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Formality theorem for differential graded manifolds ☆

Théorème de formalité pour les variétés différentielles graduées

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A R T I C L E I N F O

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A B S T R A C T

We establish a formality theorem for smooth dg manifolds. More precisely, we prove that, for any finite-dimensional dg manifold $(\mathcal{M}, Q)$, there exists an L_∞ quasi-isomorphism of dglas from $\left({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}), [Q, -], [-, -]\right)$ to $\left({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M}), [\![m + Q, -]\!], [\![-, -]\!]\right)$ whose first Taylor coefficient (1) is equal to the composition $\text{hkr} \circ (\text{td}^\nabla_{(\mathcal{M},Q)})^{\frac{1}{2}} : {}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}) \to {}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M})$ of the action of $(\text{td}^\nabla_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} \left(\Omega^k(\mathcal{M})\right)^k$ on ${}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M})$ (by contraction) with the Hochschild–Kostant–Rosenberg map and (2) preserves the associative algebra structures on the level of cohomology. As an application, we prove the Kontsevich–Shoikhet conjecture: a Kontsevich–Duflo-type theorem holds for all finite-dimensional smooth dg manifolds.

R É S U M É

Nous prouvons un théorème de formalité pour les variétés lisses différentielles graduées. Plus précisément, nous prouvons qu'il existe, pour toute variété différentielle graduée $(\mathcal{M}, Q)$, un quasi-isomorphisme L_∞ de l'algèbre de Lie différentielle graduée $\left({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}), [Q, -], [-, -]\right)$ dans l'algèbre de Lie différentielle graduée $\left({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M}), [\![m + Q, -]\!], [\![-, -]\!]\right)$, dont le premier coefficient de Taylor (1) est égal à la composée $\text{hkr} \circ (\text{td}^\nabla_{(\mathcal{M},Q)})^{\frac{1}{2}} : {}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}) \to {}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M})$ de l'action (par contraction) de $(\text{td}^\nabla_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} \left(\Omega^k(\mathcal{M})\right)^k$ sur ${}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M})$ avec l'application de Hochschild–Kostant–Rosenberg et (2) respecte les structures d'algèbres associatives en cohomologie. Comme application, nous prouvons la conjecture de Kontsevich–Shoikhet : il existe un théorème de type Kontsevich–Duflo valable pour toute variété différentielle graduée de dimension finie.

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1. Introduction

In 1997, Kontsevich revolutionized the field of deformation quantization with his formality theorem: there exists an L_∞ quasi-isomorphism from the dgla $T_{\mathrm{poly}}(M)$ of polyvector fields on a smooth manifold M to the dgla $D_{\mathrm{poly}}(M)$ of polydifferential operators on M whose first "Taylor coefficient" is the classical Hochschild–Kostant–Rosenberg map [19]. Kontsevich's formality theorem completely settled a long-standing problem [3] regarding the existence and classification of deformation quantizations for all smooth Poisson manifolds. An alternative approach to the formality theorem was developed by Tamarkin using operads [31].

Beyond deformation quantization, Kontsevich's formality construction found other important applications in several different areas of mathematics. One of them is the extension of the classical Duflo theorem. Given a finite-dimensional Lie algebra $\mathfrak{g}$, the Poincaré–Birkhoff–Witt (PBW) map is the isomorphism of $\mathfrak{g}$-modules $\mathrm{pbw}: S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})$ defined by the symmetrization map $X_1 \odot \cdots \odot X_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n} X_{\sigma(1)}\cdots X_{\sigma(n)}$. It induces an isomorphism $\mathrm{pbw}\colon S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\cong} \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ between subspaces of $\mathfrak{g}$-invariants. This isomorphism fails to intertwine the obvious multiplications on $S(\mathfrak{g})^{\mathfrak{g}}$ and $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$. Nevertheless, it can be modified so as to become an isomorphism of associative algebras. The Duflo element $J \in \widehat{S}(\mathfrak{g}^\vee)$ is the formal power series on $\mathfrak{g}$ defined by $J(x) = \det\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right)$, for all $x \in \mathfrak{g}$. Considered as a formal linear differential operator on $\mathfrak{g}^\vee$ with constant coefficients, the square root of the Duflo element defines a transformation $J^{\frac{1}{2}} : S(\mathfrak{g}) \to S(\mathfrak{g})$. A remarkable theorem due to Duflo [14] asserts that the composition $\mathrm{pbw}\circ J^{\frac{1}{2}} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$ is an isomorphism of associative algebras. Duflo's theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. Duflo's original proof is based on deep and sophisticated techniques of representation theory including Kirillov's orbit method. As an application of his formality construction, Kontsevich proposed a new proof of Duflo's theorem by means of the associative algebra structure carried by the tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich's approach [19] led to an extension of Duflo's theorem: for every finite dimensional Lie algebra $\mathfrak{g}$, the map $\mathrm{pbw}\circ J^{\frac{1}{2}} : H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \to H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ is an isomorphism of graded associative algebras. The classical Duflo theorem is simply the isomorphism between the cohomology groups of degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [29] (see also [22,23]). Furthermore, Kontsevich discovered a similar phenomenon in complex geometry [19]. Recall that the Hochschild cohomology groups $HH^\bullet(X)$ of a complex manifold X are defined as the groups $\mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Gerstenhaber–Shack [18] derived an isomorphism of cohomology groups $\mathrm{hkr}: H^\bullet(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$ from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications on the two cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [19] obtained the following theorem: the composition $\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$, where Td_X denotes the Todd class of the complex manifold X, is an isomorphism of associative algebras. The multiplications on $H^\bullet(X, \Lambda^\bullet T_X)$ and $HH^\bullet(X)$ are respectively the wedge product and the Yoneda product. Calaque–Van den Bergh [6] wrote a detailed proof of Kontsevich's theorem and showed additionally that the map $\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}}$ actually respects the Gerstenhaber algebra structures carried by the two cohomologies. A related result was also proved by Dolgushev–Tamarkin–Tsygan [12,13].

Hence Kontsevich's formality construction revealed a hidden connection between complex geometry and Lie theory. Kontsevich's discovery of this mysterious and surprising similarity between the Todd class of a complex manifold and the Duflo element of a Lie algebra — two seemingly unrelated objects — was responsible for many subsequent exciting developments. Naturally, one would wonder whether a general framework encompassing both Lie algebras and complex manifolds as special cases could be developed in which a Kontsevich–Duflo-type theorem would hold. This is indeed the main goal of this Note. We claim that *differential graded (dg) manifolds* provide the appropriate framework.

