



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Partial differential equations

On parabolic final value problems and well-posedness

*Sur les problèmes paraboliques à valeur finale bien posés*Ann-Eva Christensen¹, Jon Johnsen²

Department of Mathematics, Aalborg University, Skjernvej 4A, DK-9220 Aalborg Øst, Denmark



ARTICLE INFO

Article history:

Received 22 August 2017

Accepted after revision 29 January 2018

Available online 14 February 2018

Presented by the Editorial Board

ABSTRACT

We prove that a large class of parabolic final value problems is well posed. This results via explicit Hilbert spaces that characterise the data yielding existence, uniqueness and stability of solutions. This data space is the graph normed domain of an unbounded operator, which represents a new compatibility condition pertinent for final value problems. The framework is that of evolution equations for Lax–Milgram operators in vector distribution spaces. The final value heat equation on a smooth open set is also covered, and for non-zero Dirichlet data, a non-trivial extension of the compatibility condition is obtained by addition of an improper Bochner integral.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous prouvons que les problèmes à valeur finale sont bien posés pour une large classe d'opérateurs différentiels paraboliques. Ceci est obtenu via un espace de Hilbert qui caractérise l'existence des données impliquant l'existence, l'unicité et la stabilité des solutions. Cet espace de données est le domaine d'un opérateur non borné muni de la norme du graphe, qui représente une nouvelle condition de compatibilité pertinente pour les problèmes à valeur finale. Le cadre est celui des équations d'évolution pour des opérateurs de Lax–Milgram dans des espaces de distributions vectorielles. Nous étudions aussi le problème à valeur finale pour l'équation de la chaleur sur un ouvert lisse ; pour des données de Dirichlet non nulles, nous obtenons une extension non triviale de la condition de compatibilité par l'addition d'une intégrale de Bochner impropre.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

E-mail addresses: anneva.ch@gmail.com (A.-E. Christensen), jjohnsen@math.aau.dk (J. Johnsen).¹ Present address: Unit of Epidemiology and Biostatistics, Aalborg University Hospital, Høbrovej 18–22, DK-9000 Aalborg, Denmark.² The second author is supported by the Danish Research Council, Natural Sciences grant No. 4181-00042.

1. Introduction

Well-posedness of final value problems for a large class of parabolic differential equations is described here. That is, for suitable spaces X, Y specified below, they have *existence*, *uniqueness* and *stability* of solutions $u \in X$ for given data $(f, g, u_T) \in Y$. This should provide a basic clarification of a type of problems, which hitherto has been insufficiently understood.

As a first example, we characterise the functions $u(t, x)$ that, in a C^∞ -smooth bounded open set $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$, satisfy the following equations that constitute the final value problem for the heat equation ($\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ denotes the Laplacian):

$$\left. \begin{aligned} \partial_t u(t, x) - \Delta u(t, x) &= f(t, x) && \text{for } t \in]0, T[, x \in \Omega, \\ u(t, x) &= g(t, x) && \text{for } t \in]0, T[, x \in \partial\Omega, \\ u(T, x) &= u_T(x) && \text{for } x \in \Omega. \end{aligned} \right\} \tag{1}$$

Hereby (f, g, u_T) are the given data of the problem.

In case $f = 0, g = 0$, the first two lines of (1) are satisfied by $u(t, x) = e^{(T-t)\lambda} v(x)$ for all $t \in \mathbb{R}$, if $v(x)$ is an eigenfunction of the Dirichlet realization $-\Delta_D$ with eigenvalue λ .

Thus the homogeneous final value problem (1) has the above u as a *basic* solution if, coincidentally, the final data u_T equals the eigenfunction v . Our construction includes the set \mathcal{B} of such basic solutions u , its linear hull $\mathcal{E} = \text{span } \mathcal{B}$ and a certain completion $\overline{\mathcal{E}}$.

