C. R. Acad. Sci. Paris, Ser. I 356 (2018) 85–91

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential geometry/Mathematical problems in mechanics

$W^{2,p}$-estimates for surfaces in terms of their two fundamental forms

Estimations dans $W^{2,p}$ pour des surfaces à partir de leurs deux formes fondamentales

Philippe G. Ciarlet [a], Cristinel Mardare [b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Sorbonne Universités, Université Pierre-et-Marie-Curie, Laboratoire Jacques-Louis-Lions, 75005 Paris, France*

A R T I C L E   I N F O

Article history:
Received 7 December 2017
Accepted 7 December 2017
Available online 19 December 2017

Presented by Philippe G. Ciarlet

A B S T R A C T

Let $p > 2$. We show how the fundamental theorem of surface theory for surfaces of class $W^{2,p}_{\text{loc}}(\omega)$ over a simply-connected open subset of $\mathbb{R}^2$ established in 2005 by S. Mardare can be extended to surfaces of class $W^{2,p}(\omega)$ when ω is in addition bounded and has a Lipschitz-continuous boundary. Then we establish a nonlinear Korn inequality for surfaces of class $W^{2,p}(\omega)$. Finally, we show that the mapping that defines in this fashion a surface of class $W^{2,p}(\omega)$, unique up to proper isometries of $\mathbb{E}^3$, in terms of its two fundamental forms is locally Lipschitz-continuous.

R É S U M É

Soit $p > 2$. Nous montrons comment le théorème fondamental de la théorie des surfaces de classe $W^{2,p}_{\text{loc}}(\omega)$ sur un ouvert simplement connexe ω de $\mathbb{R}^2$ établi par S. Mardare in 2005 peut être étendu à des surfaces de classe $W^{2,p}(\omega)$ lorsque ω est de plus borné et de frontière lipschitzienne. Ensuite, nous établissons une inégalité de Korn non linéaire pour des surfaces de classe $W^{2,p}(\omega)$. Nous établissons enfin que l'application qui définit une surface de classe $W^{2,p}(\omega)$ à une isométrie propre de $\mathbb{E}^3$ près en fonction de ses deux formes fondamentales est localement lipschitzienne.

1. Preliminaries

In what follows, Greek indices and exponents, except ε and δ, vary in the set $\{1, 2\}$, Latin indices vary in the set $\{1, 2, 3\}$, and the summation convention for repeated indices and exponents is used. Boldface letters denote vector and matrix fields.

E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), mardare@ann.jussieu.fr (C. Mardare).

https://doi.org/10.1016/j.crma.2017.12.003

The three-dimensional Euclidean space is denoted $\mathbb{E}^3$. The inner product, exterior product, and norm, in $\mathbb{E}^3$ are respectively denoted $\cdot$, $\wedge$, and $|\cdot|$. The set of all proper isometries of $\mathbb{E}^3$ is denoted and defined by

$$\mathbf{Isom}_+(\mathbb{E}^3) := \{\boldsymbol{r} : \mathbb{E}^3 \to \mathbb{E}^3,\ \boldsymbol{r}(x) = \boldsymbol{R}x + \boldsymbol{a},\ x \in \mathbb{E}^3;\ \boldsymbol{R} \in \mathbb{O}^3_+,\ \boldsymbol{a} \in \mathbb{E}^3\},$$

where $\mathbb{O}^3_+$ denotes the set of all real 3×3 proper orthogonal matrices.

Remark 1. The set $\mathbf{Isom}_+(\mathbb{E}^3)$ is in effect a smooth submanifold of dimension six of the space of all 3×3 real matrices and its tangent space at the identity mapping $\boldsymbol{id} \in \mathbf{Isom}_+(\mathbb{E}^3)$ is the space of all *"infinitesimal rigid displacements of* $\mathbb{E}^3$*"*, which is denoted and defined by

$$\mathbf{Rig}(\mathbb{E}^3) = \mathcal{T}_{\boldsymbol{id}}\mathbf{Isom}_+(\mathbb{E}^3) := \left\{\boldsymbol{\zeta} : \mathbb{E}^3 \to \mathbb{E}^3,\ \boldsymbol{\zeta}(x) = \boldsymbol{A}x + \boldsymbol{b},\ x \in \mathbb{E}^3;\ \boldsymbol{A} \in \mathbb{A}^3, \boldsymbol{b} \in \mathbb{E}^3\right\},$$

where $\mathbb{A}^3$ denotes the set of all real 3×3 antisymmetric matrices. $\square$

Given an open subset ω of $\mathbb{R}^2$, we let $y = (y_\alpha)$ denote a generic point in ω, and we let $\partial_\alpha := \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$.

The space of distributions over an open subset ω of $\mathbb{R}^2$ is denoted $\mathcal{D}'(\omega)$. For each integer $m \geq 1$ and each real number $p \geq 1$, $\mathcal{C}^m(\omega)$ denotes the subspace of $\mathcal{C}^0(\omega)$ of functions that possess continuous partial derivatives up to order m, and $W^{m,p}(\omega)$ denotes the usual Sobolev space.

