C. R. Acad. Sci. Paris, Ser. I 356 (2018) 316–321

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Analytic geometry

Holomorphic Cartan geometries on complex tori

Géométries de Cartan holomorphes sur les tores complexes

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A R T I C L E   I N F O

Article history:
Received 16 October 2017
Accepted 12 February 2018
Available online 16 February 2018

Presented by Jean-Pierre Demailly

A B S T R A C T

In [6], it was asked whether all flat holomorphic Cartan geometries (G, H) on a complex torus are translation invariant. We answer this affirmatively under the assumption that the complex Lie group G is affine. More precisely, we show that every holomorphic Cartan geometry of type (G, H), with G a complex affine Lie group, on any complex torus is translation invariant.

R É S U M É

Nous démontrons que, sur les tores complexes, toutes les géométries de Cartan holomorphes modelées sur (G, H), avec G groupe de Lie complexe affine, sont invariantes par translation.

Version française abrégée

Dans cette note, nous étudions les géométries de Cartan holomorphes sur les tores complexes. Rappelons que, grâce aux résultats de [3–5], les tores complexes sont, à un revêtement fini près, les seules variétés de Calabi–Yau qui possèdent des géométries de Cartan holomorphes. Il a été conjecturé dans [6] que, sur les tores complexes, toutes les géométries de Cartan holomorphes plates sont nécessairement invariantes par translation. Cette conjecture a été prouvée dans [6] dans certains cas particuliers (par exemple, pour les tores de dimension complexe un et deux et, en toute dimension, dans le cas G nilpotent). Il a également été démontré dans [6] que, si on considère, sur chaque tore complexe, l'espace des géométries de Cartan holomorphes plates de modèle (G, H) fixé, avec G et H groupes algébriques complexes, les géométries de Cartan invariantes par translation forment un sous-ensemble ouvert et fermé (et donc une union de composantes connexes). Dans la direction de la conjecture, il a aussi été prouvé dans [10] que, sur les tores complexes, toutes les géométries de Cartan *paraboliques* sont invariantes par translation.

Dans cet article, nous démontrons que, *sur les tores complexes, toutes les géométries de Cartan holomorphes modelées sur (G, H), avec G groupe de Lie complexe affine, sont invariantes par translation.*

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https://doi.org/10.1016/j.crma.2018.02.005
1631-073X/

La démonstration de ce théorème repose fortement sur des résultats de [2], inspirés de [11]. Plus précisément, il est démontré dans [2] que les fibrés principaux holomorphes E_G, de groupe structural complexe affine G, au-dessus d'un tore complexe X, admettent des connexions holomorphes exactement quand ils sont homogènes (i.e. chaque translation dans X se relève en un isomorphisme du fibré principal E_G). Ceci est également équivalent au fait que E_G soit pseudostable, avec les deux premières classes de Chern nulles [2]. De plus, il est montré dans [2] (en adaptant des arguments de [11]) que ces fibrés principaux holomorphes admettent également une connexion canonique plate.

Dans cet article, on utilise cette connexion canonique plate sur le fibré principal E_G, associé à la géométrie de Cartan modelée sur (G, H), pour relever l'action du tore X par translation sur lui-même en une action qui préserve la classe d'isomorphisme de E_G, ainsi que sa connexion canonique plate. On démontre ensuite que cette action préserve également la connexion holomorphe de E_G et le sous-fibré principal E_H (de groupe structural H) qui caractérisent la géométrie de Cartan. Ceci implique que la géométrie de Cartan modelée sur (G, H) est invariante par translation.

1. Introduction

A classical result proved by Inoue, Kobayashi and Ochiai [9], which was done using Yau's proof of Calabi conjecture, shows that a compact complex Kähler manifold, bearing a holomorphic connection on its holomorphic tangent bundle, admits a finite unramified holomorphic covering by some compact complex torus. The pull-back of such a holomorphic connection to the covering torus is necessarily translation invariant.

This result was generalized in [3], [4], [5] for two different classes of holomorphic geometric structures: the rigid geometric structures in Gromov's sense [8], and the Cartan geometries. More precisely, any compact complex Kähler Calabi–Yau manifold bearing a holomorphic rigid geometric structure (or a holomorphic Cartan geometry) admits a finite unramified holomorphic covering by a complex torus.