Using the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the associated $L_2(\Omega)$ -orthonormal basis e_1, e_2, \dots of eigenfunctions of $-\Delta_D$, the space \mathcal{E} (that corresponds to data $u_T \in \text{span}(e_j)$) clearly consists of solutions u being *finite* sums

$$u(t, x) = \sum_j e^{(T-t)\lambda_j} (u_T | e_j) e_j(x). \tag{2}$$

So at $t = 0$ there is, by the finiteness, a vector $u(0, x)$ in $L_2(\Omega)$ fulfilling

$$\|u(0, \cdot)\|^2 = \sum_j e^{2T\lambda_j} |(u_T | e_j)|^2 < \infty. \tag{3}$$

When summation is extended to all $j \in \mathbb{N}$, condition (3) becomes very strong, as it is only satisfied for special u_T : by Weyl's law $\lambda_j = \mathcal{O}(j^{2/n})$, so a single term in (3) yields $|(u_T | e_j)| \leq c \exp(-Tj^{2/n})$; whence the L_2 -coordinates of such u_T decay rapidly for $j \rightarrow \infty$. This has been known since the 1950s; cf. the work of John [9] and Miranker [12].

More recently e.g. Isakov [7] emphasized the observation, made already in [12], that (2) gives rise to an *instability*: the sequence of data $u_{T,k} = e_k$ has length 1 for all k , but (2) gives $\|u_k(0, \cdot)\| = \|e^{T\lambda_k} e_k\| = e^{T\lambda_k} \nearrow \infty$ for $k \rightarrow \infty$. Thus (1) is not well-posed in $L_2(\Omega)$.

In general, this instability shows that the L_2 -norm is an insensitive choice. To obtain well-adapted spaces for (1) with $f = 0, g = 0$, one could depart from (3). Indeed, along with the solution space \mathcal{E} , a norm on the final data $u_T \in \text{span}(e_j)$ can be defined by (3); and $\|u_T\| = (\sum_{j=1}^\infty e^{2T\lambda_j} |(u_T | e_j)|^2)^{1/2}$ can be used as norm on the u_T that correspond to solutions u in the above completion $\overline{\mathcal{E}}$. This would give the well-posedness of the homogeneous version of (1) with $u \in \overline{\mathcal{E}}$. (Cf. [3].)

But we have first of all replaced specific eigenvalue distributions by using sesqui-linear forms, cf. Lax–Milgram's lemma, which allowed us to cover general elliptic operators A .

Secondly the *fully* inhomogeneous problem (1) is covered. Here it does not suffice to choose the norm on the data (f, g, u_T) suitably (cf. $\|u_T\|$), for one has to *restrict* (f, g, u_T) to a subspace first by imposing certain *compatibility conditions*. These have long been known for parabolic problems, but they have a new form for final value problems.

2. The abstract final value problem

Our main analysis concerns a (possibly non-selfadjoint) Lax–Milgram operator A defined in H from a bounded V -elliptic sesquilinear form $a(\cdot, \cdot)$ in a Gelfand triple, i.e. densely injected Hilbert spaces $V \hookrightarrow H \hookrightarrow V^*$ with norms $\|\cdot\|, |\cdot|$ and $\|\cdot\|_*$.

In this set-up, we consider the following general final value problem: given data $f \in L_2(0, T; V^*), u_T \in H$, determine the vector distributions $u \in \mathcal{D}'(0, T; V)$ fulfilling

$$\left. \begin{aligned} \partial_t u + Au &= f && \text{in } \mathcal{D}'(0, T; V^*), \\ u(T) &= u_T && \text{in } H. \end{aligned} \right\} \tag{4}$$

A wealth of parabolic Cauchy problems with homogeneous boundary conditions have been efficiently treated using such triples (H, V, a) and the $\mathcal{D}'(0, T; V^*)$ framework in (4); cf. works of Lions and Magenes [10], Tanabe [14], Temam [15], Amann [2]. Also recently, e.g., Almog, Grebenkov, Helffer, Henry studied variants of the complex Airy operator via such triples [1,5,4], and our results should at least extend to final value problems for those of their realisations that have non-empty spectrum.