The notation $L^p_{\mathrm{loc}}(\omega)$, resp. $W^{m,p}_{\mathrm{loc}}(\omega)$, denotes the space of functions $f : \omega \to \mathbb{R}$ such that $f|_U \in L^p(U)$, resp. $f|_U \in W^{m,p}(U)$, for all open sets $U \Subset \omega$, where $f|_U$ denotes the restriction of f to U and the notation $U \Subset \omega$ means that the closure of the set U is a compact subset of ω. Given any finite dimensional real space $\mathbb{Y}$, the notation $L^p_{\mathrm{loc}}(\omega; \mathbb{Y})$, resp. $W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{Y})$, denotes the space of $\mathbb{Y}$-valued fields with components in $L^p_{\mathrm{loc}}(\omega)$, resp. $W^{1,p}_{\mathrm{loc}}(\omega)$. Other similar notations with self-explanatory definitions will be used.

An *immersion* from ω into $\mathbb{E}^3$ is a smooth enough mapping $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$ such that the two vector fields $\partial_\alpha \boldsymbol{\theta} : \omega \to \mathbb{E}^3$ are linearly independent at each point of ω. Given an immersion $\boldsymbol{\theta} : \omega \to \mathbb{E}^3$, define the functions

$$\hat{a}_{\alpha\beta}(\boldsymbol{\theta}) := \hat{\boldsymbol{a}}_\alpha(\boldsymbol{\theta}) \cdot \hat{\boldsymbol{a}}_\beta(\boldsymbol{\theta}) \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) := \partial_\alpha \hat{\boldsymbol{a}}_\beta(\boldsymbol{\theta}) \cdot \hat{\boldsymbol{a}}_3(\boldsymbol{\theta}),$$

where

$$\hat{\boldsymbol{a}}_\alpha(\boldsymbol{\theta}) := \partial_\alpha \boldsymbol{\theta} \text{ and } \hat{\boldsymbol{a}}_3(\boldsymbol{\theta}) := \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|}.$$

The image $S = \boldsymbol{\theta}(\omega)$ is thus a *surface* in $\mathbb{E}^3$ and the functions $\hat{a}_{\alpha\beta}(\boldsymbol{\theta})$ and $\hat{b}_{\alpha\beta}(\boldsymbol{\theta})$ are the covariant components of the *first and second fundamental forms* of S.

The space of real 2×2 symmetric matrices is denoted $\mathbb{S}^2$; its subset formed by all positive-definite matrices is denoted $\mathbb{S}^2_>$.

An open subset ω of $\mathbb{R}^2$ satisfies the *uniform interior cone property* if there exists a bounded open cone $V \subset \mathbb{R}^2$ such that any point $y \in \omega$ is the vertex of a cone V_y congruent with V and contained in ω. An open subset ω of $\mathbb{R}^2$ is a *domain* if it is bounded and has a Lipschitz-continuous boundary.

Detailed proofs of the results announced here will be found in [4].

2. The fundamental theorem of surface theory in the spaces $W^{2,p}_{\mathrm{loc}}(\omega)$ and $W^{2,p}(\omega)$

The *fundamental theorem of surface theory*, which is classically established in the spaces of continuously differentiable functions (cf., e.g., [5, Theorem 3.8.8], [1, Appendix to Chapter 4], [2, Theorems 8.16-1 and 8.17-1]), has been shown to hold in function spaces with little regularity, according to the following remarkable result, due to S. Mardare [6, Theorem 9]:

Theorem 1. *Let ω be a simply-connected open subset of $\mathbb{R}^2$, let $p > 2$, and let a matrix field $(a_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and a matrix field $(b_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ be given that satisfy the Gauss and Codazzi–Mainardi equations, viz.*

$$R^\sigma_{\alpha\beta\tau} := \partial_\tau \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\sigma_{\alpha\tau} + \Gamma^\gamma_{\alpha\beta}\Gamma^\sigma_{\tau\gamma} - \Gamma^\gamma_{\alpha\tau}\Gamma^\sigma_{\beta\gamma} - b_{\alpha\beta}b^\sigma_\tau + b_{\alpha\tau}b^\sigma_\beta = 0 \text{ in } \mathcal{D}'(\omega)$$

and

$$R^3_{\alpha\beta\tau} := \partial_\tau b_{\alpha\beta} - \partial_\beta b_{\alpha\tau} + \Gamma^\gamma_{\alpha\beta} b_{\tau\gamma} - \Gamma^\gamma_{\alpha\tau} b_{\beta\gamma} = 0 \text{ in } \mathcal{D}'(\omega),$$

where the functions $\Gamma^\sigma_{\alpha\beta} \in L^p_{\mathrm{loc}}(\omega)$ and $b^\sigma_\alpha \in L^p_{\mathrm{loc}}(\omega)$ are defined by

$$\Gamma^\sigma_{\alpha\beta} := \frac{1}{2} a^{\sigma\tau}\left(\partial_\alpha a_{\beta\tau} + \partial_\beta a_{\alpha\tau} - \partial_\tau a_{\alpha\beta}\right) \text{ and } b^\sigma_\beta := a^{\sigma\tau} b_{\tau\beta}, \text{ where } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1}.$$

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{E}^3)$ *such that*

$$\hat{a}_{\alpha\beta}(\boldsymbol{\theta}) = a_{\alpha\beta} \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) = b_{\alpha\beta} \text{ a.e. in } \omega.$$

Besides, an immersion $\boldsymbol{\psi} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{E}^3)$ *satisfies*

$$\hat{a}_{\alpha\beta}(\boldsymbol{\psi}) = \hat{a}_{\alpha\beta}(\boldsymbol{\theta}) \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\psi}) = \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) \text{ a.e. in } \omega$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\theta} \text{ in } \omega. \quad \square$$

Our *first objective* (Theorem 2) consists in showing that an existence and uniqueness theorem similar to Theorem 1 holds in the spaces $W^{m,p}(\omega)$ instead of the spaces $W^{m,p}_{\mathrm{loc}}(\omega)$ if the open set ω is in addition a domain.