By a dg manifold, we mean a $\mathbb{Z}$-graded manifold endowed with a homological vector field, i.e. a vector field Q of degree $+1$ satisfying $[Q, Q] = 0$. Dg manifolds arise naturally in many situations in geometry, Lie theory, and mathematical physics. Standard examples of dg manifolds are: (1) *Lie algebras* — Given a finite-dimensional Lie algebra $\mathfrak{g}$, we write $\mathfrak{g}[1]$ to denote the dg manifold having $C^\infty(\mathfrak{g}[1]) = \wedge^\bullet\mathfrak{g}^\vee$ as its algebra of functions and the Chevalley–Eilenberg differential $Q = d_{\mathrm{CE}}$ as its homological vector field. This construction admits an up-to-homotopy version: given a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ of finite type (i.e. each $\mathfrak{g}_i$ is a finite-dimensional vector space), $\mathfrak{g}[1]$ is a dg-manifold if and only if $\mathfrak{g}$ is a curved L_∞ algebra. (2) *Complex manifolds* — Given a complex manifold X, we write $T^{0,1}_X[1]$ to denote the dg manifold having $C^\infty(T^{0,1}_X[1]) \cong \Omega^{0,\bullet}(X)$ as its algebra of functions and the Dolbeault operator $Q = \bar\partial$ as its homological vector field. (3) *Derived intersections* — Given a smooth section s of a smooth vector bundle $E \to M$, we write $E[-1]$ to denote the dg-manifold having $C^\infty(E[-1]) = \Gamma(\wedge^{-\bullet}(E^\vee))$ as its algebra of functions and the contraction operator i_s as its homological vector field.

In 1998, Shoikhet [30] proposed a conjecture, known as *Kontsevich–Shoikhet conjecture*, stating that a Kontsevich–Duflo-type formula holds for all finite-dimensional smooth dg manifolds. In this Note, we prove a formality theorem for smooth dg manifolds (Theorem 4.2) and, as an immediate consequence, we confirm the Kontsevich–Shoikhet conjecture (Theorem 4.3). Applying Theorem 4.3 to the dg manifold examples of type (1) and (2) mentioned earlier, we recover the Kontsevich–Duflo theorem for Lie algebras and Kontsevich's theorem for complex manifolds, respectively. Thus we fulfill our stated goal of conceiving a unified framework in which these two important theorems can be understood as one and the same phenomenon.

Our approach is based on the construction of *Fedosov dg Lie algebroids*, a concept of likely independent interest inspired by Dolgushev's proof of Kontsevich's global formality theorem for smooth manifolds. We start by constructing, firstly, a homological vector field D_Q^{∇} on the $\mathbb{Z}$-graded manifold $\mathcal{N} = T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}$ such that the dg manifold $\mathcal{N}_Q = (\mathcal{N}, D_Q^{\nabla})$ is weakly equivalent to $(\mathcal{M}, Q)$ and, secondly, a Fedosov dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$, i.e. a dg Lie subalgebroid of $T_{\mathcal{N}_Q} \to \mathcal{N}_Q$ — a dg foliation of the dg manifold $\mathcal{N}_Q$ — "homotopy equivalent" to the tangent dg Lie algebroid of the dg manifold $(\mathcal{M}, Q)$. Then we apply the $\mathbb{Z}$-graded version [7,32] of Kontsevich's formality quasi-isomorphism essentially leafwise on $\mathcal{N}$ w.r.t. the $\mathcal{F}$-foliation to prove our main results.

The formality theorem was transposed to the context of $\mathbb{Z}$-graded manifolds by Cattaneo–Felder in connection with their study of the quantization of coisotropic submanifolds [7]. Our formality theorem for dg manifolds reduces to Cattaneo–Felder's when the homological vector field is trivial. Conversely, subsequently applying Cattaneo–Felder's formality theorem to the $\mathbb{Z}$-graded manifold $\mathcal{M}$ and then considering the tangent L_∞ morphism between the tangent complexes at the homological vector field Q seen as a Maurer–Cartan element ought to yield the formality theorem for dg manifolds $(\mathcal{M}, Q)$. However the novelty of our approach compared to Cattaneo–Felder's resides in the explicit expression for the first Taylor coefficient of the formality quasi-isomorphism for the dg manifold $(\mathcal{M}, Q)$, which plays an essential role in our proof of the Kontsevich–Shoikhet conjecture. The precise relation between our formality theorem and Cattaneo–Felder's will be studied somewhere else. It is worth noting that, although both our and Cattaneo–Felder's approaches make use of Fedosov resolutions, the two approaches are significantly different: while Cattaneo–Felder resolve the subalgebra of functions generated by the coordinate functions on the support manifold (which is a genuine smooth manifold) and essentially ignore the virtual $\mathbb{Z}$-graded coordinates, we resolve the whole algebra of functions of $\mathcal{M}$ w.r.t. to all coordinates, both genuine and virtual, by way of the formal exponential map introduced in [20] by the first two authors.

We conclude with a short comparison of the present work with the work of Calaque–Rossi [5]. Following Shoikhet [30], Calaque–Rossi gave a detailed proof of the Kontsevich–Duflo theorem for "Q-spaces," i.e. $\mathbb{Z}_2$-graded *vector spaces* endowed with a homological vector field. Then they applied this result to the Q-space $(\mathfrak{g}[1], d_{\mathrm{CE}})$ so as to recover the Kontsevich–Duflo theorem for the finite-dimensional Lie algebra $\mathfrak{g}$. On the other hand, given a complex manifold X, they proved the analogue in complex geometry of the Duflo-type theorem by applying their "Kontsevich–Duflo theorem for Q-spaces" to the resolutions $\Omega^\bullet(X, \mathcal{O})$, $\Omega^\bullet(X, \mathcal{T}_{\mathrm{poly}})$, and $\Omega^\bullet(X, \mathcal{D}_{\mathrm{poly}})$ of the $\mathcal{O}$-modules $\Omega^{0,\bullet}(X)$, $\Omega^{0,\bullet}(X, T'_{\mathrm{poly}})$, and $\Omega^{0,\bullet}(X, D'_{\mathrm{poly}})$ constructed in [4]. These resolutions are straightforward adaptations to the context of complex manifolds of corresponding resolutions valid in the context of smooth manifolds, constructed by Dolgushev [11], and inspired by Fedosov's iterative procedure [16]. Here, however, we prove a single unified Kontsevich–Duflo-type theorem (Theorem 4.3) valid for *all* finite-dimensional smooth dg manifolds. Then we specialize this result to two important classes of dg manifolds — (1) the dg manifolds $(\mathfrak{g}[1], d_{\mathrm{CE}})$ arising from finite-dimensional Lie algebras and (2) the dg manifolds $(T_X^{0,1}[1], \bar{\partial})$ arising from complex manifolds — so as to recover (1) the Kontsevich-Duflo theorem for finite dimensional Lie algebras (Theorem 7.1) and (2) its analogue in complex geometry discovered by Kontsevich (Theorem 7.4). The correction to the HKR map is precisely the Todd class of the dg manifold, which is a function of the Atiyah class of the dg manifold [28]. The Atiyah class of a dg manifold $(\mathcal{M}, Q)$, which was introduced by Mehta and two of the authors [28], captures exactly the obstruction to the existence of a connection on $\mathcal{M}$ compatible with the homological vector field Q. This obstruction is thus clearly identified as the origin of the correction to the HKR map in the Kontsevich–Duflo-type theorem. We would like to warn the reader that, even in the complex manifold case, i.e. in the case of the dg manifold $(T_X^{0,1}[1], \bar{\partial})$, the resolutions we construct in Section 5 via the Fedosov dg Lie algebroid are totally different from those used by Calaque–Rossi [5] and Calaque–Van den Bergh [6]. In fact, in this Note, we prove a formality theorem valid for *all* finite-dimensional smooth dg manifolds (Theorem 4.2) and deduce from it our single unified Kontsevich–Duflo-type theorem as a straightforward corollary. We expect that our formality theorem will find applications to the deformation theory of dg manifolds. Furthermore, we expect to obtain a number of new and interesting Duflo type theorems by applying our unified Kontsevich–Duflo-type theorem to various classes of dg manifolds. In particular, in a forthcoming paper, we will establish a Duflo-type theorem valid for arbitrary Lie algebroids.