There are interesting examples of holomorphic rigid geometric structures on complex tori that are not translation invariant. They can be constructed using Ghys' example of holomorphic foliations on complex tori that are not translation invariant [7]. Let us recall that the main result of [7] is a classification of codimension one (nonsingular) holomorphic foliations on complex tori. The holomorphic foliations are defined by the kernel of some global holomorphic 1-form ω (and hence are translation invariant), except for those complex tori T that admit a holomorphic surjective map π to an elliptic curve E. In the last case, one can consider a global coordinate z on E, a nonconstant meromorphic function $u(z)$ on E and the pull-back to T of the meromorphic closed 1-form $u(z)\,\mathrm{d}z$. The foliation given by the kernel of $\Omega = \pi^*(u(z)\,\mathrm{d}z) + \omega$ extends to all of T as a codimension-one nonsingular holomorphic foliation; this foliation coincides with the one given by the fibration π exactly when ω vanishes on the fibers of π. This foliation is not invariant by all translations in the torus T, but only by those lying in the kernel of the linear map underlying π. Consequently, the holomorphic rigid geometric structure on T obtained by considering the previous holomorphic foliation together with the holomorphic standard flat connection of T is not translation invariant (it is invariant only by those translations lying in the kernel of the linear map underlying π).

This note deals with holomorphic Cartan geometries on complex tori. In contrast with the situation of the geometric structures in the previous example, it was conjectured in [6] that *all flat holomorphic Cartan geometries on complex tori are translation invariant*. The conjecture was proved in [6] for some particular cases — for example, when the torus is one or two dimensional, or when the structure group G of the Cartan the geometry is nilpotent. For G complex algebraic, it was also proved in [6] that, on any torus T, translation-invariant Cartan geometries form an open and closed subset (and, consequently, a union of connected components) in the space of Cartan geometries with a given model (G, H) on T. Moreover, Theorem 3 in [10] proves that every holomorphic parabolic Cartan geometry on any complex torus is translation invariant.

We prove here the following.

Every holomorphic Cartan geometry of type (G, H), with G a complex affine group, on any complex torus is translation invariant.

2. Preliminaries

2.1. Holomorphic Cartan geometries

Let G be a connected complex Lie group and $H \subset G$ a complex Lie subgroup. The Lie algebras of G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. A holomorphic Cartan geometry of type (G, H) on a complex manifold X is a principal H-bundle $f : E_H \longrightarrow X$ and a $\mathfrak{g}$ valued holomorphic 1-form $\omega : TE_H \longrightarrow E_H \times \mathfrak{g}$ on the total space of E_H, such that

(1) ω is H-equivariant for the adjoint action of H on $\mathfrak{g}$;
(2) ω is an isomorphism;
(3) the restriction of ω to any fiber of f coincides with the Maurer–Cartan form associated with the action of H on E_H.

Let $\mathrm{At}(E_H) = TE_H/H \longrightarrow X$ be the Atiyah bundle for E_H. Let

$$E_G := E_H \times^H G \longrightarrow X$$

be the holomorphic principal G-bundle obtained by extending the structure group of E_H using the inclusion of H in G. Giving a form ω satisfying the above three conditions is equivalent to giving a holomorphic isomorphism

$$\beta : \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G) := E_G \times^G \mathfrak{g}$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H) & \stackrel{i_H}{\longrightarrow} & \mathrm{At}(E_H) & \longrightarrow & TX & \longrightarrow & 0 \\ & & \| & & \downarrow \beta & & \downarrow \wr & & \\ 0 & \longrightarrow & \mathrm{ad}(E_H) & \longrightarrow & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0 \end{array} \tag{2.1}$$

where the sequence at the top is the Atiyah exact sequence for E_H (see [1] for the Atiyah exact sequence); the inclusion $\mathrm{ad}(E_H) \hookrightarrow \mathrm{ad}(E_G)$ in (2.1) is given by the inclusion of $\mathfrak{h}$ in $\mathfrak{g}$. Consider the injective homomorphism

$$\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G) \oplus \mathrm{At}(E_H), \quad v \longmapsto (v, -i_H(v)),$$

where i_H is the homomorphism in (2.1). The corresponding quotient bundle $(\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H))/\mathrm{ad}(E_H)$ is identified with the Atiyah bundle $\mathrm{At}(E_G)$ for E_G. If β is a homomorphism as above defining a holomorphic Cartan geometry on X of type (G, H), then the homomorphism

$$\beta' : \mathrm{At}(E_G) = (\mathrm{ad}(E_G) \oplus \mathrm{At}(E_H))/\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G), \quad (v, w) \longmapsto v + \beta(w)$$

has the property that the composition

$$\mathrm{ad}(E_G) \hookrightarrow \mathrm{At}(E_G) \stackrel{\beta'}{\longrightarrow} \mathrm{ad}(E_G)$$

coincides with the identity map of $\mathrm{ad}(E_G)$, where the inclusion of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_G)$ is the one occurring in the Atiyah exact sequence for E_G. Therefore, β' produces a holomorphic splitting of the Atiyah exact sequence for E_G. Hence β' is a holomorphic connection on E_G [1].