For the corresponding Cauchy problem, we recall that when solving $u' + Au = f$ so that $u(0) = u_0$ in H , for $f \in L_2(0, T; V^*)$, there is a unique solution u in the Banach space

$$X := L_2(0, T; V) \cap C([0, T]; H) \cap H^1(0, T; V^*)$$

$$\|u\|_X = \left(\int_0^T (\|u(t)\|^2 + \|u'(t)\|_*^2) dt + \sup_{0 \leq t \leq T} |u(t)|^2 \right)^{1/2}. \tag{5}$$

For (4) it would therefore be natural to expect solutions u in the same space X . This is correct, but only when the data (f, u_T) satisfy substantial further conditions.

To state these, we utilise that $-A$ generates an analytic semigroup e^{-tA} in $\mathbb{B}(H)$ and $\mathbb{B}(V^*)$, and that e^{-tA} consequently is invertible in the class of closed operators on H , resp. V^* ; cf. Proposition 2.2 in [3]. Consistently with the case when A generates a group, we set

$$(e^{-tA})^{-1} = e^{tA}. \tag{6}$$

Its domain $D(e^{tA}) = R(e^{-tA})$ is the Hilbert space normed by $\|u\| = (|u|^2 + |e^{tA}u|^2)^{1/2}$. In the common case A has non-empty spectrum, $\sigma(A) \neq \emptyset$, there is a chain of strict inclusions

$$D(e^{t'A}) \subsetneq D(e^{tA}) \subsetneq H \quad \text{for } 0 < t < t'. \tag{7}$$

At the final time, $t = T$ these domains enter the well-posedness result below, where for brevity y_f will denote the full yield of the source term f on the system, namely

$$y_f = \int_0^T e^{-(T-s)A} f(s) ds. \tag{8}$$

The map $f \mapsto y_f$ takes values in H , and it is a continuous surjection $y_f : L_2(0, T; V^*) \rightarrow H$.

Theorem 1. *The final value problem (4) has a solution u in the space X in (5) if and only if the data (f, u_T) belong to the subspace Y of $L_2(0, T; V^*) \oplus H$ defined by the condition*

$$u_T - y_f \in D(e^{TA}). \tag{9}$$

In the affirmative case, the solution u is unique in X , and it depends continuously on the data (f, u_T) in Y , that is $\|u\|_X \leq c\|(f, u_T)\|_Y$, when Y is given the graph norm

$$\|(f, u_T)\|_Y = \left(|u_T|^2 + \int_0^T \|f(t)\|_*^2 dt + |e^{TA}(u_T - y_f)|^2 \right)^{1/2}. \tag{10}$$

Condition (9) is seemingly a fundamental novelty for the final value problem (4). As for (10), it is the graph norm of $(f, u_T) \mapsto e^{TA}(u_T - y_f)$, which for $\Phi(f, u_T) = u_T - y_f$ is the unbounded operator $e^{TA} \circ \Phi$ from $L_2(0, T; V^*) \oplus H$ to H .

In fact, $e^{TA}\Phi$ is central to a rigorous treatment of (4), for (9) means that $e^{TA}\Phi$ must be defined at (f, u_T) ; i.e. the data space Y is its domain. So since $e^{TA}\Phi$ is a closed operator, Y is a Hilbert space, which by (10) is embedded into $L_2(0, T; V^*) \oplus H$.

As an inconvenient aspect, the presence of $e^{-(T-t)A}$ and the integration over $[0, T]$ make (9) non-local in space and time—exacerbated by use of the abstract domain $D(e^{TA})$, which for larger T gives increasingly stricter conditions; cf. (7).