2. Preliminaries

Throughout this Note, we use the symbol $\Bbbk$ to denote either of the fields $\mathbb{R}$ and $\mathbb{C}$. A $\mathbb{Z}$-graded manifold $\mathcal{M}$ over $\Bbbk$ is a sheaf $\mathcal{O}_{\mathcal{M}}$ of $\mathbb{Z}$-graded, graded-commutative $\Bbbk$-algebras over a smooth manifold M such that every point of M admits an open neighborhood U for which $\mathcal{O}_{\mathcal{M}}(U)$ is isomorphic to $C^\infty(U, \Bbbk) \otimes_{\Bbbk} \widehat{S}(V^\vee)$ for some fixed $\mathbb{Z}$-graded vector space V over the field $\Bbbk$. The manifold M is called the *support* of the graded manifold $\mathcal{M}$. Here $V^\vee$ denotes the $\Bbbk$-dual of V and $\widehat{S}(V^\vee)$ denotes the $\Bbbk$-algebra of formal power series on V. A $\mathbb{Z}$-graded manifold $\mathcal{M}$ is finite-dimensional if both $\dim M$ and $\dim V$ are finite. We write either $\mathcal{R}$ or $C^\infty(\mathcal{M})$ for the algebra of global functions $\mathcal{O}_{\mathcal{M}}(M)$. We refer the reader to [27, Chapter 2] for a short introduction to $\mathbb{Z}$-graded manifolds. In this Note, the word "graded" means "$\mathbb{Z}$-graded" and, unless otherwise stated, the notation $|-|$ denotes the total degree of its argument.

2.1. Polyvector fields

By a vector field on a $\mathbb{Z}$-graded manifold $\mathcal{M}$ over $\Bbbk$, we mean a $\Bbbk$-linear (graded) derivation of $\mathcal{R}$. The tangent bundle $T_{\mathcal{M}}$ of a graded manifold $\mathcal{M}$ is itself a graded manifold and is the total space of a vector bundle object in the category of graded manifolds whose space of sections $\Gamma(T_{\mathcal{M}})$ consists of all vector fields on $\mathcal{M}$.

We use the symbol $\big(\mathcal{T}^{-1}_{\text{poly}}(\mathcal{M})\big)^q$ to denote the space of smooth functions of degree q on $\mathcal{M}$ and the symbol $\big(\mathcal{T}^{p}_{\text{poly}}(\mathcal{M})\big)^q$ to denote the space $\big(\Gamma(\Lambda^{p+1}T_{\mathcal{M}})\big)^q$ of $(p+1)$-vector fields of degree q on $\mathcal{M}$. In other words, an element in $\big(\mathcal{T}^{p}_{\text{poly}}(\mathcal{M})\big)^q$ is a finite sum $\sum X_0\wedge\cdots\wedge X_p$, where $X_0,\cdots,X_p\in\Gamma(T_{\mathcal{M}})$ are homogeneous vector fields on $\mathcal{M}$ with $|X_0|+\cdots+|X_p|=q$. The bigraded left $\mathcal{R}$-module $\big(\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\big)^\bullet=\bigoplus_{\substack{p,q\in\mathbb{Z}\\ p\geqslant -1}}\big(\mathcal{T}^{p}_{\text{poly}}(\mathcal{M})\big)^q$ is called the space of polyvector fields on $\mathcal{M}$. We are most interested in the graded left $\mathcal{R}$-module ${}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})$ defined by ${}_{\oplus}\mathcal{T}^{n}_{\text{poly}}(\mathcal{M})=\bigoplus_{p+q=n}\big(\mathcal{T}^{p}_{\text{poly}}(\mathcal{M})\big)^q$.

When endowed with the graded commutator $[-,-]$, the space $\big(\mathcal{T}^{0}_{\text{poly}}(\mathcal{M})\big)^\bullet=\big(\operatorname{Der}(\mathcal{R})\big)^\bullet$ of graded derivations of $\mathcal{R}$ is a graded Lie algebra. The Lie bracket on the space $\big(\mathcal{T}^{0}_{\text{poly}}(\mathcal{M})\big)^\bullet$ of graded vector fields on $\mathcal{M}$ can be extended to the Schouten bracket $[-,-]$ on the space $\big(\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\big)^\bullet$ of graded polyvector fields on $\mathcal{M}$ in such a way that the triple $\big({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}),[-,-],\wedge\big)$ becomes a Gerstenhaber algebra:

$$[\xi,\eta_1\wedge\eta_2]=[\xi,\eta_1]\wedge\eta_2+(-1)^{|\xi||\eta_1|}\eta_1\wedge[\xi,\eta_2],$$

for $\xi\in\big(\mathcal{T}^{p_0}_{\text{poly}}(\mathcal{M})\big)^{q_0}$, $\eta_1\in\big(\mathcal{T}^{p_1}_{\text{poly}}(\mathcal{M})\big)^{q_1}$, $\eta_2\in\big(\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\big)^{\bullet}$, and $|\xi|=p_0+q_0$, $|\eta_1|=p_1+q_1$. Finally, adding the zero differential $0:{}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\to{}_{\oplus}\mathcal{T}^{\bullet+1}_{\text{poly}}(\mathcal{M})$, we obtain the dg Gerstenhaber algebra $\big({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}),0,[-,-],\wedge\big)$.

2.2. Polydifferential operators

A linear differential operator of degree q on $\mathcal{M}$ is a $\Bbbk$-linear endomorphism of $\mathcal{R}$ that can be obtained as a finite sum $\sum X_1\circ\cdots\circ X_k$ of compositions of graded derivations $X_1,\ldots,X_k$ of $\mathcal{R}$ with $|X_1|+\cdots+|X_k|=q$. We use the symbol $\big(\mathcal{D}(\mathcal{M})\big)^q$ to denote the space of linear differential operators of degree q on $\mathcal{M}$.

The space $\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})$ of $(p+1)$-differential operators on $\mathcal{M}$ admits a canonical identification with the tensor product of $(p+1)$ copies of the left $\mathcal{R}$-module $\mathcal{D}(\mathcal{M})$ of all linear differential operators on $\mathcal{M}$. We use the symbol $\big(\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})\big)^q$ to denote the space $\bigoplus_{q_0+\cdots+q_p=q}\big(\mathcal{D}(\mathcal{M})\big)^{q_0}\otimes_{\mathcal{R}}\cdots\otimes_{\mathcal{R}}\big(\mathcal{D}(\mathcal{M})\big)^{q_p}$, which must be thought of as the space of $(p+1)$-differential operators of degree q on $\mathcal{M}$, and the symbol $\big(\mathcal{D}^{-1}_{\text{poly}}(\mathcal{M})\big)^q$ to denote the space of smooth functions of degree q on $\mathcal{M}$.

