liquid crystal flows in the whole space $\mathbb{R}^2$, which describes the motion of a nonhomogeneous incompressible flow of nematic liquid crystals:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \Delta\mathbf{u} + \nabla P = -\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}), \\ \mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}, \\ \operatorname{div}\mathbf{u} = 0, \quad |\mathbf{d}| = 1 \end{cases} \tag{1.1}$$

together with the initial conditions

$$\rho(x,0) = \rho_0(x), \quad \rho\mathbf{u}(x,0) = \rho_0\mathbf{u}_0(x), \quad \mathbf{d}(x,0) = \mathbf{d}_0(x), \quad |\mathbf{d}_0(x)| = 1, \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

Here, the unknown functions $\rho = \rho(x,t)$, $\mathbf{u} = (u_1,u_2)(x,t)$, and $P = P(x,t)$ denote the density, velocity, and pressure of the fluid, respectively. $\mathbf{d} = (d_1,d_2,d_3)(x,t)$ is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow. The notation $\nabla\mathbf{d}\odot\nabla\mathbf{d}$ denotes the 2×2 matrix whose (i,j)-th entry is given by $\partial_i\mathbf{d}\cdot\partial_j\mathbf{d}$ $(1 \le i,j \le 2)$. The above system (1.1)–(1.2) is a macroscopic continuum description of the evolution for the nematic liquid crystals. It is a simplified version of the Ericksen–Leslie model [1,2], but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen–Leslie model. Some important progress has been made about the existence of weak and strong solutions to incompressible nematic liquid crystals equations for either homogeneous or nonhomogeneous fluids by many authors – refer to [4–7] and references therein.

Recently, Li–Liu–Zhong [3] established the global-in-time existence of a unique strong solution to the Cauchy problem (1.1)–(1.2) provided the initial data density and the gradient of orientation decay not too slow at infinity, and the basic energy $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla\mathbf{d}_0\|_{L^2}^2$ is small. Precisely, they showed the following result.

Theorem 1.1. *For constants $q > 2$, $a > 1$, assume that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{d}_0)$ satisfies*

$$\begin{cases} \rho_0 \ge 0, \quad \rho_0\bar{x}^a \in L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2), \quad \sqrt{\rho_0}\mathbf{u}_0 \in L^2(\mathbb{R}^2), \quad \nabla\mathbf{u}_0 \in L^2(\mathbb{R}^2), \\ \operatorname{div}\mathbf{u}_0 = 0, \quad \mathbf{d}_0 \in L^2(\mathbb{R}^2), \quad \nabla\mathbf{d}_0\bar{x}^{\frac{a}{2}} \in L^2(\mathbb{R}^2), \quad \nabla^2\mathbf{d}_0 \in L^2(\mathbb{R}^2), \quad |\mathbf{d}_0| = 1, \end{cases} \tag{1.3}$$

where

$$\bar{x} \triangleq (\mathrm{e} + |x|^2)^{\frac{1}{2}} \log^2(\mathrm{e} + |x|^2).$$

Then there is a positive constant ε_0 such that if

$$C_0 \triangleq \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla\mathbf{d}_0\|_{L^2}^2 < \varepsilon_0, \tag{1.4}$$

then the Cauchy problem (1.1)–(1.2) has a unique global strong solution $(\rho, \mathbf{u}, P, \mathbf{d})$ satisfying that for any $0 < T < \infty$,