We regard (9) as a compatibility condition on the data (f, u_T) , and thus we generalise the notion. Grubb and Solonnikov [6] made a systematic treatment of initial-boundary problems of parabolic equations with compatibility conditions, which are necessary and sufficient for well-posedness in full scales of anisotropic L_2 -Sobolev spaces—whereby compatibility conditions are decisive for the solution’s regularity. In comparison, (9) is crucial for the existence question; cf. Theorem 1.

Remark 2. Previously, uniqueness was observed by Amann [2, V.2.5.2] in a t -dependent set-up. However, the injectivity of $u(0) \mapsto u(T)$ was shown much earlier in a set-up with t -dependent sesquilinear forms by Lions and Malgrange [11].

Remark 3. Showalter [13] attempted to characterise the possible u_T in terms of Yosida approximations for $f = 0$ and A having half-angle $\pi/4$. As an ingredient, the invertibility of analytic semigroups was claimed by Showalter for such A , but his proof was flawed as A can have semi-angle $\pi/4$ even if A^2 is not accretive; cf. our example in Remark 3.15 of [3].

Theorem 1 is proved by considering the full set of solutions to the differential equation $u' + Au = f$. As indicated in (5), for fixed $f \in L_2(0, T; V^*)$ the solutions in X are parametrised by the initial state $u(0) \in H$; and they are also in this set-up necessarily given by the variation of constants formula for the analytic semigroup e^{-tA} in V^* ,

$$u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A} f(s) ds. \tag{11}$$

For $t = T$, this yields a *bijective correspondence* $u(0) \longleftrightarrow u(T)$ between the initial and terminal states—for due to the invertibility of e^{-TA} , cf. (6), one can isolate $u(0)$ here. Moreover, (11) also yields necessity of (9) at once, as the difference $u_T - y_f$ in (9) must be equal to $e^{-TA}u(0)$, which clearly belongs to the domain $D(e^{TA})$.

Moreover, $u(T)$ consists of two radically different parts, cf. (11), even when A is ‘nice’:

First, $e^{-tA}u(0)$ solves the equation for $f = 0$, and for $u(0) \neq 0$ we obtained in [3] the precise property in non-selfadjoint dynamics that the “height” function $h(t)$ is *strictly convex*. Hereby

$$h(t) = |e^{-tA}u(0)|. \tag{12}$$

This results from the injectivity of e^{-tA} when A is normal, or belongs to the class of hyponormal operators studied by Janas [8], or in case A^2 is accretive – so for such A the complex eigenvalues (if any) cannot give oscillations in the size of $e^{-tA}u(0)$, cf. the strict convexity. This stiffness from the strict convexity is consistent with the fact for analytic semigroups that $u(T) = e^{-TA}u(0)$ is confined to the dense, but very small space $\bigcap_{n \in \mathbb{N}} D(A^n)$.

In addition, $h(t)$ is strictly decreasing with $h'(0) \leq -m(A)$, where $m(A)$ denotes the lower bound; i.e. the short-time behaviour is governed by the numerical range $\nu(A)$ also for such A .

Secondly, for $u(0) = 0$ the equation is solved by the integral in (11), which has rather different properties. Its final value $y_f : L_2(0, T; V^*) \rightarrow H$ is surjective, so y_f can be *anywhere* in H . This was shown with a kind of control-theoretic argument in [3] for the case that $A = A^*$ with A^{-1} compact; and for general A by using the Closed Range Theorem.

Thus the possible final data u_T are a sum of an arbitrary $y_f \in H$ and a term $e^{-TA}u(0)$ of great stiffness, so that u_T can be prescribed anywhere in the affine space $y_f + D(e^{TA})$. As $D(e^{TA})$ is dense in H , and in general there hardly is any control over the direction of y_f (if non-zero), it is not feasible to specify u_T a priori in other spaces than H . Instead, it is by the condition $u_T - y_f \in D(e^{TA})$ that the u_T and f are properly controlled.

3. The inhomogeneous heat problem

For general data (f, g, u_T) in (1), the results in **Theorem 1** are applied with $A = -\Delta_D$. The results are analogous, but less simple to prove and state.