The Cartan geometry β is called *flat* if the curvature of the connection on E_G defined by β' vanishes identically.

2.2. Cartan geometry on a complex torus

We now take X to be a compact complex Lie group, so X is a complex torus. For any $x \in X$, let

$$t_x : X \longrightarrow X, \quad y \longmapsto y + x$$

be the translation by x. A holomorphic Cartan geometry (E_H, β) on X of type (G, H) is called translation invariant if for every $x \in X$, there is a holomorphic isomorphism of principal G-bundles

$$\delta_x : E_G \longrightarrow t_x^* E_G$$

such that

(1) $\delta_x(E_H) = E_H$, and
(2) $\delta_x^* \beta = \beta$.

A conjecture in [6] says that any flat holomorphic Cartan geometry on a complex torus is translation invariant (see the first paragraph in the introduction [6, p. 1]).

A complex Lie group G will be called *affine* if there is a holomorphic homomorphism

$$\alpha : G \longrightarrow \mathrm{GL}(N, \mathbb{C})$$

for some positive integer N, such that the corresponding homomorphism of Lie algebras

$$d\alpha : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(\mathrm{GL}(N, \mathbb{C})) = \mathrm{M}(N, \mathbb{C})$$

is injective.

We will prove that every holomorphic Cartan geometry of type (G, H), with G affine, on any complex torus is translation invariant. This would imply that they are all constructed in the following way.

2.3. Invariant Cartan geometry on a complex torus

Let $\widetilde{X}$ be the universal cover of the complex group X. The complex Lie group $\widetilde{X}$ acts holomorphically on X via translations. Let E_H be a holomorphic principal H-bundle E_H on X equipped with a holomorphic lift of the action of $\widetilde{X}$ on X such that the actions of H and $\widetilde{X}$ on E_H commute. This action of $\widetilde{X}$ on E_H produces a flat holomorphic connection on the principal H-bundle E_H. This flat connection on E_H will be denoted by ∇^H.

As before, $E_G := E_H \times^H G \longrightarrow X$ is the holomorphic principal G-bundle obtained by extending the structure group of E_G. The holomorphic connection on E_G induced by the above connection ∇^H will be denoted by ∇^G. Let $V_0 := T_0 X$ be the Lie algebra of X. Take any holomorphic section

$$\theta \in H^0(X, \operatorname{ad}(E_G) \otimes V_0^*)\,.$$

Note that θ is a holomorphic 1-form on X with values in $\operatorname{ad}(E_G)$. Therefore, $\nabla^G + \theta$ is a holomorphic connection on E_G.

Assume that θ is flat with respect to the connection on $\operatorname{ad}(E_G) \otimes V_0^*$ induced by the connection ∇^G on E_G together with the trivial connection on the trivial vector bundle $X \times V_0^*$.

Note that ∇^H defines a holomorphic 1-form on the total space of E_H with values in $\mathfrak{h}$. On the other hand, θ defines a holomorphic 1-form on E_H with values in $\mathfrak{g}$. Therefore, $\nabla^H + \theta$ is a holomorphic 1-form on E_H with values in $\mathfrak{g}$. Assume that the $\operatorname{ad}(E_G)$-valued 1-form θ satisfies the condition that this form $\nabla^H + \theta : TE_H \longrightarrow E_H \times \mathfrak{g}$ is an isomorphism.

It is evident that the pair $(E_H, \nabla^H + \theta)$ defines a Cartan geometry on X of type (G, H). It is straightforward to check that this Cartan geometry is translation invariant.

The result mentioned in Section 2.2 that every holomorphic Cartan geometry of type (G, H), with G affine, on X is translation invariant, in fact implies that if G is affine, then all holomorphic Cartan geometries of type (G, H) on X are of the type described above. This would be elaborated in Section 3.3.

