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Differential geometry

An extension of a theorem by Cheeger and Müller to spaces with isolated conical singularities

Une extension d'un résultat de Cheeger et Müller pour un espace à singularités coniques isolées

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A B S T R A C T

The aim of this note is to extend a theorem by Cheeger and Müller to spaces with isolated conical singularities by generalising the proof of Bismut and Zhang to the singular setting. The main tools in this approach are the Witten deformation and local index techniques.

R É S U M É

Le but de cette note est d'établir un théorème de Cheeger–Müller pour un espace a singularités coniques isolées en généralisant la preuve de Bismut et Zhang. Les outils utilisés dans la preuve sont les techniques d'indice local et la déformation de Witten.

1. Introduction

An important comparison theorem in global analysis is the comparison of topological and analytic torsion for smooth compact manifolds equipped with a unitary flat vector bundle. It has been conjectured by Ray and Singer [14] and has been proved independently by Cheeger [2] and Müller [12]. In [1] Bismut and Zhang combined the Witten deformation and local index techniques to generalise the result of Cheeger and Müller to arbitrary flat vector bundles with arbitrary Hermitian metrics. The study of analytic and topological torsion for spaces with conical singularities has gained a lot of interest in recent years, being first addressed by A. Dar [5]. An approach to the Cheeger–Müller theorem for spaces with isolated conical singularities, though not yet completed, is to reduce the problem, via the gluing formula of Lesch [8] and Vishik [17], to a comparison of torsions on a truncated cone. The analytic torsion of a truncated cone has been studied in [16,13,7].

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The aim of this note is to approach the Cheeger–Müller theorem for singular spaces with isolated conical singularities following a different strategy, namely by generalising the approach of Bismut and Zhang to the singular situation. The approach in the present note relies on the Witten deformation for singular spaces developed in [9–11].

2. The Main Theorem

The setting of this note is the following: (X, g^{TX}) is a space with isolated conical singularities of dimension $n \geq 2$, *i.e.*: X is a compact connected topological space containing a finite set of points $\mathrm{Sing}(X) \subset X$, such that $X_{\mathrm{sm}} := X \setminus \mathrm{Sing}(X)$ is a smooth manifold of dimension n. In the even-dimensional case, we assume in addition that X_{sm} is oriented; this assumption is needed in Theorem 4.5. The set $\mathrm{Sing}(X)$ is called the singular set of X. We denote by TX the tangent bundle of X_{sm}; g^{TX} is a Riemannian metric on X_{sm}. A neighbourhood of a singular point $p \in \mathrm{Sing}(X)$ in X can be identified with the cone over a smooth compact connected manifold L_p of dimension $n-1$; L_p is called the link of X at p. Moreover, near a singularity $p \in \mathrm{Sing}(X)$, the metric g^{TX} has the form $dr^2 + r^2 g^{TL_p}$ with r the radial coordinate and g^{TL_p} a fixed Riemannian metric on the link manifold L_p.

Let (F, ∇^F, g^F) be a unitary flat vector bundle over X_{sm} with canonical flat connection ∇^F and flat Hermitian metric g^F, *i.e.* the flat vector bundle is associated with a unitary representation of $\pi_1(X_{\mathrm{sm}})$. For $p \in \mathrm{Sing}(X)$, the restriction of (F, ∇^F, g^F) to the link L_p is a unitary flat vector bundle denoted by $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$. Moreover, in a punctured neighbourhood of $p \in \mathrm{Sing}(X)$, the unitary vector bundle (F, ∇^F, g^F) can be identified with the pull-back bundle of the unitary vector bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$. We denote by F^* the dual bundle of F and by $F^*_{L_p}$, $p \in \mathrm{Sing}(X)$, its restriction to the link L_p.