$$\begin{cases} 0 \le \rho \in C([0,T]; L^1\cap H^1\cap W^{1,q}), \\ \rho\bar{x}^a \in L^\infty(0,T; L^1\cap H^1\cap W^{1,q}) \\ \sqrt{\rho}\mathbf{u}, \nabla\mathbf{u}, \sqrt{t}\nabla\mathbf{u}, \sqrt{t}\sqrt{\rho}\mathbf{u}, \sqrt{t}\nabla P, t\nabla P, \sqrt{t}\nabla^2\mathbf{u}, t\nabla^2\mathbf{u} \in L^\infty(0,T;L^2), \\ \nabla\mathbf{d}, \nabla\mathbf{d}\bar{x}^{\frac{a}{2}}, \nabla^2\mathbf{d}, \sqrt{t}\nabla^2\mathbf{d}, \sqrt{t}\nabla\mathbf{d}_t, \sqrt{t}\nabla^3\mathbf{d}, t\nabla^3\mathbf{d} \in L^\infty(0,T;L^2), \\ \nabla\mathbf{u} \in L^2(0,T;H^1)\cap L^{\frac{q+1}{q}}(0,T;W^{1,q}), \\ \nabla P \in L^2(0,T;L^2)\cap L^{\frac{q+1}{q}}(0,T;L^q), \\ \nabla^2\mathbf{d} \in L^2(0,T;H^1), \quad \nabla\mathbf{d}_t, \nabla^2\mathbf{d}\bar{x}^{\frac{a}{2}} \in L^2(\mathbb{R}^2\times(0,T)), \\ \sqrt{\rho}\mathbf{u}_t, \sqrt{t}\nabla\mathbf{u}_t, \sqrt{t}\nabla\mathbf{d}_t, \sqrt{t}\nabla^2\mathbf{d}_t \in L^2(\mathbb{R}^2\times(0,T)), \\ \sqrt{t}\nabla\mathbf{u} \in L^2(0,T;W^{1,q}). \end{cases} \tag{1.5}$$

Moreover, the solution $(\rho, \mathbf{u}, P, \mathbf{d})$ has the following temporal decay rates, i.e., for all $t \ge 1$,

$$\|\nabla\mathbf{u}(\cdot,t)\|_{L^2}^2 + \|\nabla^2\mathbf{u}(\cdot,t)\|_{L^2} + \|\nabla P(\cdot,t)\|_{L^2} + \||\nabla\mathbf{d}||\nabla^2\mathbf{d}|(\cdot,t)\|_{L^2} + \|\nabla^2\mathbf{d}(\cdot,t)\|_{L^2}^2 \le Ct^{-1}, \tag{1.6}$$

where C depends only on C_0, $\|\rho_0\|_{L^1\cap L^\infty}$, and $\|\nabla\mathbf{u}_0\|_{L^2}$.

As pointed out in [3, Remark 1.1], it seems more involved to show the global existence of strong solutions with general initial data. This is the main reason for them to add an additional smallness condition (1.4). Although it has small energy, its oscillations can be arbitrarily large. Let us explain why Li–Liu–Zhong proposed the condition (1.4) in [3]. To prove Theorem 1.1, one has to obtain some global a priori estimates on the strong solutions to the system (1.1)–(1.2) in suitable higher norms. These lie in lower-order estimates. To show [3, Lemma 3.1], one of key steps is to get the bound on the $L^2(\mathbb{R}^2 \times (0,T))$-norm of $\nabla^2\mathbf{d}$. Combining the basic energy inequalities – see (3.4) and (3.5) in [3] – with Ladyzhenskaya's inequality, they can successfully obtain the a priori bound on the L^2-norm of $\nabla^2\mathbf{d}$ in space and time provided that (1.4) holds true (see (3.16) and (3.17) in [3]). Then, by a series of energy estimates (see Lemmas 3.2–3.6 in [3]), they establish the needed a priori estimates.

The aim of this paper is to give a sharp description of the smallness condition (1.4). Our main result reads as follows.

Theorem 1.2. *Under assumption (1.3), the Cauchy problem (1.1)–(1.2) has a unique global strong solution $(\rho, \mathbf{u}, P, \mathbf{d})$ satisfying (1.5) and (1.6) if*

$$\|\nabla\mathbf{d}_0\|_{L^2}^2 \exp\left[2\Lambda^2\left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla\mathbf{d}_0\|_{L^2}^2\right)\right] \le \frac{1}{8\Lambda^2}, \tag{1.7}$$

where the constant Λ is the best constant in Ladyzhenskaya's inequality

$$\|\nabla\mathbf{d}_0\|_{L^4}^2 \le \Lambda\|\nabla\mathbf{d}_0\|_{L^2}\|\nabla^2\mathbf{d}_0\|_{L^2}. \tag{1.8}$$

2. Proof of Theorem 1.2

We begin with the following lemma.