First of all, even though it is a linear problem, the compatibility condition (9) *destroys* the old trick of reducing to boundary data $g = 0$, for when $w \in H^1$ fulfils $w = g \neq 0$ on the curved boundary $]0, T[\times \partial\Omega$, then w *lacks* the regularity needed to test condition (9) on the resulting data $(\tilde{f}, 0, \tilde{u}_T)$ of the reduced problem.

Secondly, it therefore takes an effort to show that when the boundary data $g \neq 0$, then they *do* give rise to a correction term z_g . This means that condition (9) is replaced by

$$u_T - y_f + z_g \in D(e^{-T\Delta_D}). \tag{13}$$

Thirdly, because of the low regularity, it requires some technical diligence to show that, despite the singularity present in $\Delta e^{(T-s)\Delta_D}$ at $s = T$, the correction z_g has the structure of an improper Bochner integral converging in $L_2(\Omega)$, namely

$$z_g = \int_0^T \Delta e^{(T-s)\Delta_D} K_0 g(s) ds. \tag{14}$$

Hereby the Poisson operator $K_0 : H^{1/2}(\partial\Omega) \rightarrow Z(-\Delta)$ is chosen as the inverse of the operator, which results by restricting the boundary trace $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ to its co-image $Z(-\Delta)$ of harmonic functions in $H^1(\Omega)$; there is a direct sum $H^1(\Omega) = H_0^1(\Omega) \dot{+} Z(-\Delta)$.

It is noteworthy that the full influence of the boundary data g on the final state $u(T)$ is given in the formula for z_g above. In addition, $z_g : H^{1/2}(]0, T[\times \partial\Omega) \rightarrow L_2(\Omega)$ is bounded.

Theorem 4. For given data $f \in L_2(0, T; H^{-1}(\Omega))$, $g \in H^{1/2}(]0, T[\times \partial\Omega)$, $u_T \in L_2(\Omega)$, the final value problem (1) is solved by a function u in the Banach space X_1 , whereby

$$X_1 = L_2(0, T; H^1(\Omega)) \bigcap C([0, T]; L_2(\Omega)) \bigcap H^1(0, T; H^{-1}(\Omega)),$$

$$\|u\|_{X_1} = \left(\int_0^T (\|u(t)\|_{H^1(\Omega)}^2 + \|u'(t)\|_{H^{-1}(\Omega)}^2) dt + \sup_{0 \leq t \leq T} \|u(t)\|_{L_2(\Omega)}^2 \right)^{1/2}, \tag{15}$$

if and only if the data in terms of (8) and (14) satisfy the compatibility condition

$$u_T - y_f + z_g \in D(e^{-T\Delta_D}). \tag{16}$$

In the affirmative case, u is uniquely determined in X_1 and has the representation

$$u(t) = e^{t\Delta_D} e^{-T\Delta_D} (u_T - y_f + z_g) + \int_0^t e^{(t-s)\Delta} f(s) ds - \int_0^t \Delta e^{(t-s)\Delta_D} K_0 g(s) ds, \tag{17}$$

where the three terms all belong to X_1 as functions of t .

Clearly the space of admissible data Y_1 is here a specific subspace of

$$L_2(0, T; H^{-1}(\Omega)) \oplus H^{1/2}([0, T[\times \partial\Omega) \oplus L_2(\Omega), \tag{18}$$

for by setting $\Phi_1(f, g, u_T) = u_T - y_f + z_g$, we have

$$Y_1 = \left\{ (f, g, u_T) \mid u_T - y_f + z_g \in D(e^{-T\Delta_D}) \right\} = D(e^{-T\Delta_D} \Phi_1). \tag{19}$$