Theorem 2. *Let* ω *be a simply-connected domain in* $\mathbb{R}^2$, *let* $p > 2$, *and let a matrix field* $(a_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2_>)$ *and a matrix field* $(b_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2)$ *be given that satisfy the equations*

$$R^{\sigma}_{\alpha\beta\tau} = 0 \text{ and } R^{3}_{\alpha\beta\tau} = 0 \text{ in } \mathcal{D}'(\omega).$$

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ *such that*

$$\hat{a}_{\alpha\beta}(\boldsymbol{\theta}) = a_{\alpha\beta} \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) = b_{\alpha\beta} \text{ a.e. in } \omega.$$

Besides, an immersion $\boldsymbol{\psi} \in W^{2,p}(\omega; \mathbb{E}^3)$ *satisfies*

$$\hat{a}_{\alpha\beta}(\boldsymbol{\psi}) = \hat{a}_{\alpha\beta}(\boldsymbol{\theta}) \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\psi}) = \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) \text{ a.e. in } \omega$$

if and only if there exists an isometry $\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\theta} \text{ in } \omega. \quad \square$$

Sketch of proof. Since $p > 2$ and ω is a domain, $W^{1,p}(\omega)$ is a Banach algebra and the canonical injection from $W^{1,p}(\omega)$ into $\mathcal{C}^0(\overline{\omega})$ is continuous. Combining these two observations with the *Gauss equations*

$$\partial_\alpha \hat{\boldsymbol{a}}_\beta(\boldsymbol{\theta}) = \Gamma^{\sigma}_{\alpha\beta} \hat{\boldsymbol{a}}_\sigma(\boldsymbol{\theta}) + b_{\alpha\beta} \hat{\boldsymbol{a}}_3(\boldsymbol{\theta}) \text{ a.e. in } \omega$$

and the relations

$$|\hat{\boldsymbol{a}}_\alpha(\boldsymbol{\theta})| = \sqrt{a_{\alpha\alpha}} \text{ (no summation on } \alpha \text{ here) and } |\hat{\boldsymbol{a}}_3(\boldsymbol{\theta})| = 1 \text{ a.e. in } \omega,$$

where $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{E}^3)$ denotes the immersion found in Theorem 1 and the functions $\Gamma^{\sigma}_{\alpha\beta}$ are defined as in Theorem 1 (in effect the Christoffel symbols associated with $\boldsymbol{\theta}$), shows that the three vector fields $\hat{\boldsymbol{a}}_i(\boldsymbol{\theta})$ belong to $L^\infty(\omega; \mathbb{E}^3)$, which in turn implies that $\partial_\alpha \boldsymbol{\theta} \in L^\infty(\omega; \mathbb{E}^3)$ and $\partial_{\alpha\beta} \boldsymbol{\theta} \in L^p(\omega; \mathbb{E}^3)$. It is then an easy matter to conclude that $\boldsymbol{\theta} \in L^p(\omega; \mathbb{E}^3)$, hence that $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$. The uniqueness up to isometries follows immediately from Theorem 1. $\square$

3. A nonlinear Korn inequality for surfaces of class $W^{2,p}$

The *second objective* of this Note is to complement the *existence and uniqueness* result of Theorem 2 by a *stability* result (Theorem 3 below), showing that *the distance modulo a proper isometry between two surfaces in* $W^{2,p}$*-norm is bounded by the distance between their first fundamental forms in the* $W^{1,p}$*-norm and the distance between their second fundamental forms in the* L^p*-norm.* A notation such as $c = c(\omega, p, \varepsilon)$ means that c is a real constant that depends on ω, p and ε.

Theorem 3. *Let* ω *be a bounded and connected open subset of* $\mathbb{R}^2$ *that satisfies the uniform interior cone property. Given any* $p > 2$ *and* $\varepsilon > 0$, *let*

$$V_\varepsilon(\omega; \mathbb{E}^3) := \Big\{ \boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3);\ \|\boldsymbol{\theta}\|_{W^{2,p}(\omega;\mathbb{E}^3)} \le 1/\varepsilon \text{ and } |\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}| \ge \varepsilon \text{ in } \omega \Big\}.$$

Then there exists a constant $c = c(\omega, p, \varepsilon)$ *such that*

$$\inf_{\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)} \|\boldsymbol{\varphi} - \boldsymbol{r} \circ \boldsymbol{\psi}\|_{W^{2,p}(\omega;\mathbb{E}^3)} \le c \Big\{ \|(\hat{a}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{a}_{\alpha\beta}(\boldsymbol{\psi}))\|_{W^{1,p}(\omega;\mathbb{S}^2)} + \|(\hat{b}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{b}_{\alpha\beta}(\boldsymbol{\psi}))\|_{L^p(\omega;\mathbb{S}^2)} \Big\}$$

for all $\boldsymbol{\varphi} \in V_\varepsilon(\omega; \mathbb{E}^3)$ *and* $\boldsymbol{\psi} \in V_\varepsilon(\omega; \mathbb{E}^3)$. $\square$

