C. R. Acad. Sci. Paris, Ser. I 356 (2018) 288–292

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex analysis

Inequalities involving the multiple psi function

Inégalités mettant en jeu la fonction psi multiple

Sourav Das

Department of Mathematics, National Institute of Technology, Hamirpur, Himachal Pradesh, 177005, India

A R T I C L E   I N F O

Article history:
Received 29 September 2017
Accepted after revision 22 January 2018
Available online 1 February 2018

Presented by the Editorial Board

A B S T R A C T

In this work, multiple gamma functions of order n have been considered. The logarithmic derivative of the multiple gamma function is known as the multiple psi function. Subadditive, superadditive, and convexity properties of higher-order derivatives of the multiple psi function are derived. Some related inequalities for these functions and their ratios are also obtained.

R É S U M É

Nous considérons ici les fonctions gamma multiples d'ordre n. La dérivée logarithmique de la fonction gamma multiple est la fonction psi bien connue. Nous obtenons des propriétés additives et de convexité des dérivées d'ordre supérieur de la fonction psi multiple. Nous obtenons également quelques inégalités faisant intervenir ces fonctions et leurs quotients.

1. Introduction

Multiple gamma functions were introduced and studied systematically by E. W. Barnes [8,9], in the early 1900s. These multiple gamma functions are a generalization of Euler's gamma function. It is well known that Euler's gamma function Γ is useful to define the n-dimensional volume of the unit ball in $\mathbb{R}^n$ [15]. Multiple gamma functions are also useful to study the determinants of Laplacians on the n-dimensional unit sphere S^n [13–17]. Recently, V. S. Adamchik [2] discovered the application of multiple gamma functions in the computation of certain series. Multiple gamma functions of order n are denoted by Γ_n and defined [18, Theorem 3] as

$$\Gamma_n(1+z) = \exp[P_n(z)] \cdot \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-\binom{n+k-2}{n-1}} \exp\left[\binom{n+k-2}{n-1} \left(\sum_{j=1}^{n} \frac{(-1)^{j-1}}{j} \frac{z^j}{k^j} \right) \right] \right\}, \tag{1.1}$$

where $P_n(z)$ is a polynomial in z of degree n defined by

E-mail addresses: souravdasmath@gmail.com, sourav@nith.ac.in.

https://doi.org/10.1016/j.crma.2018.01.014
1631-073X/

$$P_n(z) := (-1)^{n-1}\left[-zA_n(1) + \sum_{k=1}^{n-1}\frac{p_k(z)}{k!}\left(f_{n-1}^{(k)}(0) - A_n^{(k)}(1)\right)\right];$$

$$f_n(z) := -zA_n(1) + \sum_{k=1}^{n-1}\frac{p_k(z)}{k!}\left[f_{n-1}^{(k)}(0) - A_n^{(k)}(1)\right] + A_n(z);$$

and

$$A_n(z) := \sum_{k=1}^{\infty}(-1)^{n-1}\binom{n+k-2}{n-1}\left[-\log\left(1+\frac{z}{k}\right) + \sum_{j=1}^{n}\frac{(-1)^{j-1}}{j}\frac{z^j}{k^j}\right];$$

$$p_n(z) = \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k}B_{n+1-k}z^k,$$

with B_n being the Bernoulli numbers and $n \in \mathbb{N}$.

$\Gamma_n(z)$ can also be expressed in a simple way by the following recurrence relations

$$\Gamma_{n+1}(1+z) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad \Gamma_1(z) = \Gamma(z), \quad \Gamma_n(1) = 1, \quad n \in \mathbb{N}, \quad z \in \mathbb{C}.$$

M. F. Vignéras [22] redefined multiple gamma functions by introducing a hierarchy of functions $G_n(z) = (\Gamma_n(z))^{(-1)^{n-1}}$ satisfying the conditions of the generalized Bohr–Mollerup theorem [15,21,22].