Lemma 2.1. *Let $T>0$ be a fixed time and $(\rho, \mathbf{u}, P, \mathbf{d})$ be the strong solution to system (1.1)–(1.2) on $\mathbb{R}^2 \times (0,T]$ with initial data $(\rho_0, \mathbf{u}_0, \mathbf{d}_0)$ satisfying (1.3) and (1.7), then there exists a positive constant C depending only on $\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}$, $\|\rho_0\|_{L^1\cap L^\infty}$, $\|\nabla\mathbf{u}_0\|_{L^2}$, $\|\nabla\mathbf{d}_0\|_{L^2}$, and $\|\nabla^2\mathbf{d}_0\|_{L^2}$ such that*

$$\sup_{0\le t\le T}\left(\|\rho\|_{L^1\cap L^\infty} + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{d}\|_{L^2}^2\right) + \int_0^T \left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\right)\mathrm{d}t \le C, \tag{2.1}$$

where $\dot{\mathbf{u}} \triangleq \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}$.

Proof. First, since $\operatorname{div}\mathbf{u} = 0$, it is easy to obtain from the equation $(1.1)_1$ that

$$\|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p} \text{ for all } p \in [1,\infty] \text{ and } t \ge 0. \tag{2.2}$$

Multiplying $(1.1)_2$ by $\mathbf{u}$ and using $(1.1)_1$, we obtain after integration by parts that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|\mathbf{u}|^2\mathrm{d}x + \int|\nabla\mathbf{u}|^2\mathrm{d}x = -\int \mathbf{u}\cdot\nabla\mathbf{d}\cdot\Delta\mathbf{d}\,\mathrm{d}x. \tag{2.3}$$

Multiplying $(1.1)_3$ by $-\Delta\mathbf{d} - |\nabla\mathbf{d}|^2\mathbf{d}$, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\mathbf{d}|^2\mathrm{d}x + \int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|\,\mathrm{d}x = \int \mathbf{u}\cdot\nabla\mathbf{d}\cdot\Delta\mathbf{d}\,\mathrm{d}x. \tag{2.4}$$

Combining (2.3) and (2.4), we get from Gronwall's inequality that

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{d}\|_{L^2}^2\right) + 2\int_0^T\!\!\int \left(|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|\right)\mathrm{d}x\,\mathrm{d}t \le \|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla\mathbf{d}_0\|_{L^2}^2. \tag{2.5}$$

It follows from (2.4), Hölder's and Ladyzhenskaya's inequalities that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\mathbf{d}|^2\mathrm{d}x + \int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|\,\mathrm{d}x &= -\int \partial_j\mathbf{u}\nabla d_i\partial_j d_i\,\mathrm{d}x \\ &\le \|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{d}\|_{L^4}^2 \\ &\le \Lambda\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{d}\|_{L^2}\|\Delta\mathbf{d}\|_{L^2} \\ &\le \frac{1}{8}\|\Delta\mathbf{d}\|_{L^2}^2 + 2\Lambda^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{d}\|_{L^2}^2.\end{aligned} \tag{2.6}$$

By the elementary inequality $(a+b)^2 \geq \frac{a^2}{2} - b^2$ and Ladyzhenskaya's inequality, we have

$$\int |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\,| dx \geq \frac{1}{2}\|\Delta \mathbf{d}\|_{L^2}^2 - \|\nabla \mathbf{d}\|_{L^4}^4 \geq \frac{1}{2}\|\Delta \mathbf{d}\|_{L^2}^2 - \Lambda^2 \|\nabla \mathbf{d}\|_{L^2}^2 \|\Delta \mathbf{d}\|_{L^2}^2, \tag{2.7}$$

which, combined with (2.6), yields

$$\frac{1}{2}\frac{d}{dt}\int |\nabla \mathbf{d}|^2 dx + \frac{3}{8}\int |\Delta \mathbf{d}|^2 dx \leq \Lambda^2 \|\nabla \mathbf{d}\|_{L^2}^2 \|\Delta \mathbf{d}\|_{L^2}^2 + 2\Lambda^2 \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{L^2}^2. \tag{2.8}$$

