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Differential geometry

Compact embedded minimal surfaces in the Berger sphere ☆

Surfaces minimales compactes intégrées dans la sphère de Berger

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A R T I C L E I N F O

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A B S T R A C T

Choe and Soret [1] constructed infinitely many compact embedded minimal surfaces in $\mathbb{S}^3$ by desingularizing Clifford tori which meet each other along a great circle at the angle of the same size. We show their method works with some modifications to construct compact embedded minimal surfaces in the Berger sphere as well.

R É S U M É

Choe et Soret [1] ont construit une infinité de surfaces minimales compactes plongées dans $\mathbb{S}^3$ en désingularisant deux tores de Clifford qui se rencontrent le long d'un grand cercle à un angle constant de la même taille. Nous montrons que leur méthode fonctionne également, avec quelques modifications, pour construire des surfaces minimales compactes plongées dans la sphère de Berger.

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1. Introduction

Lawson constructed in [2] infinitely many compact minimal surfaces in $\mathbb{S}^3$ of arbitrary genus. He first constructed a suitable geodesic polygon and then reflected the Plateau solution over this geodesic polygon along the edges to obtain compact minimal surfaces. He obtained the geodesic polygon by arranging geodesic spheres suitably. Recently this configuration was applied to the Berger sphere to give infinitely many compact minimal surfaces of arbitrary genus [5].

On the other hand, Choe and Soret constructed in [1] infinitely many compact embedded minimal surfaces in $\mathbb{S}^3$ by desingularizing Clifford tori that meet each other along a great circle C_1 at the angle of the same size. It happens that the

☆ In the memory of Professor Ok Kyung Yoon.

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Clifford tori used in the construction meet each other along the dual great circle C_2 at the angle of the same size as well. Once the Clifford tori are desingularized along C_1, there are two ways of desingularizing tori along C_2. Hence, two kinds of construction were used to give "odd" surfaces and "even" surfaces.

Their method of desingularization for odd surfaces follows the procedure:

(i) using m Clifford tori that meet each other along a great circle at the angle of π/m which makes a tesselation of $\mathbb{S}^3$ by $16m^2l$ $(m \geq 2, l \geq 1)$ pentahedra;
(ii) making the Jordan curve Γ of 6 geodesic segments, which is a subset of the 1-skeleton of the pentahedron;
(iii) finding a minimal disk D spanning Γ and extend D across Γ by the geodesic reflections.

The resulting surface is a compact embedded minimal surface of genus $1 + 4m(m-1)l$.

In this paper, we apply their method to construct compact embedded minimal surfaces in the Berger sphere. Among the main issues in applying the method to the Berger sphere are:

(i) is there a surface in the Berger sphere to substitute the Clifford torus in $\mathbb{S}^3$?
(ii) reflections along geodesics are not isometries in general in the Berger sphere.

We can apply the method successfully by using ruled minimal surfaces in the Berger sphere characterized in [4], which substitutes Clifford tori, and by using the reflections along the *horizontal* geodesics, which are isometries.

We introduce a kind of cylindrical parameterization on $\mathbb{S}^3$ in order to use the result of [4], which turns out to be more convenient to visualize the relations between the many minimal surfaces and geodesics used. This parameterization can also be effectively used in carrying out the construction in [1].

We only show the construction of odd surfaces in this paper. One can verify that the construction applies for even surfaces as well.

2. A parameterization of $\mathbb{S}^3$

We recall some facts on the Berger spheres.

2.1. $\mathbb{S}^3$ as a special unitary group

Let us identify the unit sphere $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ and the special unitary group SU(2) by the map

$$(z, w) \mapsto \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

The Lie algebra $\mathfrak{su}(2)$ is spanned by

$$X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

which generate the left-invariant vector fields:

$$X_1(z, w) = \begin{bmatrix} -w & z \\ -\bar{z} & -\bar{w} \end{bmatrix}, \quad X_2(z, w) = \begin{bmatrix} iw & iz \\ i\bar{z} & -i\bar{w} \end{bmatrix}, \quad X_3(z, w) = \begin{bmatrix} iz & -iw \\ -i\bar{w} & -i\bar{z} \end{bmatrix}.$$

Viewed as tangent vector fields on $\mathbb{S}^3$,

$$X_1(z, w) = (-w, z), \quad X_2(z, w) = (\mathrm{i}w, \mathrm{i}z), \quad X_3(z, w) = (\mathrm{i}z, -\mathrm{i}w).$$