Theorem 1.1. [15] *For all $n \in \mathbb{N}$, there exists a unique meromorphic function $G_n(z)$ satisfying each of the following properties:*

(i) $G_n(z+1) = G_n(z)G_{n-1}(z)$ *for* $z \in \mathbb{C}$;
(ii) $G_n(1) = 1$;
(iii) *For* $x \geq 0$, $G_n(x+1)$ *are infinitely differentiable and*

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\log G_n(x+1) \geq 0;$$

(iv) $G_0(z) = z$.

The function $G_2(z) = \frac{1}{\Gamma_2(z)} = G(z)$ is known as the Barnes G-function. The multiple psi function Ψ_n is defined as the logarithmic derivative of Γ_n, i.e. $\Psi_n = \frac{\Gamma_n'}{\Gamma_n}$. The poly multiple gamma function $\Psi_n^{(m)}$ is the m-th order derivative of Ψ_n, where $m, n \in \mathbb{N}$. More information on multiple gamma functions can be found in [2,8,9,15,18,20] and the references cited therein.

Finding bounds for Euler's gamma function and multiple gamma functions and their ratios have been the subject of study of many mathematicians and researchers [1–6,8–10,12,15,18,19]. Subadditivity (superadditivity) is a part of the theory of inequalities. A function f is called subadditive on a set I of real numbers if $f(x+y) \leq f(x) + f(y)$ for all $x, y \in I$ such that $x + y \in I$. If the inequality reverses, then f is called superadditive on I. If $f(xy) \leq f(x)f(y)$ holds for all $x, y \in I$ such that $xy \in I$, then f is known as submultiplicative. If the inequality reverses, then f is called supermultiplicative. These functions play vital role in number theory, in the theory of differential equations and also in the theory of convex bodies.

H. Alzer and S. Ruscheweyh [7] proved that $x \mapsto (\Gamma(x))^{\alpha}$ is subadditive on $(0, \infty)$ if and only if $\alpha^* \leq \alpha \leq 0$, where $\alpha^* \approx -0.946850\ldots$. In [4], H. Alzer derived that $\Psi(\mathrm{e}^x)$ is strictly concave on $\mathbb{R}$, where $\Psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x)$ is known as the psi (digamma) function. Recently, in 2007, H. Alzer [5] proved the subadditive and superadditive properties of Euler's gamma function, and obtained the following interesting inequality:

$$\left(\frac{\Gamma(x+y+c)}{\Gamma(x+y)}\right)^{1/\alpha} < \left(\frac{\Gamma(x+c)}{\Gamma(x)}\right)^{1/\alpha} + \left(\frac{\Gamma(y+c)}{\Gamma(y)}\right)^{1/\alpha}.$$

The above inequality holds for all $x, y > 0$ if and only if $\alpha \leq \max(1, c)$, where $0 < c \neq 1$. The reverse inequality is valid for all positive x and y if and only if $\alpha \leq \min(1, c)$. In [10,11], N. Batir obtained bounds for double gamma function and discussed the monotonicity properties of q-analogue of digamma and trigamma functions. Bounds for multiple gamma functions were derived by J. Choi and H. M. Srivastava in [18].

Motivated by the above results, subadditive, superadditive and convexity properties and related inequalities for the poly multiple gamma functions are obtained in this article.

2. Inequalities for the poly multiple gamma function

Let $m \geq n$ be any natural number. Applying logarithm on both sides of (1.1) and differentiating $m+1$ times, we have

$$\Psi_n^{(m)}(x) = (-1)^{m+1} \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{m!}{(x+k)^{m+1}}, \quad x > 0. \tag{2.1}$$

Therefore for $x > 0$, we obtain

$$\Psi_n^{(m)}(x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{a_k}{(x+k)^{m+1}}, & m \text{ is odd;} \\ \displaystyle -\sum_{k=0}^{\infty} \frac{a_k}{(x+k)^{m+1}}, & m \text{ is even,} \end{cases} \tag{2.2}$$

where $a_k = m!\binom{n+k-1}{n-1}$. Clearly, $\Psi_n^{(m)}(x)$ is positive (negative) if m is odd (even) for all $x > 0$.