Remark 2. The above inequality can indeed be seen as a *nonlinear* Korn inequality for surfaces of class $W^{2,p}$, since a *formal linearization* (such a linearization consists first in letting in the above nonlinear inequality $\boldsymbol{\varphi} := \boldsymbol{\theta} + \boldsymbol{\eta}$ and $\boldsymbol{\psi} := \boldsymbol{\theta}$, where $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ is a given immersion considered as "fixed", and $\boldsymbol{\eta} \in W^{2,p}(\omega; \mathbb{E}^3)$ is an arbitrary vector field, then in canceling all the terms that depend nonlinearly on $\boldsymbol{\eta}$) yields the following *linear* Korn inequality on the surface $S = \boldsymbol{\theta}(\omega)$: *There exists a constant* $c_0 = c_0(\boldsymbol{\theta}, \omega)$ *such that* (the space $\mathbf{Rig}(\mathbb{E}^3)$ is defined in Remark 1)

$$\inf_{\boldsymbol{\zeta} \in \mathbf{Rig}(\mathbb{E}^3)} \|\boldsymbol{\eta} - \boldsymbol{\zeta}\|_{W^{2,p}(\omega;\mathbb{E}^3)} \le c_0 \left\{ \|(\gamma_{\alpha\beta}(\boldsymbol{\eta}))\|_{W^{1,p}(\omega;\mathbb{S}^2)} + \|(\rho_{\alpha\beta}(\boldsymbol{\eta}))\|_{L^p(\omega;\mathbb{S}^2)} \right\} \text{ for all } \boldsymbol{\eta} \in W^{2,p}(\omega; \mathbb{E}^3),$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}\left[\hat{a}_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\eta}) - \hat{a}_{\alpha\beta}(\boldsymbol{\theta})\right]^{\text{lin}} \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta}) := \left[\hat{b}_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\eta}) - \hat{b}_{\alpha\beta}(\boldsymbol{\theta})\right]^{\text{lin}}$$

designate the linear parts with respect to $\boldsymbol{\eta}$ of the tensors appearing in the right-hand side of the inequality of Theorem 3. □

The proof of Theorem 3 relies on a *comparison theorem between solutions to general Pfaff systems* due to the first author and S. Mardare (see Theorem 3.1 and Remark 3.1 in [3] and Theorem 4.1 in [7]), which we state below only in the particular case needed here. The notations $\mathbb{M}^3$ and $|\cdot|$ used in the next theorem respectively denote the space of 3×3 real matrices and the Frobenius norm in this space. The notation $(\boldsymbol{a} \mid \boldsymbol{b} \mid \boldsymbol{c})$ denotes the matrix in $\mathbb{M}^3$ with column vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{E}^3$.

Theorem 4. *Let ω be a bounded and connected open subset of $\mathbb{R}^2$ that satisfies the uniform interior cone property. Given any $p > 2$, $\varepsilon > 0$, and $y_0 \in \omega$, there exists a constant $c_1 = c_1(\omega, p, \varepsilon, y_0)$ such that*

$$\|\boldsymbol{F} - \tilde{\boldsymbol{F}}\|_{W^{1,p}(\omega;\mathbb{M}^3)} \le c_1\Big(|\boldsymbol{F}(y_0) - \tilde{\boldsymbol{F}}(y_0)| + \sum_\alpha \|\boldsymbol{\Gamma}_\alpha - \tilde{\boldsymbol{\Gamma}}_\alpha\|_{L^p(\omega;\mathbb{M}^3)}\Big)$$

for all matrix fields $\boldsymbol{F}, \tilde{\boldsymbol{F}} \in W^{1,p}(\omega; \mathbb{M}^3)$ and $\boldsymbol{\Gamma}_\alpha, \tilde{\boldsymbol{\Gamma}}_\alpha \in L^p(\omega; \mathbb{M}^3)$ that satisfy

$$|\boldsymbol{F}(y_0)| + \sum_\alpha \|\boldsymbol{\Gamma}_\alpha\|_{L^p(\omega;\mathbb{M}^3)} \le \frac{1}{\varepsilon} \textit{ and } |\tilde{\boldsymbol{F}}(y_0)| + \sum_\alpha \|\tilde{\boldsymbol{\Gamma}}_\alpha\|_{L^p(\omega;\mathbb{M}^3)} \le \frac{1}{\varepsilon},$$

and

$$\partial_\alpha \boldsymbol{F} = \boldsymbol{F}\,\boldsymbol{\Gamma}_\alpha \text{ and } \partial_\alpha \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}\,\tilde{\boldsymbol{\Gamma}}_\alpha \text{ a.e. in } \omega. \quad \square$$