For the whole note, we make the following assumptions: we assume that X is a Witt space, *i.e.* either n is even, or, in case n odd, $H^{\frac{n-1}{2}}(L_p, F_{L_p}) = 0$ for $p \in \mathrm{Sing}(X)$. For Witt spaces, the intersection homology of X with upper and lower middle perversity (see [6]) do coincide. We moreover assume that the Laplacian acting on smooth compactly supported forms on X_{sm} with values in F is essentially self-adjoint. We denote its unique self-adjoint extension by Δ. For Witt spaces, this last condition can always be achieved, by a rescaling of the metrics g^{TL_p}, $p \in \mathrm{Sing}(X)$, to cg^{TL_p}, with $c > 0$ small enough.

We denote by $H^\bullet_{(2)}(X, F)$ the L^2-cohomology of X. The L^2-metric $\langle\ ,\ \rangle$ on the space of sections of $\Lambda(T^*X) \otimes F$, induced from g^{TX}, g^F, restricts to a metric on the space of L^2-harmonic forms $\mathcal{H}^\bullet_{(2)}(X, F) := \ker \Delta$. Using the L^2-Hodge isomorphism $H^\bullet_{(2)}(X, F) \simeq \mathcal{H}^\bullet(X, F)$ (see [3, Section 1 and Theorem 5.1]), we get an induced metric $|\ |^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F)}$ on the line $\det H^\bullet_{(2)}(X, F)$. We denote by $\Delta^\perp$ the restriction of Δ to $\left(\ker \Delta\right)^\perp$ and by N the number operator acting on sections of $\Lambda(T^*X) \otimes F$ by multiplication with the form degree. We denote by Tr_s the supertrace of an operator. For $s \in \mathbb{C}$, $\Re(s) > \frac{n}{2}$, set $\theta(s) := -\mathrm{Tr}_s\left[N(\Delta^\perp)^{-s}\right]$. By a result of A. Dar [5, Section 4], the function θ extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$. The Ray–Singer metric $\|\ \|^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F)}$ on the line $\det H^\bullet_{(2)}(X, F)$ is defined as

$$\|\ \|^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F)} := |\ |^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F)} \exp\left(\frac{1}{2}\theta'(0)\right). \tag{2.1}$$

Let Y be a standard anti-radial gradient-like Morse–Smale vector field on X_{sm}, *i.e.* outside a neighbourhood of $\mathrm{Sing}(X)$ it is a smooth gradient-like Morse–Smale vector field (see [15, p. 199]), with its standard form near its singular points (see [10, Definition 2.7]); in a neighbourhood of $\mathrm{Sing}(X)$, we have $Y = -r\partial_r$, where again r denotes the radial coordinate. For $p \in \mathrm{Sing}(X)$, we denote by $o(TL_p)$ the orientation bundle of the link manifold L_p. For $p \in \mathrm{Sing}(X)$ and $k \geq \frac{n}{2} + 1$, let Ξ^{n-k}_p be a set of closed forms on L_p with values in $F^*_{L_p} \otimes o(TL_p)$, whose cohomology classes form a basis of $H^{n-k}(L_p, F^*_{L_p} \otimes o(TL_p))$, hence $\mathrm{span}\, \Xi^{n-k}_p \simeq H^{n-k}(L_p, F^*_{L_p} \otimes o(TL_p))$. Let $\mathrm{Crit}(Y) \subset X_{\mathrm{sm}}$ denote the set of singular points of the vector field Y. Using the flow induced by the vector field $-Y$, one can construct a geometric complex $(C_\bullet(W^u, F^*), \partial_\bullet)$, with $C_\bullet(W^u, F^*) := \left(\bigoplus_{p \in \mathrm{Crit}(Y)} \langle [W^u(p)] \rangle \otimes F^*_p\right) \bigoplus \left(\bigoplus_{p \in \mathrm{Sing}(X), k \geq n/2+1} \mathrm{span}\, \Xi^{n-k}_p\right)$ (see [9, Section 6] and [10]). Let $(C^{\mathrm{sm}}_\bullet, \partial_\bullet)$ denote the subcomplex of $(C_\bullet(W^u, F^*), \partial_\bullet)$ generated by $\mathrm{Crit}(Y)$. There is a short exact sequence of complexes

$$0 \to (C^{\mathrm{sm}}_\bullet, \partial_\bullet) \to (C_\bullet(W^u, F^*), \partial_\bullet) \to ((C(W^u, F^*)/C^{\mathrm{sm}})_\bullet, \partial_\bullet) \to 0. \tag{2.2}$$