The bigraded left $\mathcal{R}$-module $\big(\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})\big)^\bullet=\bigoplus_{\substack{p,q\in\mathbb{Z}\\ p\geqslant -1}}\big(\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})\big)^q$ is called the space of polydifferential operators on $\mathcal{M}$. We are most interested in the graded left $\mathcal{R}$-module ${}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})$ defined by ${}_{\oplus}\mathcal{D}^{n}_{\text{poly}}(\mathcal{M})=\bigoplus_{p+q=n}\big(\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})\big)^q$.

As in the classical case, endowing the space of polydifferential operators ${}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})$ with the Gerstenhaber bracket $[\![-,-]\!]$ and the Hochschild differential $[\![m,-]\!]:\big(\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})\big)^q\to\big(\mathcal{D}^{p+1}_{\text{poly}}(\mathcal{M})\big)^q$ makes it a dgla. The tensor product of left $\mathcal{R}$-modules determines a cup product $\big(\mathcal{D}^{p}_{\text{poly}}(\mathcal{M})\big)^q\times\big(\mathcal{D}^{p'}_{\text{poly}}(\mathcal{M})\big)^{q'}\xrightarrow{\smile}\big(\mathcal{D}^{p+p'+1}_{\text{poly}}(\mathcal{M})\big)^{q+q'}$, which descends to Hochschild cohomology. When endowed with the cup product and the Gerstenhaber bracket, the cohomology of the cochain complex $({}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M}),[\![m,-]\!])$ becomes a Gerstenhaber algebra [7, Appendix].

3. Atiyah and Todd classes of a dg manifold

By a dg manifold, we mean a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a homological vector field, i.e. a vector field Q of degree $+1$ such that $[Q,Q]=0$. A dg vector bundle is a vector bundle in the category of dg manifolds. We refer the reader to [28,26] for details on dg vector bundles.

The space of sections $\Gamma(\mathcal{E})$ of a vector bundle of graded manifolds $\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ is defined to be the direct sum $\bigoplus_{j\in\mathbb{Z}}\Gamma^j(\mathcal{E})$ of the spaces $\Gamma^j(\mathcal{E})$ consisting of all degree preserving maps $s\in\operatorname{Hom}(\mathcal{M},\mathcal{E}[-j])$ such that $(\pi[-j])\circ s=\operatorname{id}_{\mathcal{M}}$, where $\pi[-j]:\mathcal{E}[-j]\to\mathcal{M}$ is the natural (graded) map determined by π – see [26] for more details. When $\mathcal{E}\to\mathcal{M}$ is a dg vector bundle, the homological vector fields on $\mathcal{E}$ and $\mathcal{M}$ naturally induce a degree $+1$ operator $\mathcal{Q}$ on $\Gamma(\mathcal{E})$, making $\Gamma(\mathcal{E})$ a dg module over $C^\infty(\mathcal{M})$. The converse is also true: since the space $C^\infty(\mathcal{M})$ and the space $\Gamma(\mathcal{E}^\vee)$ of linear functions on $\mathcal{E}$ together generate the ring $C^\infty(\mathcal{E})$, the homological vector field on $\mathcal{E}$ can be reconstructed from $\mathcal{Q}$ and the homological vector field on $\mathcal{M}$.

A dg Lie algebroid is a Lie algebroid object in the category of dg manifolds. A dg Lie algebroid $\mathcal{A}\to\mathcal{M}$ is thus a dg vector bundle together with a bundle map $\rho:\mathcal{A}\to T_{\mathcal{M}}$, called anchor, and a structure of graded Lie algebra on $\Gamma(\mathcal{A})$ with Lie bracket satisfying

$$[X,fY]=\rho(X)(f)Y+(-1)^{|X||f|}f[X,Y]$$

for all homogeneous $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$ and such that anchor and Lie bracket are 'compatible' with the homological vector fields on $\mathcal{A}$ and $\mathcal{M}$. An $\mathcal{A}$-connection on a graded vector bundle $\mathcal{E} \to \mathcal{M}$ is a degree 0 map $\nabla : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ satisfying the pair of relations

$$\nabla_{fX} s = f \nabla_X s,$$
$$\nabla_X (fs) = \big(\rho(X) f\big) s + (-1)^{|X||f|} f \nabla_X s,$$

for all homogeneous elements $f \in C^\infty(\mathcal{M})$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(\mathcal{E})$. Such connections always exist since the standard partition of unity argument holds in the context of graded manifolds. Given a dg vector bundle $\mathcal{E} \to \mathcal{M}$ and an $\mathcal{A}$-connection ∇ on it, we can consider the bundle map $\mathrm{at}^\nabla_{\mathcal{E}} : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ defined by

$$\mathrm{at}^\nabla_{\mathcal{E}}(X, s) = \mathcal{Q}(\nabla_X s) - \nabla_{\mathcal{Q}(X)} s - (-1)^{|X|} \nabla_X \big(\mathcal{Q}(s)\big), \quad \forall X \in \Gamma(\mathcal{A}), s \in \Gamma(\mathcal{E}).$$

The bundle map $\mathrm{at}^\nabla_{\mathcal{E}}$ can be regarded as a section of degree $+1$ of $\mathcal{A}^\vee \otimes \mathrm{End}\,\mathcal{E}$. It is actually a cocycle: $\mathcal{Q}(\mathrm{at}^\nabla_{\mathcal{E}}) = 0$. Furthermore, the cohomology class of $\mathrm{at}^\nabla_{\mathcal{E}}$ is independent of the choice of the connection ∇ [28]. Thus we obtain a natural cohomology class $\alpha_{\mathcal{E}} = [\mathrm{at}^\nabla_{\mathcal{E}}]$ in $H^1\big(\Gamma(\mathcal{A}^\vee \otimes \mathrm{End}\,\mathcal{E}), \mathcal{Q}\big)$ called *Atiyah class* of the dg vector bundle $\mathcal{E} \to \mathcal{M}$ relative to the dg Lie algebroid $\mathcal{A} \to \mathcal{M}$. The *Todd cocycle* $\mathrm{td}^\nabla_{\mathcal{E}}$ and the *$\widehat{A}$ cocycle* $\widetilde{\mathrm{td}}^\nabla_{\mathcal{E}}$ of the dg vector bundle $\mathcal{E} \to \mathcal{M}$ associated with the $\mathcal{A}$-connection ∇ are the elements of $\prod_{k \ge 0} \big(\Gamma(\Lambda^k \mathcal{A}^\vee)\big)^k$ defined by

$$\mathrm{td}^\nabla_{\mathcal{E}} = \mathrm{Ber}\left(\frac{\mathrm{at}^\nabla_{\mathcal{E}}}{1 - \mathrm{e}^{-\mathrm{at}^\nabla_{\mathcal{E}}}}\right) \qquad \text{and} \qquad \widetilde{\mathrm{td}}^\nabla_{\mathcal{E}} = \mathrm{Ber}\left(\frac{\mathrm{at}^\nabla_{\mathcal{E}}}{\mathrm{e}^{\frac{1}{2}\mathrm{at}^\nabla_{\mathcal{E}}} - \mathrm{e}^{-\frac{1}{2}\mathrm{at}^\nabla_{\mathcal{E}}}}\right),$$