2.2. A circle action

The orbits of the right circle's action on $\mathbb{S}^3$

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{bmatrix} \quad \text{or} \quad (z, w) \mapsto (\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{-\mathrm{i}\theta} w) \tag{1}$$

are the fibers of the Hopf fibration $H : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ given by

$$H(z, w) = \left(zw, \frac{1}{2}(|z|^2 - |w|^2) \right).$$

The fiber over the general point (z, w) is the circle $\{(\mathrm{e}^{\mathrm{i}\theta} z, \mathrm{e}^{-\mathrm{i}\theta} w)\}$ to which the vector field $X_3(z, w) = (iz, -iw)$ is tangent.

2.3. A parameterization

Let $P = \{(s, t, \theta) : 0 \le s \le 2\pi, 0 \le t \le \pi/2, 0 < \theta \le 2\pi\}$ and consider the map $\Phi : P \to \mathbb{S}^3$

$$\Phi(s, t, \theta) = (e^{is}\cos t, e^{i(s+\theta)}\sin t),$$

which is one-to-one on P', the interior of P, and yields coordinates on $\Phi(P')$. Since

$$\Phi(s, 0, \theta) = (e^{is}, 0), \quad \Phi(s, \pi/2, \theta) = (0, e^{i(s+\theta)}),$$

we have the following proposition.

Proposition 1. *For any θ, the image of the curve $s \mapsto \Phi(s, 0, \theta)$ is the Hopf fiber over $(1, 0)$ and the image of the curve $s \mapsto \Phi(s, \pi/2, \theta)$ is the Hopf fiber over $(0, 1)$.*

Note also that, when $t = \pi/2$, since $\Phi(s, \pi/2, \theta) = \Phi(s', \pi/2, \theta')$ if and only if $s + \theta = s' + \theta' \mod 2\pi$, those points $s + \theta = s' + \theta' \mod 2\pi$ correspond to a single point on the fiber $\{(0, e^{is})\}$ over $(0, 1)$.

Let

$$T_1 = \{(s, t, \theta) : 0 \le s \le 2\pi, 0 \le t \le \pi/4, 0 < \theta \le 2\pi\},$$
$$T_2 = \{(s, t, \theta) : 0 \le s \le 2\pi, \pi/4 \le t \le \pi/2, 0 < \theta \le 2\pi\}.$$

Then one has trivially $T_1 \cup T_2 = P$, $T_1 \cap T_2 = \{t = \pi/4\}$. Now, since $\Phi(0, t, \theta) = \Phi(2\pi, 0, \theta)$, one can see that the set $\Phi(T_1)$ is a solid torus along the circle $\Phi(s, 0, \theta)$ over $(z, w) = (1, 0)$ and the set $\Phi(T_2)$ is a solid torus along the circle $\Phi(s, \pi/2, \theta)$ over $(z, w) = (0, 1)$ and that the set $\Phi(T_1 \cap T_2)$ is a torus. Moreover, since the boundaries are the same,

$$\partial\Phi(T_1) = \partial\Phi(T_2) = \Phi(T_1 \cap T_2).$$

Thus the map Φ yields the usual topological picture of $\mathbb{S}^3$ as the union of two solid tori with their boundaries identified.

3. The Berger sphere

Now consider the left-invariant Riemannian metrics g_δ on $\mathbb{S}^3 = \mathrm{SU}(2)$ given in terms of the left-invariant vector fields X_1, X_2, X_3 by

$$X_i \cdot X_j = 0,\ i \neq j,$$
$$X_1 \cdot X_1 = X_2 \cdot X_2 = \delta^2, \quad X_3 \cdot X_3 = 1.$$

The Berger sphere is the Riemannian manifold $(\mathbb{S}^3, g_\delta)$. When $\delta = 1$, the Berger sphere is the standard round sphere $\mathbb{S}^3$.

Since it is a left-invariant metric, one can see that the left multiplication of SU(2)

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad a\bar{a} + b\bar{b} = 1,$$

that is,

$$(z, w) \mapsto (az - b\bar{w}, b\bar{z} + aw)$$

is an isometry. Note that the circle action (1) is also an isometry. Then the maps

$$(z, w) \mapsto (e^{i\theta}z, w), \quad (z, w) \mapsto (z, e^{i\theta}w) \tag{2}$$

are isometries as well.