Hence, $\Psi_n^{(m)}(x)$ is decreasing if m is odd and increasing if m is even for all $x > 0$. For our next results, we consider $m \geq n$, where $m, n \in \mathbb{N}$. The following results are immediate. Proofs are omitted.

Theorem 2.1. *$\Psi_n^{(m)}(x)$ is convex (concave) on $\mathbb{R}$ if m is odd (even).*

Corollary 2.2. *$\Psi_n^{(m)}(e^x)$ is convex (concave) on $\mathbb{R}$ if m is odd (even).*

Next results deal with subadditive (superadditive) properties of $\Psi_n^{(m)}(x)$.

Theorem 2.3. *For $a \geq 0$, $x, y > 0$ and $m \geq n \geq 1$, the following inequalities hold:*

$$\Psi_n^{(m)}(a+x+y) < \Psi_n^{(m)}(a+x) + \Psi_n^{(m)}(a+y), \quad m \text{ is odd;}$$
$$\Psi_n^{(m)}(a+x+y) > \Psi_n^{(m)}(a+x) + \Psi_n^{(m)}(a+y), \quad m \text{ is even.}$$

Proof. Let

$$g_m(x) = \frac{1}{(a+x+k)^{m+1}} + \frac{1}{(a+y+k)^{m+1}} - \frac{1}{(a+x+y+k)^{m+1}}.$$

Then keeping y fixed, we have

$$g_m'(x) = -\frac{m+1}{(a+x+k)^{m+2}} + \frac{m+1}{(a+x+y+k)^{m+2}} < 0,$$

which implies that $g_m(x)$ is decreasing and $\lim_{x\to\infty} g_m(x) > 0$. Therefore, $g_m(x) > 0$.

Now,

$$\begin{aligned} &\Psi_n^{(m)}(a+x) + \Psi_n^{(m)}(a+y) - \Psi_n^{(m)}(a+x+y) \\ &= (-1)^{m+1} \sum_{k=0}^{\infty} a_k \left[\frac{1}{(a+x+k)^{m+1}} + \frac{1}{(a+y+k)^{m+1}} - \frac{1}{(a+x+y+k)^{m+1}} \right] \\ &= (-1)^{m+1} \sum_{k=0}^{\infty} a_k g_m(x), \end{aligned}$$

which proves the theorem. □

Now we have the following corollary explaining the subadditivity (superadditivity) of $\Psi_n^{(m)}(x)$ for $x > 0$.

Corollary 2.4. *$\Psi_n^{(m)}(x)$ is subadditive (superadditive) if m is odd (even) for $x > 0$.*

Proof. $a = 0$ in Theorem 2.3 gives the proof of the corollary. □

Theorem 2.5. *Let $m \geq n \geq 1$ be any integer, α be a real number, and*

$$f_\alpha(x) = x^\alpha |\Psi_n^{(m)}(x)|, \quad x > 0.$$

Then f_α is strictly increasing on $(0, \infty)$ if $\alpha \geq m+1$.

Proof. Let $x > 0$. By differentiation, we obtain

$$\begin{aligned} f'_\alpha(x) &= -x^\alpha |\Psi_n^{(m+1)}(x)| + \alpha x^{\alpha-1} |\Psi_n^{(m)}(x)| \\ &= x^{\alpha-1} m! \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \frac{(x(\alpha-m-1)+k\alpha)}{(x+k)^{m+2}}. \end{aligned}$$

If $\alpha \geq m+1$, then $f'_\alpha(x) > 0$. Hence the theorem is proved. □

Theorem 2.6. *Let a, x and y be positive real numbers. Then for all $m \geq n \geq 1$, the following inequalities hold*

$$\left[\Psi_n^{(m)}(a+x+y)\right]^2 < \Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y), \quad \text{if } m \text{ is odd}, \tag{2.3}$$

$$\left[\Psi_n^{(m)}(a+x+y)\right]^2 > \Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y), \quad \text{if } m \text{ is even}. \tag{2.4}$$