where Ber denotes the Berezinian [25]. They satisfy the cocycle condition $\mathcal{Q}(\mathrm{td}^\nabla_{\mathcal{E}}) = 0 = \mathcal{Q}(\widetilde{\mathrm{td}}^\nabla_{\mathcal{E}})$. Note that every element of $\big(\Gamma(\Lambda^k \mathcal{A}^\vee)\big)^k$ is a finite sum $\sum \alpha_1 \wedge \cdots \wedge \alpha_k$ with $\alpha_1, \cdots, \alpha_k \in \Gamma(\mathcal{A}^\vee)$ homogeneous and satisfying the degree condition $|\alpha_1| + \cdots + |\alpha_k| = k$. The respective cohomology classes $\mathrm{Td}_{\mathcal{E}}$ and $\widetilde{\mathrm{Td}}_{\mathcal{E}}$ in $\prod_{k \ge 0} H^k\big((\Gamma(\Lambda^k \mathcal{A}^\vee))^\bullet, \mathcal{Q}\big)$ of the cocycles $\mathrm{td}^\nabla_{\mathcal{E}}$ and $\widetilde{\mathrm{td}}^\nabla_{\mathcal{E}}$ are independent of the choice of the connection ∇ and are respectively called the *Todd class* and *$\widehat{A}$ class* of the dg vector bundle $\mathcal{E} \to \mathcal{M}$ relative to the dg Lie algebroid $\mathcal{A}$. When $\mathcal{A} = T_{\mathcal{M}}$, we will often write $\Omega^k(\mathcal{M})$ instead of $\Gamma(\Lambda^k T^\vee_{\mathcal{M}})$.

Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. Its tangent bundle $T_{\mathcal{M}}$ is naturally a dg vector bundle; the homological vector field $\widehat{Q}$ on $T_{\mathcal{M}}$ is the tangent lift of Q and the differential on $\Gamma(\mathcal{M}; T_{\mathcal{M}})$ determined by the homological vector fields $\widehat{Q}$ and Q is the Lie derivative $\mathcal{L}_Q$ [28]. Actually, $T_{\mathcal{M}}$ is naturally a dg Lie algebroid. Its Atiyah class, relative to $T_{\mathcal{M}}$ itself, is called the Atiyah class of the dg manifold $(\mathcal{M}, Q)$. More precisely, an affine connection on a graded manifold $\mathcal{M}$ is a $T_{\mathcal{M}}$-connection on the graded vector bundle $T_{\mathcal{M}}$. The torsion of an affine connection ∇ is the tensor $T^\nabla : \Gamma(T_{\mathcal{M}}) \times \Gamma(T_{\mathcal{M}}) \to \Gamma(T_{\mathcal{M}})$ defined by the formula

$$T^\nabla(X, Y) = \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y],$$

for all homogeneous vector fields $X, Y \in \Gamma(T_{\mathcal{M}})$. We say that an affine connection is torsion-free if its torsion vanishes. A torsion-free affine connection always exists since the standard existence argument still holds in the context of graded manifolds. Given a dg manifold $(\mathcal{M}, Q)$ and an affine connection ∇, the Atiyah cocycle $\mathrm{at}^\nabla_{(\mathcal{M},Q)} \in \Gamma(T^\vee_{\mathcal{M}} \otimes \mathrm{End}\, T_{\mathcal{M}})$ is the section of degree $+1$ characterized by

$$\mathrm{at}^\nabla_{(\mathcal{M},Q)}(X, Y) = \mathcal{L}_Q(\nabla_X Y) - \nabla_{\mathcal{L}_Q(X)} Y - (-1)^{|X|} \nabla_X \big(\mathcal{L}_Q(Y)\big), \quad \forall X, Y \in \Gamma(T_{\mathcal{M}}).$$

The Atiyah class of $(\mathcal{M}, Q)$ is the cohomology class $\alpha_{(\mathcal{M},Q)} = [\mathrm{at}^\nabla_{(\mathcal{M},Q)}] \in H^1(\Gamma(T^\vee_{\mathcal{M}} \otimes \mathrm{End}\, T_{\mathcal{M}}), L_Q)$, which is independent of the choice of the connection ∇. We define the *Todd cocycle* $\mathrm{td}^\nabla_{(\mathcal{M},Q)}$ of the dg manifold $(\mathcal{M}, Q)$ associated with the affine connection ∇ by

$$\mathrm{td}^\nabla_{(\mathcal{M},Q)} = \mathrm{Ber}\left(\frac{\mathrm{at}^\nabla_{(\mathcal{M},Q)}}{1 - \mathrm{e}^{-\mathrm{at}^\nabla_{(\mathcal{M},Q)}}}\right) \in \prod_{k \ge 0} \big(\Gamma(\Lambda^k T^\vee_{\mathcal{M}})\big)^k \cong \prod_{k \ge 0} \big(\Omega^k(\mathcal{M})\big)^k.$$

Its cohomology class $\mathrm{Td}_{(\mathcal{M},Q)} \in \prod_{k \ge 0} H^k\big((\Gamma(\Lambda^k T^\vee_{\mathcal{M}}))^\bullet, \mathcal{L}_Q\big)$, which is independent of the choice of the connection ∇, will be referred to as the *Todd class* of the dg manifold $(\mathcal{M}, Q)$.

4. Formality and Kontsevich–Duflo-type theorem for dg manifolds

Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. Since Q is a homological vector field of degree $+1$, it is a Maurer–Cartan element in the dgla of polyvector fields $(\oplus \mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}), 0, [-,-])$. Therefore, we can consider the tangent dgla at the homological vector field Q:

$$\big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)_Q = \big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [Q,-], [-,-]\big).$$

The associated cohomology $\mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), [Q,-]\big)$ is a Gerstenhaber algebra with the associative multiplication induced by the wedge product. Likewise, we can consider the tangent dgla at the Maurer–Cartan element Q of the dgla of polydifferential operators:

$$\big({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)_Q = \big({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M}), [\![m+Q,-]\!], [\![-,-]\!]\big).$$

The corresponding cohomology $\mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), [\![m+Q,-]\!]\big)$ is a Gerstenhaber algebra with the cup product as associative multiplication.

The Hochschild–Kostant–Rosenberg map $\mathrm{hkr}: \big(\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet} \hookrightarrow \big(\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}$ is the natural inclusion of $\big(\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}$ into $\big(\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}$ defined by skew-symmetrization:

$$\mathrm{hkr}(X_0 \wedge \cdots \wedge X_p) = \sum_{\sigma \in S_{p+1}} \kappa(\sigma)\, X_{\sigma(0)} \otimes \cdots \otimes X_{\sigma(p)}$$

for all homogeneous vector fields $X_0, \cdots, X_p \in \big(\mathcal{T}^{0}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}$ – the skew Koszul sign $\kappa(\sigma)$ is the scalar defined by the relation $X_0 \wedge \cdots \wedge X_p = \kappa(\sigma)\, X_{\sigma(0)} \wedge \cdots \wedge X_{\sigma(p)}$. The Hochschild–Kostant–Rosenberg map is a morphism of double complexes

$$\mathrm{hkr}: \big(\big(\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}, 0, [Q,-]\big) \to \big(\big(\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)^{\bullet}, [\![m,-]\!], [\![Q,-]\!]\big). \tag{1}$$

The following Hochschild–Kostant–Rosenberg theorem for dg manifolds follows from the HKR theorem for graded manifolds [7, Lemma A.2] and a spectral sequence argument.