3.1. Geodesics

A geodesic in the Berger sphere is called *horizontal* if it is orthogonal to the Hopf fibers everywhere and is called *vertical* if it is tangent to the Hopf fibers everywhere. One can see that the curve

$$t \mapsto (e^{it}z, e^{-it}w)$$

is a vertical geodesic passing through the point (z, w) (whose whole image is the Hopf fiber through (z, w)) and that the curve

$$t \mapsto (z\cos t - e^{-i\theta}w\sin t, e^{i\theta}z\sin t + w\cos t)$$

which is the integral curve of the vector field $\cos\theta X_1 + \sin\theta X_2$ passing through the point (z, w) is a horizontal geodesic.

3.2. Ruled minimal surfaces

It was shown in [4] that every ruled minimal surface in the Berger sphere is congruent to an open subset of the image of

$$X(s,t) = (\mathrm{e}^{\mathrm{i}\alpha s}\cos t, \mathrm{e}^{\mathrm{i}s}\sin t), \quad \alpha \in \mathbb{R}.$$

Since one can show by computation that these surfaces are in fact minimal (cf. [5]), we have the following proposition.

Proposition 2. *A surface in the Berger sphere is a ruled minimal surface if and only it is congruent to the parametric surface*

$$X(s,t) = (\mathrm{e}^{\mathrm{i}\alpha s}\cos t, \mathrm{e}^{\mathrm{i}s}\sin t), \quad \alpha \in \mathbb{R}.$$

4. The Berger metric in terms of the parameterization Φ

Now let

$$\partial_s := \frac{\partial \Phi}{\partial s}(s,t,\theta), \quad \partial_t := \frac{\partial \Phi}{\partial t}(s,t,\theta), \quad \partial_\theta := \frac{\partial \Phi}{\partial \theta}(s,t,\theta).$$

One has

$$\begin{aligned}
\partial_s &= 2\cos t\sin t(-\sin\theta X_1 + \cos\theta X_2) + (\cos^2 t - \sin^2 t)X_3 \\
&= (\mathrm{i}\,\mathrm{e}^{\mathrm{i}s}\cos t, \mathrm{i}\,\mathrm{e}^{\mathrm{i}(s+\theta)}\sin t), \\
\partial_t &= \cos\theta X_1 + \sin\theta X_2 \\
&= (-\mathrm{e}^{\mathrm{i}s}\sin t, \mathrm{e}^{\mathrm{i}(s+\theta)}\cos t), \\
\partial_\theta &= \cos t\sin t(-\sin\theta X_1 + \cos\theta X_2) - \sin^2 t X_3 \\
&= (0, \mathrm{i}\,\mathrm{e}^{\mathrm{i}(s+\theta)}\sin t).
\end{aligned}$$

Note also that X_3 is written as

$$X_3 = \partial_s - 2\partial_\theta$$

at every point.

Proposition 3. *The following curves are horizontal geodesics.*

(i) *the* t*-curve* $H_{(s_0,\theta_0)}(t) := \Phi(s_0, t, \theta_0)$, $\quad 0 < s_0 < 2\pi, \quad 0 < \theta_0 < 2\pi$,
(ii) *the* s*-curve* $V_{\theta_0}(s) := \Phi(s, \pi/4, \theta_0)$, $\quad 0 \le \theta_0 < 2\pi$.

The t-curves are called H-geodesics and the s-curves V-geodesics.

It was shown in [5] that the geodesic reflection across a horizontal or a vertical geodesic is an isometry of the Berger sphere. In particular, we have the following:

Proposition 4. *For* $0 < s < 2\pi$, $0 < t < \pi/2$, *the following two maps*

$$\begin{aligned}
R_H &: \Phi(s,t,\theta) \mapsto \Phi(2s_0 - s, t, 2\theta_0 - \theta), \\
R_V &: \Phi(s,t,\theta) \mapsto \Phi(s + \theta - \theta_0, \pi/2 - t, 2\theta_0 - \theta)
\end{aligned}$$

which represent reflections along segments of geodesics $H_{(s_0,\theta_0)}(t)$ *and* $V_{\theta_0}(s)$*, respectively, are isometries.*

One can also see that the rotation

$$\mathrm{Rot}_{\theta_0} : \Phi(s,t,\theta) \mapsto \Phi(s,t,\theta+\theta_0)$$

with respect to the Hopf fiber over the point $(1,0)$ is an isometry.