Proof. Let $m \geq n$ be any odd natural number. Then

$$\Psi_n^{(m)}(a+x) = \sum_{k=0}^{\infty} \frac{a_k}{(k+a+x)^{m+1}} > \sum_{k=0}^{\infty} \frac{a_k}{(k+a+x+y)^{m+1}} = \Psi_n^{(m)}(a+x+y) > 0. \tag{2.5}$$

Similarly,

$$\Psi_n^{(m)}(a+y) > \Psi_n^{(m)}(a+x+y) > 0. \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y) > \left[\Psi_n^{(m)}(a+x+y)\right]^2.$$

If m is any even natural number, then the inequality (2.4) can also be proved in a similar way. Hence the proof is complete. □

Theorem 2.7. *Let $a \geq 0$, $m \geq n \geq 1$ and $0 < x, y < 1$. Then*

$$\left[\Psi_n^{(m)}(a+xy)\right]^2 > \Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y).$$

Proof. Let $f(x) = \left[\Psi_n^{(m)}(a+x)\right]^2$. Then

$$f'(x) = 2\Psi_n^{(m)}(a+x)\Psi_n^{(m+1)}(a+x) < 0,$$

which implies that $f(x)$ is a decreasing on $(0, 1)$. Therefore,

$$\left[\Psi_n^{(m)}(a+xy)\right]^2 > \left[\Psi_n^{(m)}(a+x)\right]^2 \tag{2.7}$$

and

$$\left[\Psi_n^{(m)}(a+xy)\right]^2 > \left[\Psi_n^{(m)}(a+y)\right]^2. \tag{2.8}$$

Combining the above inequalities (2.7) and (2.8), we have

$$\left[\Psi_n^{(m)}(a+xy)\right]^4 > \left[\Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y)\right]^2.$$

Since, $\Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y) > 0$. Therefore,

$$\left[\Psi_n^{(m)}(a+xy)\right]^2 > \Psi_n^{(m)}(a+x)\Psi_n^{(m)}(a+y),$$

which completes the proof. □

The next result describes the bounds for the ratio of poly multiple gamma functions.

Theorem 2.8. *Let a, b, c, d, e, f be real numbers and $f(x)$ be a function defined as*

$$f(x) = \frac{\Psi_n^{(m)}(a+bx)^c}{\Psi_n^{(m)}(d+ex)^f}, \quad x \geq 0, \quad m \geq n \geq 1.$$

(a) *If $a, b, d, e > 0$, $c \leq 0$, $f \geq 0$, then $f(x)$ is increasing in $[0, \infty)$ and for all $x \in [0, 1]$, the following inequality holds:*

$$\frac{\Psi_n^{(m)}(a)^c}{\Psi_n^{(m)}(d)^f} \leq \frac{\Psi_n^{(m)}(a+bx)^c}{\Psi_n^{(m)}(d+ex)^f} \leq \frac{\Psi_n^{(m)}(a+b)^c}{\Psi_n^{(m)}(d+e)^f}.$$

(b) *If $a, b, d, e > 0$, $c \geq 0$, $f \leq 0$, then $f(x)$ is decreasing in $[0, \infty)$ and for all $x \in [0, 1]$, the following inequality holds:*

$$\frac{\Psi_n^{(m)}(a+b)^c}{\Psi_n^{(m)}(d+e)^f} \leq \frac{\Psi_n^{(m)}(a+bx)^c}{\Psi_n^{(m)}(d+ex)^f} \leq \frac{\Psi_n^{(m)}(a)^c}{\Psi_n^{(m)}(d)^f}.$$

Proof. Let $g(x) = \ln f(x)$. Then

$$g'(x) = \frac{bc\Psi_n^{(m)}(d+ex)\Psi_n^{(m+1)}(a+bx) - ef\Psi_n^{(m)}(a+bx)\Psi_n^{(m+1)}(d+ex)}{\Psi_n^{(m)}(a+bx)\Psi_n^{(m)}(d+ex)}. \tag{2.9}$$

From (2.2), we have

$$\Psi_n^{(m)}(d+ex)\Psi_n^{(m+1)}(a+bx) < 0, \; \Psi_n^{(m+1)}(d+ex)\Psi_n^{(m)}(a+bx) < 0$$
$$\text{and } \Psi_n^{(m)}(a+bx)\Psi_n^{(m)}(d+ex) > 0.$$

Hence, using the conditions of part (a), we have $g'(x) \geq 0$, which implies that $g(x)$ is increasing in $[0, \infty)$. Consequently, $f(x)$ is increasing in $[0, \infty)$ and for $0 \leq x \leq 1$, $f(0) \leq f(x) \leq f(1)$, which proves part (a) of the theorem.