For $p \in \mathrm{Sing}(X)$ we denote by cL_p the infinite cone over L_p and by Z_p the infinite cone with the cone tip removed. The unitary flat vector bundle $(F_{L_p}, \nabla^{F_{L_p}}, g^{F_{L_p}})$ can be extended in a trivial way to a unitary flat vector bundle over Z_p, which we still denote by F. Similarly we still denote by F^* the extension of the flat dual bundle $F^*_{L_p}$ to Z_p. We denote by $IH_\bullet(cL_p, L_p, F^*)$ the relative intersection homology of the cone cL_p with middle perversity and coefficients in F^*. Using the local calculation for intersection homology [6, Section 2.4] and Poincaré duality on the link manifold L_p, we have $H_\bullet((C(W^u, F^*)/C^{\mathrm{sm}})_\bullet, \partial_\bullet) \simeq \oplus_{p \in \mathrm{Sing}(X)} IH_\bullet(cL_p, L_p, F^*)$. Moreover, a straightforward generalisation of [10, Theorem 6.2] shows that the complex $(C_\bullet(W^u, F^*), \partial_\bullet)$ computes the intersection homology of X with middle perversity and coefficients in F^*, $IH_\bullet(X, F^*)$. Hence, from the exact sequence of complexes (2.2), we get natural isomorphisms

$$\begin{aligned} \det H_\bullet(C^{\mathrm{sm}}_\bullet, \partial_\bullet) \otimes \det(\oplus_{p \in \mathrm{Sing}(X)} IH_\bullet(cL_p, L_p, F)) &\simeq \det H_\bullet(C_\bullet(W^u, F^*), \partial_\bullet) \\ &\simeq \det IH_\bullet(X, F^*). \end{aligned} \tag{2.3}$$

We now explain the construction of a metric on $\det H^\bullet(\mathrm{Hom}((C_\bullet(W^u, F^*), \partial_\bullet), \mathbb{C}))$: for $p \in \mathrm{Sing}(X)$, we equip Z_p with the conic metric. In view of our assumptions, the Laplacian acting on smooth compactly supported sections of $\Lambda(T^*Z_p) \otimes F$ admits a unique closed self-adjoint extension $\mathbf{\Delta}^p$. The model Witten Laplacian at $p \in \mathrm{Sing}(X)$ is the operator defined as $\mathbf{\Delta}^p_T := \mathbf{\Delta}^p - T(2N-n) + T^2 r^2$, $T > 0$. By [11] the model Witten Laplacian is a discrete operator. By [9, Theorem 4.2], we have $\ker \mathbf{\Delta}^p_T \simeq IH_\bullet(cL_p, L_p, F^*)^*$. The L^2-metric on sections of $\Lambda(T^*Z_p) \otimes F$ restricts to a metric on $\ker \mathbf{\Delta}^p_T$. We denote the induced metric on the line $\det IH_\bullet(cL_p, L_p, F^*)^*$ by $|\ |^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*,T}$. We denote by $\mathbf{\Delta}^{p,\perp}_T$ the restriction of $\mathbf{\Delta}^p_T$ to $(\ker \mathbf{\Delta}^p_T)^\perp$. By [11, Theorem I], for $\Re(s) > \frac{n}{2}$, the zeta function $s \mapsto \zeta^p_T(s) := -\mathrm{Tr_s}[N(\mathbf{\Delta}^{p,\perp}_T)^{-s}]$ is a well-defined holomorphic function. Moreover, ζ^p_T extends to a meromorphic function on the whole complex plane, which is holomorphic at $s = 0$. For $p \in \mathrm{Sing}(X)$, set $\zeta_p := \zeta^p_1$ and $|\ |^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*} := |\ |^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*,1}$. The Ray–Singer metric on the line $\det IH_\bullet(cL_p, L_p, F^*)^*$ is defined as