Proposition 4.1. *Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. The Hochschild–Kostant–Rosenberg map (1) induces an isomorphism of vector spaces*

$$\mathrm{hkr}: \mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), [Q,-]\big) \xrightarrow{\cong} \mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), [\![m+Q,-]\!]\big)$$

on the cohomology level.

Remark. Proposition 4.1 holds for direct sum total cohomology. The analogous assertion for direct product total cohomology is false; a counterexample can be found in [9].

The following theorem is the main result of this Note.

Theorem 4.2 *(Formality theorem for dg manifolds). Let $(\mathcal{M}, Q)$ be a finite-dimensional dg manifold. Given a torsion-free affine connection ∇ on $\mathcal{M}$, there exists an L_∞ quasi-isomorphism of dglas*

$$\mathcal{I}: \big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)_Q \rightsquigarrow \big({}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)_Q$$

with first Taylor coefficient $\mathcal{I}_1 : {}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M}) \to {}_{\oplus}\mathcal{D}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ satisfying the following two properties:

(1) *$\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) on the level of cohomology;*
(2) *$\mathcal{I}_1 = \mathrm{hkr} \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}}$, where $(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} \big(\Gamma(\Lambda^k T^{\vee}_{\mathcal{M}})\big)^k \cong \prod_{k\geq 0} \big(\Omega^k(\mathcal{M})\big)^k$ acts on ${}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})$ by contraction.*

Remark. Given a pair of torsion-free affine connections ∇ and ∇' on $(\mathcal{M}, Q)$ with corresponding Todd cocycles $\mathrm{td}^{\nabla}_{(\mathcal{M},Q)}$ and $\mathrm{td}^{\nabla'}_{(\mathcal{M},Q)}$, there exists, according to Proposition 6.8, an L_∞ automorphism of the dgla $\big({}_{\oplus}\mathcal{T}^{\bullet}_{\mathrm{poly}}(\mathcal{M})\big)_Q$ having the operator $\big(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\nabla'}_{(\mathcal{M},Q)}\big)^{\frac{1}{2}}$ as first Taylor coefficient.

As an immediate consequence, we obtain the following confirmation of the Kontsevich–Shoikhet conjecture:

Theorem 4.3 *(Kontsevich–Duflo-type theorem for dg manifolds [30]). For any finite-dimensional dg manifold $(\mathcal{M}, Q)$, the composition*

$$\mathrm{hkr} \circ (\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}} : \mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), [Q,-]\big) \xrightarrow{\cong} \mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), [\![m+Q,-]\!]\big)$$

of (1) the action of $(\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} H^k\big((\Omega^k(\mathcal{M}))^{\bullet}, \mathcal{L}_Q\big)$ on $\mathbb{H}^{\bullet}\big({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), [Q,-]\big)$, by contraction, with (2) the Hochschild–Kostant–Rosenberg map (on cohomology) is an isomorphism of Gerstenhaber algebras.

5. Fedosov dg Lie algebroids

Our proof of Theorem 4.2 is based on the construction of Fedosov dg Lie algebroids, objects that are very much inspired by the resolutions of smooth manifolds devised by Dolgushev for his construction of the global Kontsevich formality quasi-isomorphism [11] — see [8] for a different approach to the globalization of Kontsevich's formality quasi-isomorphism. Dolgushev constructed his resolutions by applying Fedosov's famous gluing technique — an effective procedure for assembling global objects out of local building blocks. Fedosov's method was itself motivated by and is, in some sense, essentially equivalent to formal geometry [16]. The version of Fedosov's method that is relevant to our purpose is the one developed by Emmrich–Weinstein [15] and later refined by Dolgushev [11] specifically for arbitrary ordinary smooth manifolds. Originally, the Dolgushev–Fedosov differentials were constructed by Fedosov's iterative method [15,11]. Recently, the first two authors proposed a different construction based on formal exponential maps [20].

Let us recall briefly the construction of [20]. Let $\mathcal{M}$ be a finite-dimensional $\mathbb{Z}$-graded manifold, let $(x_i)_{i\in\{1,\dots,n\}}$ be a set of local coordinates on $\mathcal{M}$ and let $(y_j)_{j\in\{1,\dots,n\}}$ be the induced local frame of $T^\vee_{\mathcal{M}}$ regarded as fiberwise linear functions on $T_{\mathcal{M}}$. We define

$$\delta : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p+1}(\mathcal{M}, S^{q-1} T^\vee_{\mathcal{M}})$$

and

$$h : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1} T^\vee_{\mathcal{M}})$$

by

$$\delta = \sum_{i=1}^{n} \mathrm{d}x_i \otimes \frac{\partial}{\partial y_i} \qquad \text{and} \qquad h = \frac{1}{p+q} \sum_{i=1}^{n} i_{\frac{\partial}{\partial x_i}} \otimes y_i$$

or, more precisely,

$$\delta(\omega \otimes f) = \sum_{i=1}^{n} (-1)^{\left|\frac{\partial}{\partial y_i}\right||\omega|} \mathrm{d}x_i \wedge \omega \otimes \frac{\partial}{\partial y_i}(f)$$

and

$$h(\omega \otimes f) = \tfrac{1}{p+q} \sum_{i=1}^{n} (-1)^{|y^i||\omega|} i_{\frac{\partial}{\partial x_i}} \omega \otimes y_i \cdot f$$

for all homogeneous $\omega \in \Omega^p(\mathcal{M})$ and for all $f \in \Gamma(S^q T^\vee_{\mathcal{M}})$. The operators δ and h are well defined, i.e. independent of the choice of local coordinates, and can be extended to $\Omega^\bullet\big(\mathcal{M}, \mathrm{End}(\widehat{S}(T^\vee_{\mathcal{M}}))\big)$. The operator δ has degree $+1$ while the operator h has degree -1.

An affine connection ∇ on the tangent bundle $T_{\mathcal{M}}$ of a graded manifold $\mathcal{M}$ determines a connection $\Gamma(T_{\mathcal{M}}) \times \Gamma(S(T_{\mathcal{M}})) \to \Gamma(S(T_{\mathcal{M}}))$ on $S(T_{\mathcal{M}})$, also denoted ∇ by abuse of notation. We use the symbol d^∇ to denote the covariant differential of the induced connection on the dual vector bundle $\widehat{S}(T^\vee_{\mathcal{M}})$.

The following proposition was proved in [20].

Proposition 5.1 *([20]). Given a torsion-free affine connection ∇ on the tangent bundle $T_{\mathcal{M}}$ of a $\mathbb{Z}$-graded manifold $\mathcal{M}$, there exists a unique element*

$$A^\nabla = \sum_{i=1}^{n} \sum_{\substack{J\in\mathbb{Z}^n_{\geqslant 0} \\ |J|\geqslant 2}} \sum_{k=1}^{n} A^i_{J,k} \mathrm{d}x_i \otimes y^J \frac{\partial}{\partial y_k}$$

of degree +1 in $\Omega^1\big(\mathcal{M}, \widehat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}\big)$ such that $h(A^\nabla) = 0$ and the operator

$$D^\nabla : \Omega^\bullet\big(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}})\big) \to \Omega^{\bullet+1}\big(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}})\big)$$

of degree $+1$ defined by $D^\nabla = -\delta + d^\nabla + A^\nabla$ satisfies $D^\nabla \circ D^\nabla = 0$.