If the initial datum $\mathbf{d}_0$ satisfies

$$\|\nabla \mathbf{d}_0\|_{L^2}^2 \leq \frac{1}{8\Lambda^2},$$

then, by continuity, there exists a value of T_1 such that, for any $t \in [0, T_1]$,

$$\|\nabla \mathbf{d}(t)\|_{L^2}^2 \leq \frac{1}{4\Lambda^2}. \tag{2.9}$$

We denote by T_1^* the maximal time such that (2.9) holds on $[0, T_1^*]$. Then, it follows from (2.8) and (2.9) that

$$\frac{d}{dt}\int |\nabla \mathbf{d}|^2 dx + \frac{1}{4}\int |\Delta \mathbf{d}|^2 dx \leq 4\Lambda^2 \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{L^2}^2, \tag{2.10}$$

which together with (2.5) and Gronwall's inequality implies that

$$\begin{aligned} \sup_{0\leq t\leq T_1^*} \|\nabla \mathbf{d}\|_{L^2}^2 + \frac{1}{4}\int_0^{T_1^*} \|\Delta \mathbf{d}\|_{L^2}^2 dt &\leq \|\nabla \mathbf{d}_0\|_{L^2}^2 \exp\left(4\Lambda^2 \int_0^{T_1^*} \|\nabla \mathbf{u}\|_{L^2}^2 dt\right) \\ &\leq \|\nabla \mathbf{d}_0\|_{L^2}^2 \exp\left[2\Lambda^2 \left(\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 + \|\nabla \mathbf{d}_0\|_{L^2}^2\right)\right] \leq \frac{1}{8\Lambda^2}, \end{aligned} \tag{2.11}$$

which leads to $T_1^* = T$ if (1.7) holds true. Hence, we derive from (2.5) and (2.11) that

$$\sup_{0\leq t\leq T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{d}\|_{L^2}^2\right) + \int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta \mathbf{d}\|_{L^2}^2\right) dt \leq C. \tag{2.12}$$

Now, multiplying $(1.1)_2$ by $\dot{\mathbf{u}}$ and then integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\int \rho|\dot{\mathbf{u}}|^2 dx = \int \Delta \mathbf{u}\cdot\dot{\mathbf{u}}\, dx - \int \nabla P\cdot\dot{\mathbf{u}}\, dx - \int \operatorname{div}(\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot\dot{\mathbf{u}}\, dx \triangleq I_1 + I_2 + I_3. \tag{2.13}$$

By the same arguments as in [3], one has

$$\begin{aligned} I_1 &\leq -\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla^2 \mathbf{u}\|_{L^2}, \\ |I_2| &\leq C\|\nabla P\|_{L^2}\|\nabla \mathbf{u}\|_{L^2}^2. \end{aligned}$$

To bound the term I_3, we first apply ∇ on $(1.1)_3$ to get

$$\nabla \mathbf{d}_t - \Delta\nabla \mathbf{d} = -\nabla(\mathbf{u}\cdot\nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}), \tag{2.14}$$