Let m and k be positive integers and p, q nonnegative integers. For notational convenience, let $H_{p,q} := H_{(\pi p/k, \pi q/m)}$, $V_q := V_{\pi q/m}$ and let $R_{p,q}$ denote the reflection across the horizontal geodesic $H_{p,q}$ and let R^q denote the reflection across the horizontal geodesic V_q. Then computations give

$$R_{p-1,q} \circ R_{p,q} : \Phi(s,t,\theta) \mapsto \Phi(s - 2\pi/k, t, \theta),$$
$$R_{p,q-1} \circ R_{p,q} : \Phi(s,t,\theta) \mapsto \Phi(s, t, \theta - 2\pi/m),$$
$$R^i \circ R^0 : \Phi(s,t,\theta) \mapsto \Phi(s - 2l\pi i/k, t, \theta + 2\pi i/m)$$

and

$$R_{p-1,q} \circ R_{p-1,q+1} \circ R_{p,q+1} = R_{p,q}. \tag{3}$$

Proposition 2 and (1), (2) give the following:

Proposition 5. *The following surfaces are ruled minimal surfaces.*

$$(t,\theta) \mapsto \Phi(s_0,t,\theta) = (\mathrm{e}^{\mathrm{i}s_0}\cos t, \mathrm{e}^{\mathrm{i}(s_0+\theta)}\sin t),$$
$$(s,t) \mapsto \Phi(s,t,\theta_0) = (\mathrm{e}^{\mathrm{i}s}\cos t, \mathrm{e}^{\mathrm{i}(s+\theta_0)}\sin t).$$

Since $\Phi(s,\pi/2,\theta) = \Phi(s',\pi/2,\theta')$ if and only if $s+\theta = s'+\theta' \mod 2\pi$, one can see that the second surfaces are tori, which will play the role of Clifford tori in the construction of Choe and Soret [1].

Proposition 6. *The surface*

$$\mathcal{T} := \Phi\left(s, \frac{\pi}{4}, \theta\right) = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}(s+\theta)})$$

is a doubly ruled minimal torus.

Proof. Computations show that $\mathcal{T}$ is minimal. Since

$$\partial_s|_{\Phi(s,\pi/4,\theta)} = \frac{1}{\sqrt{2}}(\mathrm{i}\,\mathrm{e}^{\mathrm{i}s}, \mathrm{i}\,\mathrm{e}^{\mathrm{i}(s+\theta)}) = (-\sin\theta X_1 + \cos\theta X_2)|_{\Phi(s,\pi/4,\theta)},$$

one can see that the s-parameter curves are horizontal geodesics. On the other hand, since $\partial_\theta\Phi(s,\pi/4,\theta) = \frac{1}{\sqrt{2}}(0, \mathrm{i}\,\mathrm{e}^{\mathrm{i}(s+\theta)})$, one has

$$(\partial_s - 2\partial_\theta)|_{\Phi(s,\pi/4,\theta)} = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}s}, -\mathrm{i}\,\mathrm{e}^{\mathrm{i}(s+\theta)}) = X_3(\Phi(s,\pi/4,\theta)),$$

which shows that the surface is ruled by vertical geodesics as well. Furthermore, since $\mathcal{T} = \Phi(T_1 \cap T_2)$, it is a torus. □

5. Construction of odd surfaces

Note that the surface $\mathcal{T}$ divides the Berger sphere into two regions $\{\Phi(s,t,\theta) : t < \pi/4)\}$ and $\{\Phi(s,t,\theta) : t > \pi/4\}$ and that the two regions are congruent; in fact, any reflection along a V-geodesic gives the congruence. We first construct a minimal surface $\mathcal{D}_-$ embedded in $\{\Phi(s,t,\theta) : t \le \pi/4\}$, whose boundary components consist of V-geodesics, and then reflect the surface $\mathcal{D}_-$ with respect to a V-geodesic to get the minimal surface $\mathcal{D}_+$ embedded in $\{\Phi(s,t,\theta) : t \ge \pi/4\}$. Then we show that the embedded minimal surface $\mathcal{D} := \mathcal{D}_- \cup \mathcal{D}_+$ is smooth without boundary.