Sketch of the proof of Theorem 3. With any immersion $\boldsymbol{\varphi} \in W^{2,p}(\omega; \mathbb{E}^3)$, we associate: the proper isometry $\boldsymbol{r}(\boldsymbol{\varphi}, y_0)$ of $\mathbb{E}^3$ defined by

$$\boldsymbol{r}(\boldsymbol{\varphi}, y_0)(x) := (\boldsymbol{B}^T\boldsymbol{B})^{1/2}\boldsymbol{B}^{-1}(x - \boldsymbol{\varphi}(y_0)) \text{ for all } x \in \mathbb{E}^3,$$

where

$$\boldsymbol{B} := \big(\hat{\boldsymbol{a}}_1(\boldsymbol{\varphi})(y_0) \mid \hat{\boldsymbol{a}}_2(\boldsymbol{\varphi})(y_0) \mid \hat{\boldsymbol{a}}_3(\boldsymbol{\varphi})(y_0)\big);$$

the immersion

$$\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0) := \boldsymbol{r}(\boldsymbol{\varphi}, y_0) \circ \boldsymbol{\varphi} \in W^{2,p}(\omega; \mathbb{E}^3);$$

and the matrix fields

$$\boldsymbol{F}(\boldsymbol{\varphi}, y_0) := \big(\hat{\boldsymbol{a}}_1(\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0)) \mid \hat{\boldsymbol{a}}_2(\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0)) \mid \hat{\boldsymbol{a}}_3(\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0))\big)$$

and

$$\boldsymbol{A}(\boldsymbol{\varphi}) := \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi}) := \begin{pmatrix} \Gamma^1_{\alpha 1} & \Gamma^1_{\alpha 2} & -b^1_\alpha \\ \Gamma^2_{\alpha 1} & \Gamma^2_{\alpha 2} & -b^2_\alpha \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix},$$

where

$$a_{\alpha\beta} := \hat{a}_{\alpha\beta}(\boldsymbol{\varphi}),\ b_{\alpha\beta} := \hat{b}_{\alpha\beta}(\boldsymbol{\varphi}),\ b^\alpha_\beta := a^{\alpha\sigma} b_{\sigma\beta},\ (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1},$$

and

$$\Gamma^\sigma_{\alpha\beta} := \frac{1}{2} a^{\sigma\tau}(\partial_\alpha a_{\beta\tau} + \partial_\beta a_{\alpha\tau} - \partial_\tau a_{\alpha\beta}).$$

These matrix fields satisfy the *Pfaff system*

$$\partial_\alpha \boldsymbol{F}(\boldsymbol{\varphi}, y_0) = \boldsymbol{F}(\boldsymbol{\varphi}, y_0)\,\boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi}) \text{ a.e. in } \omega,$$

and the *"initial condition"*

$$(\boldsymbol{F}(\boldsymbol{\varphi}, y_0))(y_0) = (\boldsymbol{A}(\boldsymbol{\varphi})(y_0))^{1/2} \in \mathbb{S}^3_>.$$

Note in passing that the above Pfaff system is equivalent to the *equations of Gauss and Weingarten* associated with the immersion $\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0)$.

In addition, if $\boldsymbol{\varphi} \in V_\varepsilon(\omega; \mathbb{E}^3)$ for some $\varepsilon > 0$ (the set $V_\varepsilon(\omega; \mathbb{E}^3)$ is defined in the statement of Theorem 3), then

$$\boldsymbol{F}(\boldsymbol{\varphi}, y_0) \in W^{1,p}(\omega; \mathbb{S}^3) \text{ and } \boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi}) \in L^p(\omega; \mathbb{M}^3),$$

and there exists a constant $c_1 = c_1(\omega, p, \varepsilon)$ such that

$$|(\boldsymbol{F}(\boldsymbol{\varphi}, y_0))(y_0)| + \|\boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^3)} \le c_1.$$

This allows us to apply Theorem 4 and to deduce that there exists a constant $c_2 = c_2(\omega, y_0, p, \varepsilon)$ such that

$$\|\boldsymbol{F}(\boldsymbol{\varphi}, y_0) - \boldsymbol{F}(\boldsymbol{\psi}, y_0)\|_{W^{1,p}(\omega;\mathbb{M}^3)} \le c_2\Big(|(\boldsymbol{A}(\boldsymbol{\varphi}))(y_0) - (\boldsymbol{A}(\boldsymbol{\psi}))(y_0)| + \sum_\alpha \|\boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi}) - \boldsymbol{\Gamma}_\alpha(\boldsymbol{\psi})\|_{L^p(\omega;\mathbb{M}^3)}\Big)$$

for all immersions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ that belong to the set $V_\varepsilon(\omega; \mathbb{E}^3)$.

Next, using the expressions of the matrix fields appearing in the right-hand side of the above inequality in terms of the fundamental forms associated with the immersions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, we deduce after a series of straightforward, but somewhat technical, computations that there exist two constants $c_3 = c_3(\omega, p, \varepsilon)$ and $c_4 = c_4(\omega, p, \varepsilon)$ such that

$$|(\boldsymbol{A}(\boldsymbol{\varphi}))(y_0) - (\boldsymbol{A}(\boldsymbol{\psi}))(y_0)| \le c_3 \|(\hat{a}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{a}_{\alpha\beta}(\boldsymbol{\psi}))\|_{W^{1,p}(\omega;\mathbb{S}^2)},$$

and

$$\|\boldsymbol{\Gamma}_\alpha(\boldsymbol{\varphi}) - \boldsymbol{\Gamma}_\alpha(\boldsymbol{\psi})\|_{L^p(\omega;\mathbb{M}^3)} \le c_4\Big(\|(\hat{a}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{a}_{\alpha\beta}(\boldsymbol{\psi}))\|_{W^{1,p}(\omega;\mathbb{S}^2)} + \|(\hat{b}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{b}_{\alpha\beta}(\boldsymbol{\psi}))\|_{L^p(\omega;\mathbb{S}^2)}\Big).$$