which combined with Hölder's and Gagliardo–Nirenberg inequalities leads to

$$\begin{aligned} I_3 &= \int (\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot\nabla \mathbf{u}_t\, dx - \int \operatorname{div}(\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot(\mathbf{u}\cdot\nabla \mathbf{u})\, dx \\ &= \frac{d}{dt}\int (\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot\nabla \mathbf{u}\, dx - \int (\nabla \mathbf{d}_t\odot\nabla \mathbf{d})\cdot\nabla \mathbf{u}\, dx - \int (\nabla \mathbf{d}\odot\nabla \mathbf{d}_t)\cdot\nabla \mathbf{u}\, dx \\ &\quad - \int \operatorname{div}(\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot(\mathbf{u}\cdot\nabla \mathbf{u})\, dx \\ &= \frac{d}{dt}\int (\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot\nabla \mathbf{u}\, dx - \int [(\Delta\nabla \mathbf{d} - \nabla(\mathbf{u}\cdot\nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}))\odot\nabla \mathbf{d}]\cdot\nabla \mathbf{u}\, dx \\ &\quad - \int [\nabla \mathbf{d}\odot(\Delta\nabla \mathbf{d} - \nabla(\mathbf{u}\cdot\nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}))]\cdot\nabla \mathbf{u}\, dx - \int \operatorname{div}(\nabla \mathbf{d}\odot\nabla \mathbf{d})\cdot(\mathbf{u}\cdot\nabla \mathbf{u})\, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x - \int [(\Delta\nabla\mathbf{d} - \nabla\mathbf{u}\cdot\nabla\mathbf{d} + \nabla(|\nabla\mathbf{d}|^2\mathbf{d}))\odot\nabla\mathbf{d}]\cdot\nabla\mathbf{u}\,\mathrm{d}x \\
&\quad + \int u_k\partial_k\partial_i d_\ell\partial_j d_\ell\partial_j u_i\,\mathrm{d}x - \int [\nabla\mathbf{d}\odot(\Delta\nabla\mathbf{d} - \nabla\mathbf{u}\cdot\nabla\mathbf{d} + \nabla(|\nabla\mathbf{d}|^2\mathbf{d}))]\cdot\nabla\mathbf{u}\,\mathrm{d}x \\
&\quad + \int \partial_i d_\ell u_k\partial_k\partial_j d_\ell\partial_j u_i\,\mathrm{d}x + \int \partial_i d_\ell\partial_j d_\ell\partial_j(u_k\partial_k u_i)\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x - \int [(\Delta\nabla\mathbf{d} - \nabla\mathbf{u}\cdot\nabla\mathbf{d} + \nabla(|\nabla\mathbf{d}|^2\mathbf{d}))\odot\nabla\mathbf{d}]\cdot\nabla\mathbf{u}\,\mathrm{d}x \\
&\quad - \int [\nabla\mathbf{d}\odot(\Delta\nabla\mathbf{d} - \nabla\mathbf{u}\cdot\nabla\mathbf{d} + \nabla(|\nabla\mathbf{d}|^2\mathbf{d}))]\cdot\nabla\mathbf{u}\,\mathrm{d}x + \int \partial_i d_\ell\partial_j d_\ell\partial_j u_k\partial_k u_i\,\mathrm{d}x \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + C\|\nabla^3\mathbf{d}\|_{L^2}\|\nabla\mathbf{d}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + C\|\nabla\mathbf{d}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^3 \\
&\quad + C\|\nabla\mathbf{d}\|_{L^6}^4\|\nabla\mathbf{u}\|_{L^3} + C\|\nabla^2\mathbf{d}\|_{L^3}\|\nabla\mathbf{d}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3} \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + \frac{\varepsilon}{4}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^3}^3 + C\|\nabla\mathbf{d}\|_{L^6}^6 + C\|\nabla^2\mathbf{d}\|_{L^3}^{\frac{3}{2}}\|\nabla\mathbf{d}\|_{L^6}^3 \text{ (with } \varepsilon > 0) \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + \frac{\varepsilon}{4}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^3}^3 + C\|\nabla\mathbf{d}\|_{L^6}^6 + C\|\nabla^3\mathbf{d}\|_{L^2}^{\frac{3}{4}}\|\nabla\mathbf{d}\|_{L^6}^{\frac{15}{4}} \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^3}^3 + C\|\nabla\mathbf{d}\|_{L^6}^6 \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2} + C\|\nabla\mathbf{d}\|_{L^2}^2\|\nabla^2\mathbf{d}\|_{L^2}^4 \\
&\le \frac{\mathrm{d}}{\mathrm{d}t}\int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x + \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2} + C\|\nabla^2\mathbf{d}\|_{L^2}^4,
\end{aligned}
$$