$$\|\ \|^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*} := |\ |^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*} \exp\left(\frac{1}{2}\zeta_p'(0)\right). \tag{2.4}$$

The metrics g^{F_p} on the fibre F_p, $p \in \mathrm{Crit}(Y)$, induce a metric on $\det H^\bullet(\mathrm{Hom}((C^{\mathrm{sm}}_\bullet, \partial_\bullet), \mathbb{C}))$ (see [1, Section 1 (d)]). Via the natural isomorphism (2.3) the metric on $\det H^\bullet(\mathrm{Hom}((C^{\mathrm{sm}}_\bullet, \partial_\bullet), \mathbb{C}))$ and the Ray–Singer metrics $\|\ \|^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*}$, $p \in \mathrm{Sing}(X)$, induce a metric on $\det IH_\bullet(X, F^*)^*$. We denote by $\|\ \|^{Y,g}_{\det H^\bullet_{(2)}(X,F)}$ the metric on the line $\det H^\bullet_{(2)}(X, F)$ induced via the natural de Rham isomorphism $IH_\bullet(X, F^*)^* \simeq H^\bullet_{(2)}(X, F)$. The superscript g indicates that, by construction, the metric $\|\ \|^{Y,g}_{\det H^\bullet_{(2)}(X,F)}$ does depend on the metrics $(g^{TL_p}, g^{F_{L_p}})$, $p \in \mathrm{Sing}(X)$, and on $g^F_{|\mathrm{Crit}(Y)}$.

Note that in case that L_p is the standard sphere, *i.e.* for a smooth point, the metric $\|\ \|^{\mathrm{RS}}_{\det IH_\bullet(cL_p,L_p,F^*)^*}$ is trivial. Hence the definition of the metric $\|\ \|^{Y,g}_{\det H^\bullet_{(2)}(X,F)}$ is consistent with the definition of the smooth Milnor metric.

Main Theorem. *The following identity holds:* $\|\ \|^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F)} = \|\ \|^{Y,g}_{\det H^\bullet_{(2)}(X,F)}$.

In case of a smooth compact manifold, the statement of the Main Theorem is equivalent to the smooth Cheeger–Müller theorem. In case of an even-dimensional oriented space with isolated conical singularities, $\|\ \|^{Y,g}_{\det H^\bullet_{(2)}(X,F)}$ is equal to the intersection Reidemeister metric, and we recover the equality between Ray–Singer metric and intersection Reidemeister metric proved by A. Dar (see [5, Theorem 2.17 and Theorem 4.5], both metrics are trivial in this case).

3. Strategy of proof

The proof of the Main Theorem follows closely the proof of the extension of the Cheeger–Müller theorem given by Bismut and Zhang [1]. The proof can be reduced to the case, where the standard anti-radial gradient-like Morse–Smale vector field Y is the gradient vector field of an anti-radial Morse function $f : X \to \mathbb{R}$ (as defined in [9]).