Finally, the definition of the immersions $\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0)$ and $\boldsymbol{\theta}(\boldsymbol{\psi}, y_0)$ implies that the vector field

$$\boldsymbol{\eta} := \boldsymbol{\theta}(\boldsymbol{\varphi}, y_0) - \boldsymbol{\theta}(\boldsymbol{\psi}, y_0) \in W^{2,p}(\omega; \mathbb{E}^3)$$

satisfies the *Poincaré system* (the notation $[\,\cdot\,]_\alpha$ denotes the α-th column vector of the matrix appearing between the brackets)

$$\partial_\alpha \boldsymbol{\eta} = [\boldsymbol{F}(\boldsymbol{\varphi}, y_0) - \boldsymbol{F}(\boldsymbol{\psi}, y_0)]_\alpha \text{ in } \omega$$

and the *"initial condition"*

$$\boldsymbol{\eta}(y_0) = \boldsymbol{0}.$$

Using an *inequality of Poincaré's type*, we infer from the above system and initial condition that there exists a constant $c_5 = c_5(\omega, p)$ such that

$$\|\boldsymbol{\eta}\|_{W^{2,p}(\omega;\mathbb{E}^3)} \le c_5 \|\boldsymbol{F}(\boldsymbol{\varphi}, y_0) - \boldsymbol{F}(\boldsymbol{\psi}, y_0)\|_{W^{1,p}(\omega;\mathbb{M}^3)}.$$

The conclusion follows by combining the above inequalities and by noting that, thanks to the invariance under rotations of the Euclidean and Frobenius norms,

$$\|\boldsymbol{\eta}\|_{W^{2,p}(\omega;\mathbb{E}^3)} = \|\boldsymbol{\theta}(\boldsymbol{\varphi}, y_0) - \boldsymbol{\theta}(\boldsymbol{\psi}, y_0)\|_{W^{2,p}(\omega;\mathbb{E}^3)} \ge \inf_{\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)} \|\boldsymbol{\varphi} - \boldsymbol{r} \circ \boldsymbol{\psi}\|_{W^{2,p}(\omega;\mathbb{E}^3)}. \quad \square$$

4. Local Lipschitz-continuity of the mapping defining a surface of class $W^{2,p}$, $p > 2$, in terms of its fundamental forms

Let ω be an open subset of $\mathbb{R}^2$. Given two symmetric matrix fields

$$\boldsymbol{A} = (a_{\alpha\beta}) \in W^{1,p}_{\rm loc}(\omega; \mathbb{S}^2) \text{ and } \boldsymbol{B} = (b_{\alpha\beta}) \in L^p_{\rm loc}(\omega; \mathbb{S}^2),\ p > 2,$$

such that $\boldsymbol{A}(y) \in \mathbb{S}^2_>$ for all $y \in \overline{\omega}$, define the distributions

$$R^{\sigma}_{\ \alpha\beta\tau}(\boldsymbol{A}, \boldsymbol{B}) := \partial_\tau \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\sigma_{\alpha\tau} + \Gamma^\gamma_{\alpha\beta}\Gamma^\sigma_{\tau\gamma} - \Gamma^\gamma_{\alpha\tau}\Gamma^\sigma_{\beta\gamma} - b_{\alpha\beta} b^\sigma_\tau + b_{\alpha\tau} b^\sigma_\beta \in \mathcal{D}'(\omega),$$

$$R^{3}_{\ \alpha\beta\tau}(\boldsymbol{A}, \boldsymbol{B}) := \partial_\tau b_{\alpha\beta} - \partial_\beta b_{\alpha\tau} + \Gamma^\gamma_{\alpha\beta} b_{\tau\gamma} - \Gamma^\gamma_{\alpha\tau} b_{\beta\gamma} \in \mathcal{D}'(\omega),$$

where

$$\Gamma^\sigma_{\alpha\beta} = \Gamma^\sigma_{\alpha\beta}(\boldsymbol{A}) := \frac{1}{2} a^{\sigma\tau} \left(\partial_\alpha a_{\beta\tau} + \partial_\beta a_{\alpha\tau} - \partial_\tau a_{\alpha\beta}\right) \in L^p_{\rm loc}(\omega),$$
$$b^\sigma_\beta := a^{\sigma\tau} b_{\tau\beta} \in L^p_{\rm loc}(\omega), \text{ and } (a^{\sigma\tau}) := (a_{\alpha\beta})^{-1} \in W^{1,p}_{\rm loc}(\omega).$$

Remark 3. The above regularity assumptions on the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ are the minimal possible in order that the definitions of the distributions $R^{j}_{\ \alpha\beta\tau}(\boldsymbol{A}, \boldsymbol{B})$ make sense: combined with the Sobolev embedding $W^{1,p}_{\rm loc}(\omega) \subset \mathcal{C}^0(\omega)$, they ensure that $\det \boldsymbol{A}$ is a continuous positive function over ω, which in turn implies that $a^{\sigma\tau} \in \mathcal{C}^0(\omega)$ and so the products appearing in the definitions of $\Gamma^\sigma_{\alpha\beta}$ and b^σ_α belong to $L^p_{\rm loc}(\omega)$; this allows to define the partial derivatives of $\Gamma^\sigma_{\alpha\beta}$ and b^σ_α appearing in the above definition of $R^{j}_{\ \alpha\beta\tau}(\boldsymbol{A}, \boldsymbol{B})$ as distributions in $\mathcal{D}'(\omega)$. □

The *third objective* of this Note is to establish, as a consequence of the nonlinear Korn inequality of Theorem 3, the following *"existence, uniqueness, and stability theorem"* for the reconstruction of a surface from its fundamental forms in the spaces $W^{1,p}(\omega; \mathbb{S}^2)$ and $L^p(\omega; \mathbb{S}^2)$.