where in the last inequality we have used (2.12). Inserting the above estimates of I_i $(i = 1, 2, 3)$ into (2.13), and then using (2.12), it holds that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\mathbf{u}\|_{L^2}^2 - \int (\nabla d\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x\right) + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \\
&\le \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla^2\mathbf{d}\|_{L^2}^4 + C(\|\nabla^2\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2})\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned} \tag{2.15}
$$

On the other hand, since $(\rho, \mathbf{u}, P, \mathbf{d})$ satisfies the following Stokes system

$$
\begin{cases}
-\Delta\mathbf{u} + \nabla P = -\rho\,\dot{\mathbf{u}} - \operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}), & x\in\mathbb{R}^2, \\
\operatorname{div}\mathbf{u} = 0, & x\in\mathbb{R}^2, \\
\mathbf{u}(x)\to\mathbf{0}, & |x|\to\infty,
\end{cases}
$$

applying the standard L^p-estimate to the above system, and using the identity $\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d}) = \nabla\mathbf{d}\cdot\Delta\mathbf{d}$ and (2.2) gives that

$$
\begin{aligned}
\|\nabla^2\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2} &\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + \||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}\right) \\
&\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}\right) \\
&\le C\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{d}\|_{L^2}^{\frac{1}{2}}\|\nabla^2\mathbf{d}\|_{L^2}\|\nabla^3\mathbf{d}\|_{L^2}^{\frac{1}{2}}\right),
\end{aligned} \tag{2.16}
$$

which combined with (2.15) and Young's inequality leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t}B(t) + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \le \varepsilon\|\nabla^3\mathbf{d}\|_{L^2}^2 + \varepsilon\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^4 + C\|\nabla^2\mathbf{d}\|_{L^2}^4, \tag{2.17}
$$

where

$$
B(t) \triangleq \frac{1}{2}\|\nabla\mathbf{u}\|_{L^2}^2 - \int (\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}\,\mathrm{d}x
$$

satisfies

$$
\frac{1}{4}\|\nabla\mathbf{u}\|_{L^2}^2 - C_1\|\nabla^2\mathbf{d}\|_{L^2}^2 \le B(t) \le C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{d}\|_{L^2}^2 \tag{2.18}
$$

owing to the following estimate

$$\left|\int (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \cdot \nabla \mathbf{u}\, dx\right| \le \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla \mathbf{d}\|_{L^4}^4 \le \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla \mathbf{d}\|_{L^2}^2\|\nabla^2 \mathbf{d}\|_{L^2}^2$$
$$\le \frac{1}{4}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla^2 \mathbf{d}\|_{L^2}^2.$$