Let us consider $2m$ ruled minimal surfaces

$$\mathcal{V}_q := \Phi\left(s, t, \frac{\pi q}{m}\right) = \mathrm{e}^{\mathrm{i}s}(\cos t, \mathrm{e}^{\mathrm{i}\frac{\pi q}{m}}\sin t), \quad q = 0, 1, \cdots, 2m-1$$

all of which meet along the Hopf fiber over the point $(1,0)$, and $2k$ ruled minimal surfaces

$$\mathcal{H}_p := \Phi\left(\frac{\pi p}{k}, t, \theta\right) = \mathrm{e}^{\mathrm{i}\frac{\pi p}{k}}(\cos t, \mathrm{e}^{\mathrm{i}\theta}\sin t), \quad p = 0, 1, \cdots, 2k-1$$

where $k = 2ml$ for some integer $l \ge 1$. Now, for $p = 0, 1, \ldots, 2k-1$ and $q = 0, 1, \ldots, 2m-1$, we consider the pentahedral regions

$$P_{p,q} := \Phi(s,t,\theta), \quad \frac{\pi p}{k} < s < \frac{\pi(p+1)}{k},\ 0 < t < \frac{\pi}{4},\ \frac{\pi q}{m} < \theta < \frac{\pi(q+1)}{m}$$

bounded by five ruled minimal surfaces $\mathcal{H}_p, \mathcal{H}_{p+1}, \mathcal{V}_q, \mathcal{V}_{q+1}$ and $\mathcal{T}$, which are mean-convex by [3], (see Fig. 1).

Let $\bar{P}_{p,q}$ be the closure of $P_{p,q}$. Since $\mathrm{Rot}_{\pi/m}(P_{p,q}) = P_{p,q+1}$ and since $R_{p,q}(P_{p,q}) = P_{p-1,q-1}$, one can see that all the pentahedral regions $P_{p,q}$ are congruent to each other. Moreover, one can see

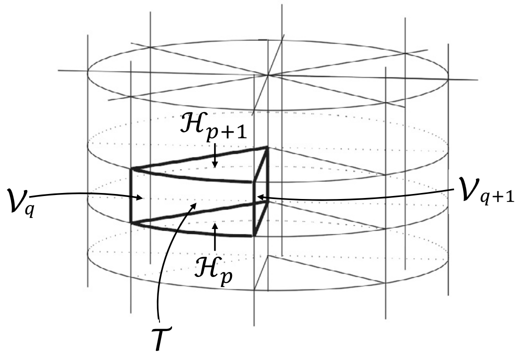

Fig. 1. $P_{p,q}$.

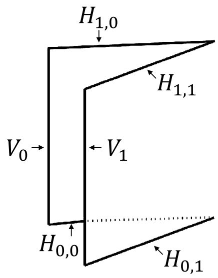

Fig. 2. Γ.

$$\bigcup_{p=0}^{2k-1}\bigcup_{q=0}^{2m-1}\bar{P}_{p,q}=\{\Phi(s,t,\theta):t\le\pi/4\}.$$

That is, $\bar{P}_{p,q},\ p=0,1,\dots,2k-1, q=0,1,\dots,2m-1$, yields a tesselation of the half of the Berger sphere $\{\Phi(s,t,\theta):t\le\pi/4\}$.

Now let us begin the construction. We first construct a minimal surface embedded in $\{\Phi(s,t,\theta):t\le\pi/4\}$, whose boundary lies on V-geodesics. In this construction, we use only the reflections along H-geodesics. By abuse of notation, we will denote the segment $\{H_{p,q}(t):0<t<\pi/4\}$ by $H_{p,q}$.

Let $\Gamma\subset\partial P_{0,0}$ be the piecewise geodesic polygon of six segments of horizontal geodesics $H_{0,0},H_{1,0},H_{1,1},H_{0,1}$ and V_0,V_1, (see Fig. 2), which is a subset of the 1-skeleton of $P_{0,0}$.

Then Γ spans an embedded minimal disk $D_{0,0}$ that lies inside of $P_{0,0}$ since $P_{0,0}$ is mean-convex. Now $D_{0,0}$ can be analytically extended across the boundary segments $H_{0,0},H_{1,0},H_{1,1}$ and $H_{0,1}$ by reflections to get four minimal discs $D_{2k-1,2m-1},D_{2k-1,1},D_{1,1}$ and $D_{1,2m-1}$ in $P_{2k-1,2m-1},P_{2k-1,1},P_{1,1}$ and $P_{1,2m-1}$ respectively. Continuing such reflections about the boundary segments of H-geodesics, we get minimal discs $D_{p,q}$ in each region $P_{p,q}$ with $p+q=\text{even}$. Now, we set

$$\mathcal{D}_-=\bigcup_{p+q=\text{even}}D_{p,q}.$$

The well-definedness and the smoothness of $\mathcal{D}_-$ can be checked as follows.