In Theorem 5 below, the set $\dot{W}^{2,p}(\omega; \mathbb{E}^3)$ is the quotient set of the space $W^{2,p}(\omega; \mathbb{E}^3)$ by the *equivalence relation* between isometrically equivalent immersions, and the set $\mathbb{T}(\omega)$ is the subset of the space $W^{1,p}(\omega; \mathbb{S}^2) \times L^p(\omega; \mathbb{S}^2)$ formed by all pairs of a positive-definite symmetric matrix field and a symmetric matrix field that satisfy together the *equations of Gauss and Codazzi–Mainardi* in the distributional sense. As such, the sets $\dot{W}^{2,p}(\omega; \mathbb{E}^3)$ and $\mathbb{T}(\omega)$ are *metric spaces* equipped respectively with the distances defined by

$$\mathrm{dist}_{\dot{W}^{2,p}(\omega;\mathbb{E}^3)}(\dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\psi}}) := \inf_{\tilde{\boldsymbol{\theta}} \in \dot{\boldsymbol{\theta}},\ \tilde{\boldsymbol{\psi}} \in \dot{\boldsymbol{\psi}}} \|\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\psi}}\|_{W^{2,p}(\omega;\mathbb{E}^3)} = \inf_{\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)} \|\boldsymbol{\theta} - \boldsymbol{r} \circ \boldsymbol{\psi}\|_{W^{2,p}(\omega;\mathbb{E}^3)}$$

for all $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\psi}}$ in $\dot{W}^{2,p}(\omega; \mathbb{E}^3)$, and by

$$\mathrm{dist}_{\mathbb{T}(\omega)}\big((\boldsymbol{A}, \boldsymbol{B}), (\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})\big) := \|\boldsymbol{A} - \tilde{\boldsymbol{A}}\|_{W^{1,p}(\omega;\mathbb{S}^2)} + \|\boldsymbol{B} - \tilde{\boldsymbol{B}}\|_{L^p(\omega;\mathbb{S}^2)}$$

for all $(\boldsymbol{A}, \boldsymbol{B})$ and $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})$ in $\mathbb{T}(\omega)$.

Theorem 5. *Let ω be a domain in $\mathbb{R}^2$. Given any $p > 2$, define the sets*

$$\dot{W}^{2,p}(\omega; \mathbb{E}^3) := \left\{\dot{\boldsymbol{\theta}} = \{\boldsymbol{r} \circ \boldsymbol{\theta};\ \boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)\};\ \boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)\right\}$$

and

$$\mathbb{T}(\omega) := \left\{(\boldsymbol{A}, \boldsymbol{B}) \in W^{1,p}(\omega; \mathbb{S}^2) \times L^p(\omega; \mathbb{S}^2);\ \boldsymbol{A}(y) \in \mathbb{S}^2_> \text{ at each } y \in \overline{\omega},\ R^{j}_{\ \alpha\beta\tau}(\boldsymbol{A}, \boldsymbol{B}) = 0 \text{ in } \mathcal{D}'(\omega)\right\}.$$

Then the following assertions are true:

(a) *Two matrix fields $\boldsymbol{A} = (a_{\alpha\beta})$ and $\boldsymbol{B} = (b_{\alpha\beta})$ satisfy*

$$(\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{T}(\omega)$$

if and only if there exists an immersion $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ such that

$$\hat{a}_{\alpha\beta}(\boldsymbol{\theta}) = a_{\alpha\beta} \text{ in } \omega \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) = b_{\alpha\beta} \text{ a.e. in } \omega.$$

(b) *Two immersions $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ and $\boldsymbol{\psi} \in W^{2,p}(\omega; \mathbb{E}^3)$ satisfy the relations*

$$\hat{a}_{\alpha\beta}(\boldsymbol{\theta}) = \hat{a}_{\alpha\beta}(\boldsymbol{\psi}) \text{ in } \omega \text{ and } \hat{b}_{\alpha\beta}(\boldsymbol{\theta}) = \hat{b}_{\alpha\beta}(\boldsymbol{\psi}) \text{ a.e. in } \omega$$

if and only if there exists a proper isometry $\boldsymbol{r}$ *of* $\mathbb{E}^3$ *such that*

$$\boldsymbol{\psi} = \boldsymbol{r} \circ \boldsymbol{\theta} \text{ in } \omega.$$

(c) *The mapping defined by* (a) *and* (b)*, namely*

$$\boldsymbol{\mathcal{G}} : (\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{T}(\omega) \to \boldsymbol{\mathcal{G}}((\boldsymbol{A}, \boldsymbol{B})) := \dot{\boldsymbol{\theta}} \in \dot{W}^{2,p}(\omega; \mathbb{E}^3),$$

where $\boldsymbol{\theta} \in W^{2,p}(\omega; \mathbb{E}^3)$ *is any immersion that satisfies*

$$(\hat{a}_{\alpha\beta}(\boldsymbol{\theta})) = \boldsymbol{A} \text{ and } (\hat{b}_{\alpha\beta}(\boldsymbol{\theta})) = \boldsymbol{B} \text{ a.e. in } \omega,$$

is locally Lipschitz-continuous. □

Sketch of proof. Parts (a) and (b) are just a re-statement of Theorem 2. Otherwise, the rest of the proof follows a strategy introduced by the first author and S. Mardare in [3]. More precisely, part (c) of Theorem 5 is deduced from Theorem 3 as follows.