Now, multiplying (2.14) by $-\nabla\Delta\mathbf{d}$ and then integrating by parts over $\mathbb{R}^2$, it follows from Hölder's and Gagliardo–Nirenberg's inequalities, (2.12) and (2.16), that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla^2\mathbf{d}\|_{L^2}^2 + \|\nabla\Delta\mathbf{d}\|_{L^2}^2\\
&= \int \nabla(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla\Delta\mathbf{d}\,dx - \int \nabla(|\nabla\mathbf{d}|^2\mathbf{d})\nabla\Delta\mathbf{d}\,dx\\
&= \int (\nabla\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla\Delta\mathbf{d}\,dx + \int u_i\partial_i\partial_j d_\ell\partial_i\partial_{kk}d_\ell\,dx - \int \nabla(|\nabla\mathbf{d}|^2\mathbf{d})\nabla\Delta\mathbf{d}\,dx\\
&= \int (\nabla\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla\Delta\mathbf{d}\,dx - \int \partial_k u_i\partial_i\partial_j d_\ell\partial_i\partial_k d_\ell dx - \int \nabla(|\nabla\mathbf{d}|^2\mathbf{d})\cdot\nabla\Delta\mathbf{d}\,dx\\
&\le \frac{\varepsilon}{4}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^3}^3 + C\|\nabla\mathbf{d}\|_{L^6}^6 + C\|\nabla^2\mathbf{d}\|_{L^3}^3 + C\|\nabla^2\mathbf{d}\|_{L^3}^2\|\nabla\mathbf{d}\|_{L^6}^2\\
&\le \frac{\varepsilon}{4}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^3}^3 + C\|\nabla\mathbf{d}\|_{L^6}^6 + C\|\nabla^3\mathbf{d}\|_{L^2}^{\frac{3}{2}}\|\nabla\mathbf{d}\|_{L^2}^{\frac{3}{2}} + C\|\nabla^3\mathbf{d}\|_{L^2}\|\nabla\mathbf{d}\|_{L^6}^3\\
&\le \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{u}\|_{L^2} + C\|\nabla\mathbf{d}\|_{L^2}^2\|\nabla^2\mathbf{d}\|_{L^2}^4\\
&\le \frac{\varepsilon}{2}\|\nabla^3\mathbf{d}\|_{L^2}^2 + \frac{\varepsilon}{2}(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2) + C\|\nabla^2\mathbf{d}\|_{L^2}^4 + C\|\nabla\mathbf{u}\|_{L^2}^4\\
&\le \varepsilon\|\nabla^3\mathbf{d}\|_{L^2}^2 + \varepsilon\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla^2\mathbf{d}\|_{L^2}^4 + C\|\nabla\mathbf{u}\|_{L^2}^4.
\end{aligned} \tag{2.19}$$

Multiplying (2.19) by $2(C_1+1)$, then adding the resulting inequality with (2.17) and choosing ε suitably small, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(B(t) + (C_1+1)\|\nabla^2\mathbf{d}\|_{L^2}^2\right) + \frac{1}{2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\\
&\le C\left(\|\nabla^2\mathbf{d}\|_{L^2}^2\|\nabla^2\mathbf{d}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\right).
\end{aligned} \tag{2.20}$$

This along with (2.12), (2.18), and Gronwall's inequality yields

$$\sup_{0\le t\le T}\left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{d}\|_{L^2}^2\right) + \int_0^T \left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\right)dt \le C. \tag{2.21}$$

Combining (2.2) with (2.21), we derive (2.1). This completes the proof of Lemma 2.1. □

Proof of Theorem 1.2. With Lemma 2.1 at hand, we can follow the same proof procedure as in [3] to show Theorem 1.2. Hence, we omit the detailed proof here. □

References

[1] J.L. Ericksen, Hydrostatic theory of liquid crystal, Arch. Ration. Mech. Anal. 9 (1962) 371–378.
[2] F. Leslie, Theory of flow phenomenum in liquid crystals, in: The Theory of Liquid Crystals, vol. 4, Academic Press, London–New York, 1979, pp. 1–81.
[3] L. Li, Q. Liu, X. Zhong, Global strong solutions to the two-dimensional density-dependent nematic liquid crystal flows with vacuum, Nonlinearity 30 (2017) 4062–4088.
[4] X. Li, D. Wang, Global strong solution to the density-dependent incompressible flow of liquid crystals, Trans. Amer. Math. Soc. 367 (2015) 2301–2338.
[5] F. Lin, J. Lin, C. Wang, Liquid crystal flow in two dimensions, Arch. Ration. Mech. Anal. 197 (2010) 297–336.
[6] F. Lin, C. Wang, Global existence of weak solutions of the nematic liquid crystal flow in dimensions three, Commun. Pure Appl. Math. 69 (2016) 1532–1571.
[7] Q. Liu, S. Liu, W. Tan, X. Zhong, Global well-posedness of the 2D nonhomogeneous incompressible nematic liquid crystal flows with vacuum, J. Differ. Equ. 261 (2016) 6521–6569.