From now on we assume that $f : X \to \mathbb{R}$ is an anti-radial Morse function and $Y = \nabla_{g^{TX}} f =: \nabla f$ is a standard anti-radial Morse–Smale vector field. This means, in particular, that the metric g^{TX} is flat in a neighbourhood of the smooth critical points of f. We denote by δ the adjoint of the outer differential d with respect to $\langle\ ,\ \rangle$. Let $D := \mathrm{d} + \delta$. For $T \geq 0$, let $\langle\ ,\ \rangle_T$ be the twisted L^2-metric $\langle \alpha, \beta\rangle_T := \int_X \langle \alpha, \beta\rangle_{\Lambda(T^*X)\otimes F} \mathrm{e}^{-2Tf(x)} \mathrm{dvol}_X(x)$. Let δ'_T be the adjoint of d with respect to $\langle\ ,\ \rangle_T$. Let $D_T := \mathrm{d} + \delta'_T$ and $\Delta_T := D_T^2$. Since for $l \in \mathbb{N}$ and $T \geq 0$, $\mathrm{dom}(D^l) = \mathrm{dom}(D^l_T)$, one can proceed as in [1, Theorem 5.6] to prove that the form $\omega := \frac{\mathrm{d}t}{2t}\mathrm{Tr_s}[N\exp(-tD_T^2)] - \mathrm{d}T\,\mathrm{Tr_s}[f\exp(-tD_T^2)]$ is a closed form on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \ni (T, t)$.

Let $\epsilon, A, T_0 \in \mathbb{R}$ with $0 < \epsilon < A < \infty$, $0 < T_0 < \infty$. Let Γ be the boundary of the rectangle $\{(T, t) \mid 0 \leq T \leq T_0,\ \epsilon \leq t \leq A\} \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ oriented anti-clockwise. Denote by

$$\Gamma_1 := \{(T,t) \mid T = T_0,\ \epsilon \leq t \leq A\},\ \Gamma_2 := \{(T,t) \mid 0 \leq T \leq T_0,\ t = A\},$$
$$\Gamma_3 := \{(T,t) \mid T = 0,\ \epsilon \leq t \leq A\},\ \Gamma_4 := \{(T,t) \mid 0 \leq T \leq T_0,\ t = \epsilon\},$$

the oriented faces of the rectangle. For $k = 1, \dots, 4$, set $I_k := \int_{\Gamma_k} \omega$. Since the form ω is closed,

$$\sum_{k=1}^{4} I_k = 0. \tag{3.1}$$

As in [1, Section 7], we study each I_k ($k = 1, \dots, 4$) individually, by taking in succession the limits $A \to \infty$, $T_0 \to \infty$ and $\epsilon \to 0$. The Main Theorem follows from (3.1), and the nine intermediary results of the next section.

4. The nine intermediary results

Denote by $\mathrm{Crit}(f_{|X_{\mathrm{sm}}})$ the set of critical points of the smooth Morse function $f_{|X_{\mathrm{sm}}}$ and by $c_k(f_{|X_{\mathrm{sm}}})$ the number of critical points of $f_{|X_{\mathrm{sm}}}$ of index k. For $k=0,\dots,n-1$, we denote by $b^k(L_p,F_{L_p}) := \dim H^k(L_p,F_{L_p})$. Set

$$c_k(f) := c_k(f,F) := \begin{cases} \mathrm{rk}(F)\cdot c_k(f_{|X_{\mathrm{sm}}}) + \sum\limits_{p\in\mathrm{Sing}(X)} b^{k-1}(L_p,F_{L_p}) & \text{for } k \geq \frac{n}{2}+1, \\ \mathrm{rk}(F)\cdot c_k(f_{|X_{\mathrm{sm}}}) & \text{else.} \end{cases}$$

For $k=0,\dots,n$, we denote by $b^k_{(2)}(X,F) := \dim H^k_{(2)}(X,F)$. We denote the L^2-Euler characteristic of X with coefficients in F by $\chi := \chi_{(2)}(X,F) := \sum_{k=0}^n (-1)^k b^k_{(2)}(X,F)$. By the spectral gap theorem for the Witten Laplacian ([9, Theorem I]) $\chi = \sum_{k=0}^n (-1)^k c_k(f)$. Set