Here $e^{-T\Delta_D} \Phi_1$ is an unbounded operator from the space in (18) to H . Therefore Y_1 is a hilbertable Banach space when endowed with the corresponding graph norm

$$\begin{aligned} \|(f, g, u_T)\|_{Y_1}^2 &= \|u_T\|_{L_2(\Omega)}^2 + \|g\|_{H^{1/2}([0, T[\times \partial\Omega)}^2 + \|f\|_{L_2(0, T; H^{-1}(\Omega))}^2 \\ &\quad + \int_{\Omega} \left| e^{-T\Delta_D} \left(u_T - \int_0^T e^{-(T-s)\Delta} f(s) ds + \int_0^T \Delta e^{(T-s)\Delta_D} K_0 g(s) ds \right) \right|^2 dx. \end{aligned} \tag{20}$$

Using this the solution operator $(f, g, u_T) \mapsto u$ is bounded $Y_1 \rightarrow X_1$, that is,

$$\|u\|_{X_1} \leq c \|(f, g, u_T)\|_{Y_1}. \tag{21}$$

This can be shown by exploiting the bijection $u(0) \longleftrightarrow u(T)$ to invoke the classical estimates of the initial value problem, which in the present low-regularity setting has no compatibility conditions and therefore allows a reduction to the case $g = 0$. So, in combination with Theorem 4, we have:

Theorem 5. *The final value Dirichlet heat problem (1) is well posed in the spaces X_1 and Y_1 ; cf. (15) and (18)–(20).*

The full proofs of the results in this note can be found in our exposition [3].

References

[1] Y. Almog, B. Helffer, On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent, *Commun. Partial Differ. Equ.* 40 (8) (2015) 1441–1466.
 [2] H. Amann, *Linear and Quasilinear Parabolic Problems, Vol. I, Abstract Linear Theory, Monographs in Mathematics, vol. 89, Birkhäuser Boston, Inc., Boston, MA, USA, 1995.*
 [3] A.-E. Christensen, J. Johnsen, Final value problems for parabolic differential equations and their well-posedness, preprint, arXiv:1707.02136, 2017.
 [4] D.S. Grebenkov, B. Helffer, On the spectral properties of the Bloch–Torrey operator in two dimensions, arXiv:1608.01925, 2016.
 [5] D.S. Grebenkov, B. Helffer, R. Henry, The complex Airy operator on the line with a semipermeable barrier, *SIAM J. Math. Anal.* 49 (3) (2017) 1844–1894.
 [6] G. Grubb, V.A. Solonnikov, Solution of parabolic pseudo-differential initial-boundary value problems, *J. Differ. Equ.* 87 (1990) 256–304.
 [7] V. Isakov, *Inverse Problems for Partial Differential Equations, Applied Mathematical Sciences, vol. 127, Springer-Verlag, New York, 1998.*
 [8] J. Janas, On unbounded hyponormal operators III, *Stud. Math.* 112 (1994) 75–82.
 [9] F. John, Numerical solution of the equation of heat conduction for preceding times, *Ann. Mat. Pura Appl.* (4) 40 (1955) 129–142.
 [10] J.-L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications, Vol. I, Die Grundlehren der mathematischen Wissenschaften, vol. 181, Springer-Verlag, New York–Heidelberg, 1972, translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.*
 [11] J.-L. Lions, B. Malgrange, Sur l’unicité rétrograde dans les problèmes mixtes parabolic, *Math. Scand.* 8 (1960) 227–286.
 [12] W.L. Miranker, A well posed problem for the backward heat equation, *Proc. Amer. Math. Soc.* 12 (1961) 243–247.
 [13] R.E. Showalter, The final value problem for evolution equations, *J. Math. Anal. Appl.* 47 (1974) 563–572.
 [14] H. Tanabe, *Equations of Evolution, Monographs and Studies in Mathematics, vol. 6, Pitman, Boston, Mass, 1979, translated from the Japanese by N. Mugibayashi and H. Haneda.*
 [15] R. Temam, *Navier–Stokes Equations, Theory and Numerical Analysis, 3rd edition, Elsevier Science Publishers B.V., Amsterdam, 1984.*