Note that the condition (b) reverses the inequalities given in part (a). Hence, proceeding similarly like part (a), part (b) of the theorem can be established. □

Remark. It can be observed that the case $n = 1$ in Theorem 2.5 leads to be a part of Lemma 2.3 of [4].

Acknowledgements

The author wishes to thank the reviewers for suggestions that helped to improve the paper.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington, DC, 1964. For sale by the Superintendent of Documents.
[2] V.S. Adamchik, The multiple gamma function and its application to computation of series, Ramanujan J. 9 (3) (2005) 271–288.
[3] H. Alzer, Mean-value inequalities for the polygamma functions, Aequ. Math. 61 (1–2) (2001) 151–161.
[4] H. Alzer, Sharp inequalities for the digamma and polygamma functions, Forum Math. 16 (2) (2004) 181–221.
[5] H. Alzer, Sub- and superadditive properties of Euler's gamma function, Proc. Amer. Math. Soc. 135 (11) (2007) 3641–3648 (electronic).
[6] H. Alzer, O.G. Ruehr, A submultiplicative property of the psi function, J. Comput. Appl. Math. 101 (1–2) (1999) 53–60.
[7] H. Alzer, S. Ruscheweyh, A subadditive property of the gamma function, J. Math. Anal. Appl. 285 (2) (2003) 564–577.
[8] E.W. Barnes, The theory of the G-function, Q. J. Math. 31 (1899) 264–314.
[9] E.W. Barnes, On the theory of the multiple gamma function, Trans. Camb. Philos. Soc. 19 (1904) 374–439.
[10] N. Batir, Inequalities for the double gamma function, J. Math. Anal. Appl. 351 (1) (2009) 182–185.
[11] N. Batir, Monotonicity properties of q-digamma and q-trigamma functions, J. Approx. Theory 192 (2015) 336–346.
[12] C.-P. Chen, Inequalities associated with Barnes G-function, Expo. Math. 29 (1) (2011) 119–125.
[13] J. Choi, Determinant of Laplacian on S^3, Math. Jpn. 40 (1) (1994) 155–166.
[14] J. Choi, Determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$, Adv. Differ. Equ. 2013 (2013) 236.
[15] J. Choi, Multiple gamma functions and their applications, in: Analytic Number Theory, Approximation Theory, and Special Functions, Springer, New York, 2014, pp. 93–129.
[16] J. Choi, H.M. Srivastava, An application of the theory of the double gamma function, Kyushu J. Math. 53 (1) (1999) 209–222.
[17] J. Choi, H.M. Srivastava, Certain classes of series associated with the zeta function and multiple gamma functions, J. Comput. Appl. Math. 118 (1–2) (2000) 87–109.
[18] J. Choi, H.M. Srivastava, Some two-sided inequalities for multiple gamma functions and related results, Appl. Math. Comput. 219 (20) (2013) 10343–10354.
[19] A.Sh. Shabani, Some inequalities for the gamma function, J. Inequal. Pure Appl. Math. 8 (2) (2007) 49.
[20] H.M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier, Inc., Amsterdam, 2012.
[21] K. Ueno, M. Nishizawa, The multiple gamma function and its q-analogue, in: Quantum Groups and Quantum Spaces, Warsaw, 1995, in: Banach Cent. Publ., vol. 40, Polish Acad. Sci., Warsaw, 1997, pp. 429–441.
[22] M.F. Vignéras, L'équation fonctionnelle de la fonction zeta de Selberg de groupe modulaire PSL(2; Z), Astérisque 61 (1979) 235–249.