$$\widetilde{\chi}' := \widetilde{\chi}'(X,F) := \sum_{k=0}^n (-1)^k k c_k(f),$$

$$\mathrm{Tr}_s[f,F] := \mathrm{rk}(F) \sum_{p\in\mathrm{Crit}(f_{|X_{\mathrm{sm}}})} (-1)^{\mathrm{ind}(p)} f(p) + \sum_{p\in\mathrm{Sing}(X)} f(p)\cdot \sum_{k\geq\frac{n}{2}+1}^{n} (-1)^k b^{k-1}(L_p,F_{L_p}),$$

$$\chi_{\mathrm{sm}} := \mathrm{rk}(F)\sum_{k=0}^n (-1)^k c_k(f_{|X_{\mathrm{sm}}}), \quad \widetilde{\chi}'_{\mathrm{sm}} := \mathrm{rk}(F)\sum_{k=0}^n (-1)^k k c_k(f_{|X_{\mathrm{sm}}}).$$

For $T\geq 0$, we denote by $(\mathcal{S}^\bullet_T, \mathrm{d}, \langle\ ,\ \rangle_T)$ the complex generated by the eigenforms of the Laplacian $\Delta_T = D_T^2$ to eigenvalues in $[0,1]$. We denote by $P_T^{[0,1]}$ the orthogonal projection to $\mathcal{S}^\bullet_T$ w.r.t. $\langle\ ,\ \rangle_T$. Set $P_T^{]1,\infty[} = 1 - P_T^{[0,1]}$. We denote by $D_T^{2,\,]0,1]}$ (resp. $D_T^{2,[0,1]}$) the restriction of Δ_T to the eigenspace of Δ_T associated with eigenvalues in the interval $]0,1]$ (resp. in the interval $[0,1]$). The twisted L^2-metric $\langle\ ,\ \rangle_T$ restricted to $\ker \Delta_T^{(k)} \simeq H^k_{(2)}(X,F)$ induces a metric on the line $\det H^\bullet_{(2)}(X,F)$, which we denote by $|\ \ |^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F),T}$.

The following nine intermediary results are the analogues of [1, Theorem 7.6–7.14].

Theorem 4.1. *The following identity holds, for $T\to\infty$,*

$$\mathrm{Tr}_s\left[N\log\left(D_T^{2,\,]0,1]}\right)\right] - \log\left(\frac{\|\ \ \|^{\nabla f,g}_{\det H^\bullet_{(2)}(X,F)}}{|\ \ |^{\mathrm{RS}}_{\det H^\bullet_{(2)}(X,F),T}}\right)^2 + 2T\mathrm{Tr}_s[f,F] + \left(\frac{n}{2}\chi - \widetilde{\chi}'\right)\log(T)$$
$$-\left(\frac{n}{2}\chi_{\mathrm{sm}} - \widetilde{\chi}'_{\mathrm{sm}}\right)\log(\pi) + \sum_{p\in\mathrm{Sing}(X)} \zeta'_p(0) = \mathcal{O}(\exp(-cT)).$$

Theorem 4.2. *Given ϵ, A with $0<\epsilon<A<\infty$, there exists $C>0$ such that if $t\in[\epsilon,A]$, $T\geq 1$, then $\left|\mathrm{Tr}_s[N\exp(-tD_T^2)] - \widetilde{\chi}'\right| \leq \frac{C}{\sqrt{T}}$.*

Theorem 4.3. *For any $t>0$, $\lim_{T\to\infty}\mathrm{Tr}_s\left[N\exp(-tD_T^2)P_T^{]1,\infty[}\right]=0$. Moreover there exist $c>0$, $C>0$ such that for $t\geq 1$, $T\geq 0$: $\left|\mathrm{Tr}_s\left[N\exp(-tD_T^2)P_T^{]1,\infty[}\right]\right| \leq C\exp(-ct)$.*