Let $\mathcal{N} = T_{\mathcal{M}}[1] \oplus T_{\mathcal{M}}$ be the $\mathbb{Z}$-graded manifold with support T_M characterized by the function space $C^\infty(\mathcal{N}) = \Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}}))$. According to Proposition 5.1, $(\mathcal{N}, D^\nabla)$ is a dg manifold. It turns out that $(\mathcal{N}, D^\nabla)$ is in fact weakly equivalent to the $\mathbb{Z}$-graded manifold $\mathcal{M}$ (endowed with the zero homological vector field), i.e. $(C^\infty(\mathcal{N}), D^\nabla)$ is quasi-isomorphic to $C^\infty(\mathcal{M})$ seen as a cochain complex concentrated in degree 0. Indeed, it was proved in [20] that we have a contraction

$$\big(C^\infty(\mathcal{M}),0\big) \underset{\sigma}{\overset{\check{\tau}}{\rightleftarrows}} \big(\Omega^\bullet(\mathcal{M},\widehat{S}(T^\vee_{\mathcal{M}})),D^\nabla\big) \circlearrowright \check{h}$$

with homotopy operator $\check{h}:\Omega^\bullet(\mathcal{M},\widehat{S}(T^\vee_{\mathcal{M}}))\to\Omega^{\bullet-1}(\mathcal{M},\widehat{S}(T^\vee_{\mathcal{M}}))$ defined by the convergent series $\check{h}=\sum_{n=0}^\infty\big(h\circ(d^\nabla+A^\nabla)\big)^n\circ h$. Any such dg manifold $(\mathcal{N},D^\nabla)$ is called a *Fedosov dg manifold* associated with $\mathcal{M}$.

More precisely we have, for each $q\in\mathbb{Z}$, a contraction

$$\big(K^{\bullet,q},0\big) \underset{\sigma}{\overset{\check{\tau}}{\rightleftarrows}} \big(W^{\bullet,q},D^\nabla\big) \circlearrowright \check{h}\,, \tag{2}$$

where $K^{r,q}$ is the homogeneous component of degree q of $C^\infty(\mathcal{M})$ if $r=0$ and is zero otherwise while $W^{r,q}$ is the homogeneous component of degree $r+q$ of the subspace $\Gamma(\Lambda^r(T^\vee_{\mathcal{M}})\otimes\widehat{S}(T^\vee_{\mathcal{M}}))$ of $C^\infty(\mathcal{N})$.

Consider the surjective submersion $\mathcal{N}\twoheadrightarrow\mathcal{M}$. Let $\mathcal{F}\to\mathcal{N}$ denote the pullback of the vector bundle $T_{\mathcal{M}}\to\mathcal{M}$ through $\mathcal{N}\twoheadrightarrow\mathcal{M}$. It is a graded vector bundle whose total space $\mathcal{F}$ is a graded manifold with support T_M. Its space of sections $\Gamma(\mathcal{N};\mathcal{F})$ is the $C^\infty(\mathcal{N})$-module $C^\infty(\mathcal{N})\otimes_{\mathcal{R}}\mathfrak{X}(\mathcal{M})=\Omega^\bullet(\mathcal{M},\widehat{S}(T^\vee_{\mathcal{M}})\otimes T_{\mathcal{M}})$. It can be identified canonically to a $C^\infty(\mathcal{N})$-submodule of $\mathfrak{X}(\mathcal{N})$ as follows. Let $\partial_1,\dots,\partial_m$ and $\chi_1,\dots,\chi_m$ denote the dual local frames for $T_{\mathcal{M}}$ and $T^\vee_{\mathcal{M}}$ arising from a choice of local coordinates $(x_1,\dots,x_m)$ on $\mathcal{M}$. To $1\otimes\partial_k\in C^\infty(\mathcal{N})\otimes_{\mathcal{R}}\mathfrak{X}(\mathcal{M})=\Gamma(\mathcal{N};\mathcal{F})$ we associate the (graded) derivation of $C^\infty(\mathcal{N})$ mapping $\chi_j\in\Omega^0\big(\mathcal{M},S^1(T^\vee_{\mathcal{M}})\big)\subset C^\infty(\mathcal{N})$ to δ_{kj} and $\omega\in\Omega^p\big(\mathcal{M},S^0(T^\vee_{\mathcal{M}})\big)\subset C^\infty(\mathcal{N})$ to 0. Thus $\mathcal{F}\to\mathcal{N}$ is a vector subbundle of $T_{\mathcal{N}}\to\mathcal{N}$.

Lemma 5.2. *The pullback bundle $\mathcal{F}\to\mathcal{N}$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_{\mathcal{N}}\to\mathcal{N}$ of the Fedosov dg manifold $(\mathcal{N},D^\nabla)$.*

In other words, $\mathcal{F}$ is a dg foliation of the dg manifold $(\mathcal{N},D^\nabla)$. Each leaf of this foliation is essentially diffeomorphic to a fixed formal $\mathbb{Z}$-graded vector space: the standard fiber of the vector bundle $T_{\mathcal{M}}\to\mathcal{M}$. Each such dg Lie algebroid $\mathcal{F}\to\mathcal{N}$ is called a *Fedosov dg Lie algebroid* associated with the $\mathbb{Z}$-graded manifold $\mathcal{M}$.

Since the dg Lie algebroid $\mathcal{F}\to\mathcal{N}$ is a (dg) foliation on $\mathcal{N}$, its universal enveloping algebra $\mathcal{U}(\mathcal{F})$ can be thought of as the algebra of leafwise differential operators on $\mathcal{N}$ — see [33] for the general theory of the universal enveloping algebra of a Lie algebroid and [21, Appendix] for an instance of the extension of the concept to *dg* Lie algebroids. The universal enveloping algebra $\mathcal{U}(\mathcal{F})$ of the Fedosov dg Lie algebroid $\mathcal{F}\to\mathcal{N}$ can be identified in a natural way with the $C^\infty(\mathcal{N})$-module $\Gamma(\Lambda(T^\vee_{\mathcal{M}})\otimes\widehat{S}(T^\vee_{\mathcal{M}})\otimes S(T_{\mathcal{M}}))$.