3. Principal bundles with holomorphic connection over a torus

3.1. A canonical flat connection

As before, X is a compact complex torus. Let G be a connected complex affine Lie group and E_G a holomorphic principal G-bundle over X. In [2], the following was proved.

If E_G admits a holomorphic connection, then it admits a flat holomorphic connection; see [2, p. 41, Theorem 4.1].

It should be clarified that in [2, Theorem 4.1], it is assumed that G admits a holomorphic embedding into some linear group $\mathrm{GL}(N, \mathbb{C})$. However, if $p : G \longrightarrow G'$ is a holomorphic homomorphism of complex Lie groups that produces an isomorphism of Lie algebras, then the holomorphic connections on a holomorphic principal G-bundle E_G are in bijection with the holomorphic connections on the associated holomorphic principal G'-bundle $E_{G'} := E_G \times^G G'$. Indeed, this follows immediately from the fact that the Atiyah bundle and the Atiyah exact sequence for E_G are canonically identified with the Atiyah bundle and the Atiyah exact sequence respectively for $E_{G'}$. The above bijection of holomorphic connections evidently takes flat connections to flat connections. Therefore, the above-mentioned result of [2, Theorem 4.1] for $E_{G'}$ implies that it also holds for E_G.

Assume now that E_G admits a holomorphic connection. In [2, p. 41, Theorem 4.1], it was proved that E_G admits a canonical flat connection. Indeed, E_G is pseudostable and its characteristic classes of degree one and two vanish (see the fourth statement in [2, p. 41, Theorem 4.1]). Now setting the zero Higgs field on E_G, from [2, p. 20, Theorem 1.1] we conclude that E_G has a canonical flat connection; this canonical connection on E_G will be denoted by ∇^{E_G}. This flat connection ∇^{E_G} enjoys the following properties.

Let $G \longrightarrow M$ be a holomorphic homomorphism of affine groups, and let

$$E_M := E_G \times^G M \longrightarrow X$$

be the associated holomorphic principal M-bundle. Then the canonical connection ∇^{E_M} on E_M coincides with the one induced by ∇^{E_G}. Now take $M = \mathrm{GL}(n, \mathbb{C})$; the holomorphic connection, induced by ∇^{E_M}, on the rank n vector bundle E_n associated with E_M by the standard representation of $\mathrm{GL}(n, \mathbb{C})$ will be denoted by ∇^{E_n}. If V is a pseudostable vector bundle on X with $c_1(V) = 0 = c_2(V)$, and

$$\phi : V \longrightarrow E_n$$

is any holomorphic homomorphism of vector bundles, then ϕ is flat with respect to ∇^{E_n} and the canonical connection on V (see [11], [2]). Below we briefly recall from [11] and [2].

A pseudostable vector bundle W over X with vanishing Chern classes has a canonical flat connection that is obtained by setting the zero Higgs field on W [11, p. 36, Lemma 3.5]. This canonical connection on such vector bundles is compatible with the operations of direct sum, tensor product, dualization, coherent sheaf homomorphisms, etc. From these properties, it can be deduced that any pseudostable principal bundle with vanishing characteristic classes has a canonical flat connection [2, p. 20, Theorem 1.1].

3.2. *Flat connection and translation invariance*

Let $p : \widetilde{X} \longrightarrow X$ be the universal cover, so $\widetilde{X}$ is a complex Lie group isomorphic to $\mathbb{C}^d$, where $d = \dim_{\mathbb{C}} X$. Let

$$\varphi : \widetilde{X} \times X \longrightarrow X$$

be the holomorphic map defined by $(y, x) \longmapsto x + p(y)$. Define the map

$$\widetilde{\varphi} : \mathbb{R} \times \widetilde{X} \times X \longrightarrow X, \quad (\lambda, y, x) \longmapsto x + p(\lambda \cdot y).$$