Theorem 4.4. *For $T>0$ large enough and $k=0,\dots,n$: $\dim \mathcal{S}^k_T = c_k(f)$. Moreover, $\lim\limits_{T\to\infty}\mathrm{Tr}\left[D_T^{2,[0,1]}\right]=0$.*

Let $e_1,\dots,e_n$ be an orthonormal basis of TX, $e^1,\dots,e^n$ the corresponding dual basis of the cotangent bundle T^*X. Let ∇^{TX} denote the Levi-Civita connection of (X,g^{TX}). We denote by $\dot{R}^{TX}$ it curvature seen as a section of $\Lambda^2(T^*X)\widehat{\otimes}\Lambda^2(T^*X)$. We decorate elements in the second factor of $\Lambda^2(T^*X)\widehat{\otimes}\Lambda^2(T^*X)$ with a $\widehat{\ }$. Let W be the smooth section of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$ defined by $W := \frac{1}{2}\sum_{i=1}^n e^i\wedge\widehat{e^i}$. We denote by $o(TX)$ the orientation bundle of X. The Berezin integral formalism yields a map $\int^B$ from sections of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$ to section of $\Lambda(T^*X)\otimes o(TX)$ (see [1, Section 3]).

Theorem 4.5. *As $t \searrow 0$, the following asymptotic expansion holds, in case n is odd,*

$$\mathrm{Tr}_s\big[N\exp(-tD^2)\big] = \frac{\mathrm{rk}(F)}{\sqrt{t}} \int_X \int^B W \exp\left(-\frac{\dot{R}^{TX}}{2}\right) + \frac{n}{2}\cdot \chi_{(2)}(X,F) + \mathcal{O}(\sqrt{t}).$$

In case n is even, X oriented, for $t>0$: $\mathrm{Tr}_s\big[N\exp(-tD^2)\big] = \frac{n}{2}\cdot\chi_{(2)}(X,F)$.

Theorem 4.6. *Let $0<t<1$ be small enough. Then there exists $c>0$ such that, as $T\to\infty$,*

$$\mathrm{Tr}_s\big[f\exp(-tD_T^2)\big] = \mathrm{Tr}_s[f,F] + \left(\frac{n}{4}\chi - \frac{1}{2}\widetilde{\chi}'\right)\frac{1}{T} + \mathcal{O}(\exp(-cT)).$$

For $t>0$ and $p\in\mathrm{Sing}(X)$, we denote by $Q_t^p(x,x')$, $x,x'\in Z_p$, the kernel of the operator $\exp\big(-t\mathbf{\Delta}^p\big)$ with respect to dvol_{Z_p}. The integral $\gamma_p(F) := \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t}\int_L \mathrm{Tr}_s\big[Q_t^p((1,y),(1,y))\big]\mathrm{dvol}_{L_p}$ is well defined (see [4]). For $T\geq 0$, let B_T be the smooth section of $\Lambda(T^*X)\widehat{\otimes}\Lambda(T^*X)$ defined by $B_T := \frac{\dot{R}^{TX}}{2} + \sqrt{T}\sum_{i=1}^n e^i\wedge \widehat{\nabla_{e_i}^{TX}\nabla f} + T|\mathrm{d}f|^2$. We denote by $\widehat{c}(f)$ the Clifford multiplication operator, $\widehat{c}(f) = \mathrm{d}f\wedge\ + \iota_{\nabla f}$.

Theorem 4.7. *For any $d>0$, there exists $C>0$ such that for $0<t\leq 1$, $0\leq T\leq \frac{d}{t}$,*

$$\left|\mathrm{Tr}_s\left[f\exp\left(-\big(tD+T\widehat{c}(\nabla f)\big)^2\right)\right] - \mathrm{rk}(F)\int_X f\int^B \exp(-B_{T^2}) - \sum_{p\in\mathrm{Sing}(X)} f(p)\gamma_p(F)\right| \leq Ct^2.$$