On the one hand, the Sobolev embedding $W^{1,p}(\omega) \subset \mathcal{C}^0(\overline{\omega})$ implies that, given any $(\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{T}(\omega)$, there exists $\delta = \delta(\boldsymbol{A}, \boldsymbol{B}) > 0$ such that the set

$$\mathbb{T}_\delta(\omega) := \left\{(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) \in \mathbb{T}(\omega);\ \det \tilde{\boldsymbol{A}} \geq \delta \text{ in } \omega,\ \|\tilde{\boldsymbol{A}}\|_{W^{1,p}(\omega;\mathbb{S}^2)} \leq 1/\delta, \text{ and } \|\tilde{\boldsymbol{B}}\|_{L^p(\omega;\mathbb{S}^2)} \leq 1/\delta\right\}$$

is a neighborhood of $(\boldsymbol{A}, \boldsymbol{B})$ in the metric space $\mathbb{T}(\omega)$. It also implies that

$$\mathbb{T}(\omega) = \bigcup_{\delta>0} \mathbb{T}_\delta(\omega).$$

Besides, for each $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that

$$\boldsymbol{\mathcal{G}}(\mathbb{T}_\delta(\omega)) \subset \{\dot{\boldsymbol{\theta}} \in \dot{W}^{2,p}(\omega; \mathbb{E}^3);\ \boldsymbol{\theta} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)\},$$

where $\boldsymbol{\mathcal{G}}$ denotes the mapping defined in part (c) of the statement of the theorem and $V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ is defined as in Theorem 3.

On the other hand, Theorem 3 implies that there exists a constant $c = c(\omega, p, \varepsilon(\delta))$ such that

$$\inf_{\boldsymbol{r} \in \mathbf{Isom}_+(\mathbb{E}^3)} \|\boldsymbol{\varphi} - \boldsymbol{r} \circ \boldsymbol{\psi}\|_{W^{2,p}(\omega;\mathbb{E}^3)} \leq c \left\{\|(\hat{a}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{a}_{\alpha\beta}(\boldsymbol{\psi}))\|_{W^{1,p}(\omega;\mathbb{S}^2)} + \|(\hat{b}_{\alpha\beta}(\boldsymbol{\varphi}) - \hat{b}_{\alpha\beta}(\boldsymbol{\psi}))\|_{L^p(\omega;\mathbb{S}^2)}\right\}$$

for all mappings $\boldsymbol{\varphi} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ and $\boldsymbol{\psi} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ (note that Theorem 3 can be applied under the assumptions of Theorem 5 since a domain satisfies the uniform interior cone property).

We then infer from the observations above that, given any mappings $\boldsymbol{\varphi} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ and $\tilde{\boldsymbol{\varphi}} \in V_{\varepsilon(\delta)}(\omega; \mathbb{E}^3)$ such that $\dot{\boldsymbol{\varphi}} = \boldsymbol{\mathcal{G}}(\boldsymbol{A}, \boldsymbol{B})$ and $\dot{\tilde{\boldsymbol{\varphi}}} = \boldsymbol{\mathcal{G}}(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})$ for some $(\boldsymbol{A}, \boldsymbol{B}) \in \mathbb{T}_\delta(\omega)$ and $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}) \in \mathbb{T}_\delta(\omega)$,

$$\operatorname{dist}_{\dot{W}^{2,p}(\omega;\mathbb{E}^3)}(\dot{\boldsymbol{\varphi}}, \dot{\tilde{\boldsymbol{\varphi}}}) \leq c \operatorname{dist}_{\mathbb{T}(\omega)}((\boldsymbol{A}, \boldsymbol{B}), (\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})).$$

This shows that the restriction of the mapping $\boldsymbol{\mathcal{G}}$ to the set $\mathbb{T}_\delta(\omega)$ is Lipschitz-continuous. □

Acknowledgements

The work described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. 9042388, CityU 11305716].

References

[1] M.P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, 1976.
[2] P.G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, SIAM, Philadelphia, 2013.
[3] P.G. Ciarlet, S. Mardare, Nonlinear Korn inequalities in $\mathbb{R}^n$ and immersions in $W^{2,p}$, $p > n$, considered as functions of their metric tensors in $W^{1,p}$, J. Math. Pures Appl. 105 (2016) 873–906.
[4] P.G. Ciarlet, C. Mardare, A surface in $W^{2,p}$ is a locally Lipschitz-continuous function of its fundamental forms in $W^{1,p}$ and L^p, $p > 2$, in preparation.
[5] W. Klingenberg, Eine Vorlesung über Differentialgeometrie, Springer, Berlin, 1973. English translation: A Course in Differential Geometry, Springer, Berlin, 1978.
[6] S. Mardare, On Pfaff systems with L^p coefficients and their applications in differential geometry, J. Math. Pures Appl. 84 (2005) 1659–1692.
[7] S. Mardare, On systems of first order linear partial differential equations with L^p coefficients, Adv. Differ. Equ. 12 (2007) 301–360.