By reflecting $D_{p,q}$ along the H-geodesics $H_{p,q+1}$, we get the surface $D_{p-1,q+1}$; by reflecting $D_{p-1,q+1}$ along $H_{p-1,q}$, we get the surface $D_{p-2,q}$ and by reflecting $D_{p-2,q}$ along $H_{p-1,q}$ we get the surface $D_{p-1,q-1}$. Let us consider the surface $D_{p,q}\cup D_{p-1,q-1}\cup D_{p-2,q}\cup D_{p-1,q+1}$ (see Fig. 3, where the vertical lines represent V-geodesics and the dotted parallel lines represent intersections of the minimal surfaces $\mathcal{H}_p$ and $\mathcal{T}$).

It is smooth along $H_{p,q+1},H_{p-1,q+1}$ and $H_{p-1,q}$ because the surfaces obtained after reflection are conformal immersions of a disk that extend continuously up to the boundary and a piece of the boundary is sent to a piece of the line of the reflection. The surfaces $D_{p,q}$ and $D_{p-1,q-1}$ have common boundary $H_{p,q}$. Since $R_{p-1,q}\circ R_{p-1,q+1}\circ R_{p,q+1}=R_{p,q}$ by (3), we have $D_{p-1,q-1}=R_{p,q}(D_{p,q})$, which implies that $D_{p-1,q-1}$ is the analytic continuation of $D_{p,q}$ about the boundary $H_{p,q}$ by reflection. This implies that the surface $D_{p,q}\cup D_{p-1,q-1}\cup D_{p-2,q}\cup D_{p-1,q+1}$ is smooth along $H_{p,q}$ as well. Hence one can see that $\mathcal{D}_-$ is well defined and smooth. By construction, one can check that the surface $\mathcal{D}_-$ is invariant under the following two transformations:

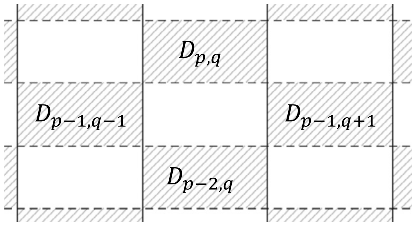

Fig. 3. The surface $D_{p,q} \cup D_{p-1,q-1} \cup D_{p-2,q} \cup D_{p-1,q+1}$.

$$\Phi(s,t,\theta) \mapsto \Phi(s - 2\pi/k, t, \theta),$$
$$\Phi(s,t,\theta) \mapsto \Phi(s, t, \theta - 2\pi/m).$$

Hence the surface $\mathcal{D}_-$ is invariant under the transformation $R^i \circ R^0$. Note that $\partial\mathcal{D}_-$, the boundary of $\mathcal{D}_-$ is

$$\partial\mathcal{D}_- = \bigcup_{q=0}^{2m-1} V_q.$$

Now let $\mathcal{D}_+ = R^0(\mathcal{D}_-)$, then $\mathcal{D}_+$ is a smooth minimal surface embedded in $\{\Phi(s,t,\theta) : t \geq \pi/4\}$. Since $R^0(V_q) = V_{2m-q}$, the boundary of $\mathcal{D}_+$ also consists of V-geodesics $V_0, V_1, \ldots, V_{2m-1}$. Along the common boundary V_i of $\mathcal{D}_-$ and $\mathcal{D}_+$, we have

$$R^i(\mathcal{D}_+) = R^i(R^0(\mathcal{D}_-)) = \mathcal{D}_-$$

since the surface $\mathcal{D}_-$ is invariant under the transformation $R^i \circ R^0$. This implies that $\mathcal{D}_+$ is the reflection of $\mathcal{D}_-$ along each common boundary V_i and hence we have a smooth surface

$$\mathcal{D} := \mathcal{D}_- \cup \mathcal{D}_+$$

without boundary, which is a compact minimal surface embedded in the Berger sphere. This completes the construction.

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