Theorem 4.8. *For any $T>0$, the following identity holds,*

$$\lim_{t\to 0}\frac{1}{t^2}\left\{\mathrm{Tr}_s\left[f\exp\left(-\left(tD+\frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{Tr}_s[f,F]\right\} = \left(\frac{n}{4}\chi_{\mathrm{sm}} - \frac{1}{2}\widetilde{\chi}'_{\mathrm{sm}}\right)\frac{1}{T\tanh(T)}$$
$$-\frac{1}{2T}\sum_{p\in\mathrm{Sing}(X)}\left(\mathrm{Tr}_s[N\exp(-\mathbf{\Delta}_T^{p,\perp})] - \sum_{k\geq\frac{n}{2}+1}(-1)^k b^{k-1}(L_p,F_{L_p})\left(\frac{n}{2}-k\right)\right).$$

Theorem 4.9. *There exist constants $t_0, c, C>0$, such that for $t\in]0,t_0]$ and $T\geq 1$,*

$$\left|\frac{1}{t^2}\left\{\mathrm{Tr}_s\left[f\exp\left(-\left(tD+\frac{T}{t}\widehat{c}(\nabla f)\right)^2\right)\right] - \mathrm{Tr}_s[f,F] - \frac{t^2}{T}\left(\frac{n}{4}\chi - \frac{1}{2}\widetilde{\chi}'\right)\right\}\right| \leq C\exp(-cT).$$

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References

[1] J.-M. Bismut, W. Zhang, An Extension of a Theorem by Cheeger and Müller, With an appendix by François Laudenbach, Astérisque, vol. 205, Société mathématique de France, Paris, 1992.
[2] J. Cheeger, Analytic torsion and the heat equation, Ann. of Math. (2) 109 (1979) 259–322.
[3] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds, in: Geometry of the Laplace Operator, in: Proc. Sympos. Pure Math., vol. XXXVI, American Mathematical Society, Providence, RI, USA, 1980, pp. 91–146.
[4] J. Cheeger, Spectral geometry of singular Riemannian spaces, J. Differ. Geom. 18 (4) (1983) 575–657.
[5] A. Dar, Intersection R-torsion and analytic torsion for pseudomanifolds, Math. Z. 194 (1987) 193–216.
[6] M. Goresky, R. MacPherson, Intersection homology II, Invent. Math. 72 (1983) 77–129.
[7] L. Hartmann, M. Spreafico, On the Cheeger–Müller theorem for an even-dimensional cone, St. Petersburg Math. J. 27 (1) (2016) 137–154.
[8] M. Lesch, A gluing formula for the analytic torsion on singular spaces, Anal. PDE 6 (1) (2013) 221–256.
[9] U. Ludwig, Comparison between two complexes on a singular space, J. Reine Angew. Math. 724 (2017) 1–52.
[10] U. Ludwig, A complex in Morse theory computing intersection homology, Ann. Inst. Fourier 67 (1) (2017) 197–236.
[11] U. Ludwig, An index theorem for the intersection Euler characteristic of the infinite cone, C. R. Acad. Sci. Paris, Ser. I 355 (1) (2017) 94–98.

[12] W. Müller, Analytic torsion and R-torsion of Riemannian manifolds, Adv. Math. 28 (1978) 233–305.
[13] W. Müller, B. Vertman, The metric anomaly of analytic torsion on manifolds with conical singularities, Commun. Partial Differ. Equ. 39 (1) (2014) 146–191.
[14] D.B. Ray, I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. 7 (1971) 145–210.
[15] S. Smale, On gradient dynamical systems, Ann. of Math. (2) 74 (1961) 199–206.
[16] B. Vertman, Analytic torsion of a bounded generalized cone, Commun. Math. Phys. 290 (3) (2009) 813–860.
[17] S.M. Vishik, Generalized Ray–Singer conjecture. I: a manifold with a smooth boundary, Commun. Math. Phys. 167 (1) (1995) 1–102.