Let E_G be a holomorphic principal G-bundle on X equipped with a flat connection ∇^G. Consider the flat principal G-bundle $(\widetilde{\varphi}^* E_G, \widetilde{\varphi}^* \nabla^G)$ on $\mathbb{R} \times \widetilde{X} \times X$. The flat principal G-bundles on $\widetilde{X} \times X$ given by $(\widetilde{\varphi}^* E_G, \widetilde{\varphi}^* \nabla^G)|_{\{0\} \times \widetilde{X} \times X}$ and $(\widetilde{\varphi}^* E_G, \widetilde{\varphi}^* \nabla^G)|_{\{1\} \times \widetilde{X} \times X}$ are canonically identified by taking parallel translations along the paths $\lambda \longmapsto (\lambda, y, x)$, $\lambda \in [0, 1]$, in $\mathbb{R} \times \widetilde{X} \times X$. Note that $(\widetilde{\varphi}^* E_G, \widetilde{\varphi}^* \nabla^G)|_{\{1\} \times \widetilde{X} \times X}$ is identified with the pullback $(\varphi^* E_G, \varphi^* \nabla^G)$, while $(\widetilde{\varphi}^* E_G, \widetilde{\varphi}^* \nabla^G)|_{\{0\} \times \widetilde{X} \times X}$ is identified with the pullback $(p_X^* E_G, p_X^* \nabla^G)$, where p_X is the natural projection of $\widetilde{X} \times X$ to X.

It is straightforward to check that the above isomorphism between $(\varphi^* E_G, \varphi^* \nabla^G)$ and $(p_X^* E_G, p_X^* \nabla^G)$ produces a translation invariance structure on E_G that preserves the connection ∇^G.

3.3. *Translation invariance of Cartan geometries*

Let G be a complex affine Lie group. Let (E_H, β) be a holomorphic Cartan geometry of type (G, H) on a compact complex torus X. Let ∇^G be the holomorphic connection on the principal G-bundle $E_G = E_H \times^H G$ defined by β. Let $\nabla^{G,0}$ be the canonical flat connection on E_G. So we have

$$\theta := \nabla^G - \nabla^{G,0} = H^0(X, \mathrm{ad}(E_G) \otimes \Omega_X^1), \tag{3.1}$$

where $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ is the adjoint bundle for E_G.

Lemma 3.1. *The translation invariance structure on E_G given by the flat connection $\nabla^{G,0}$ preserves the holomorphic connection ∇^G.*

Proof. Let ∇^{ad} be the flat holomorphic connection on the adjoint vector bundle $\mathrm{ad}(E_G)$ induced by the flat holomorphic connection $\nabla^{G,0}$ on E_G. Let $\widetilde{\nabla}^{\mathrm{ad}}$ be the connection on $\mathrm{ad}(E_G) \otimes \Omega_X^1$ given by the above connection ∇^{ad} on $\mathrm{ad}(E_G)$ and the unique trivial connection on Ω_X^1. To prove the lemma, it suffices to show that the section θ in (3.1) is flat with respect to this connection $\widetilde{\nabla}^{\mathrm{ad}}$ on $\mathrm{ad}(E_G) \otimes \Omega_X^1$.

To prove that θ is flat, first note that $\widetilde{\nabla}^{\mathrm{ad}}$ is the canonical flat connection on $\mathrm{ad}(E_G) \otimes \Omega_X^1$, because ∇^{ad} is the canonical flat connection on $\mathrm{ad}(E_G)$ and the trivial connection on Ω_X^1 is the canonical flat connection on Ω_X^1. Therefore, any holomorphic section of $\mathrm{ad}(E_G) \otimes \Omega_X^1$ is flat with respect to the connection $\widetilde{\nabla}^{\mathrm{ad}}$. In particular, the section θ in (3.1) is flat with respect to $\widetilde{\nabla}^{\mathrm{ad}}$. □

Lemma 3.2. *The translation invariance structure on E_G given by the flat connection $\nabla^{G,0}$ preserves the reduction $E_H \subset E_G$.*

Proof. We know that E_G is pseudostable and its characteristic classes vanish [2]. Let

$$\mathrm{ad}(E_H) \subset \mathrm{ad}(E_G)$$

be the adjoint bundle for E_H. From (2.1) we know that the quotient bundle $\mathrm{ad}(E_G)/\mathrm{ad}(E_H)$ is isomorphic to the holomorphic tangent bundle TX of the torus and, consequently, it is trivial. In particular, $\mathrm{ad}(E_G)/\mathrm{ad}(E_H)$ is pseudostable and its Chern classes vanish. Therefore, the sub-vector bundle $\mathrm{ad}(E_H)$ is also pseudostable and its Chern classes vanish. From these, it follows that E_H is pseudostable and its characteristic classes vanish.

Let $\nabla^{H,0}$ be the canonical flat connection on E_H. The canonical connection $\nabla^{G,0}$ on E_G is induced by $\nabla^{H,0}$. This implies that the reduction $E_H \subset E_G$ is preserved by $\nabla^{G,0}$. □

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