For all $p\in\mathbb{Z}_{\geqslant-1}$, $q\in\mathbb{Z}$, and $r\in\mathbb{Z}_{\geqslant0}$, let $\mathcal{K}^{r,q,p}$ be $\big(\mathcal{T}^p_{\text{poly}}(\mathcal{M})\big)^q$ if $r=0$ and zero otherwise; let $\mathcal{W}^{r,q,p}$ be the homogeneous component of degree $r+q$ of the subspace $\Gamma(\Lambda^r(T^\vee_{\mathcal{M}})\otimes\widehat{S}(T^\vee_{\mathcal{M}})\otimes\Lambda^{p+1}T_{\mathcal{M}})$ of $\Gamma(\mathcal{N};\Lambda^{p+1}\mathcal{F})$; let $\mathscr{K}^{r,q,p}$ be $\big(\mathcal{D}^p_{\text{poly}}(\mathcal{M})\big)^q$ if $r=0$ and zero otherwise; and let $\mathscr{W}^{r,q,p}$ be the homogeneous component of degree $r+q$ of the subspace $\Gamma(\Lambda^r(T^\vee_{\mathcal{M}})\otimes\widehat{S}(T^\vee_{\mathcal{M}})\otimes(S(T_{\mathcal{M}}))^{\otimes p+1})$ of the tensor power $\mathcal{U}(\mathcal{F})^{\otimes p+1}$ of p+1 copies of the left (graded) $C^\infty(\mathcal{N})$-module $\mathcal{U}(\mathcal{F})$. Then, for each $p\in\mathbb{Z}_{\geqslant-1}$ and $q\in\mathbb{Z}$, there is a pair of contractions

$$\big(\mathcal{K}^{\bullet,q,p},0\big) \underset{\sigma_\natural}{\overset{\check{\tau}_\natural}{\rightleftarrows}} \big(\mathcal{W}^{\bullet,q,p},[D^\nabla,-]\big) \circlearrowright \check{h}_\natural \tag{3}$$

and

$$\big(\mathscr{K}^{\bullet,q,p},0\big) \underset{\sigma_\natural}{\overset{\check{\tau}_\natural}{\rightleftarrows}} \big(\mathscr{W}^{\bullet,q,p},[\![D^\nabla,-]\!]\big) \circlearrowright \check{h}_\natural \tag{4}$$

analogous to the contraction (2) above.

We note that

$$\oplus\mathcal{T}^n_{\text{poly}}(\mathcal{M})=\bigoplus_{\substack{p,q\in\mathbb{Z}\\p+q=n\\p\geqslant-1}}\mathcal{K}^{0,q,p}\qquad\text{and}\qquad\oplus\mathcal{D}^n_{\text{poly}}(\mathcal{M})=\bigoplus_{\substack{p,q\in\mathbb{Z}\\p+q=n\\p\geqslant-1}}\mathscr{K}^{0,q,p}.$$

Defining

$$\oplus\mathscr{T}^n_{\text{poly}}(\mathcal{F})=\bigoplus_{\substack{p,q,r\in\mathbb{Z}\\p+q+r=n\\p\geqslant-1,\ r\geqslant0}}\mathcal{W}^{r,q,p}\qquad\text{and}\qquad\oplus\mathscr{D}^n_{\text{poly}}(\mathcal{F})=\bigoplus_{\substack{p,q,r\in\mathbb{Z}\\p+q+r=n\\p\geqslant-1,\ r\geqslant0}}\mathscr{W}^{r,q,p},$$

we obtain two cochain complexes

$$\big(\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F}),[D^\nabla,-]\big)\qquad\text{and}\qquad\big(\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F}),[\![m+D^\nabla,-]\!]\big).$$

Taking direct sums of the contractions (3) and (4), we prove

Proposition 5.3. *Given a finite-dimensional $\mathbb{Z}$-graded manifold $\mathcal{M}$ and a torsion-free affine connection ∇ on $\mathcal{M}$, let $\mathcal{F} \to \mathcal{N}$ be the corresponding Fedosov dg Lie algebroid and let D^{∇} be the corresponding Fedosov homological vector field on $\mathcal{N}$. Then there are contractions*

(1) *at the level of polyvector fields:*

$$\left(\oplus \mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}), 0\right) \underset{\sigma_{\natural}}{\overset{\breve{\tau}_{\natural}}{\rightleftarrows}} \left(\oplus \mathscr{T}^{\bullet}_{\text{poly}}(\mathcal{F}), [D^{\nabla}, -]\right) \circlearrowleft \breve{h}_{\natural}$$

(2) *at the level of polydifferential operators:*

$$\left(\oplus \mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M}), [\![m, -]\!]\right) \underset{\sigma_{\natural}}{\overset{\breve{\tau}_{\natural}}{\rightleftarrows}} \left(\oplus \mathscr{D}^{\bullet}_{\text{poly}}(\mathcal{F}), [\![m + D^{\nabla}, -]\!]\right) \circlearrowleft \breve{h}_{\natural}$$

(3) *and at the level of tensor fields of any type (k, l):*

$$\left(\left(\Gamma(\mathcal{M}; T^{\otimes k}_{\mathcal{M}} \otimes (T^{\vee}_{\mathcal{M}})^{\otimes l})\right)^{\bullet}, 0\right) \underset{\sigma^{k,l}_{\natural}}{\overset{\breve{\tau}^{k,l}_{\natural}}{\rightleftarrows}} \left(\left(\Gamma(\mathcal{N}; \mathcal{F}^{\otimes k} \otimes (\mathcal{F}^{\vee})^{\otimes l})\right)^{\bullet}, \mathcal{L}_{D^{\nabla}}\right) \circlearrowleft \breve{h}^{k,l}_{\natural}$$

Now we move to the case of a finite-dimensional *dg* manifold $(\mathcal{M}, Q)$. Let $\mathcal{F} \to \mathcal{N}$ be a Fedosov dg Lie algebroid over a Fedosov dg manifold $(\mathcal{N}, D^{\nabla})$ associated with the graded manifold $\mathcal{M}$. By $\mathcal{N}_Q$, we denote the dg manifold $(\mathcal{N}, D^{\nabla} + \breve{\tau}_{\natural}(Q))$. We write $\mathcal{F}_Q$ to denote the dg manifold structure on the $\mathbb{Z}$-graded manifold $\mathcal{F}$ characterized by the following property: $\mathcal{F}_Q \to \mathcal{N}_Q$ is a dg vector space such that the induced operator on $\Gamma(\mathcal{N}; \mathcal{F})$ determined by the homological vector fields of $\mathcal{F}_Q$ and $\mathcal{N}_Q$ is $\mathcal{L}_{D^{\nabla} + \breve{\tau}_{\natural}(Q)}$.

Lemma 5.4. *The dg vector bundle $\mathcal{F}_Q \to \mathcal{N}_Q$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_{\mathcal{N}_Q} \to \mathcal{N}_Q$.*

In other words, $\mathcal{F}_Q$ is a dg foliation of the dg manifold $\mathcal{N}_Q = (\mathcal{N}, D^{\nabla} + \breve{\tau}_{\natural}(Q))$. Such a dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$ is called a *Fedosov dg Lie algebroid* associated with the dg manifold $(\mathcal{M}, Q)$.

The cochain complexes appearing in Proposition 5.3 admit exhaustive and complete filtrations compatible with the contraction data:

$$F^k\left(\oplus \mathcal{T}^n_{\text{poly}}(\mathcal{M})\right) = \bigoplus_{\substack{p,q \in \mathbb{Z} \\ p+q=n \\ p \geqslant -1,\ q \geqslant k}} \mathcal{K}^{0,q,p} \qquad F^k\left(\oplus \mathscr{T}^n_{\text{poly}}(\mathcal{F})\right) = \bigoplus_{\substack{p,q,r \in \mathbb{Z} \\ p+q+r=n \\ p \geqslant -1,\ q \geqslant k,\ r \geqslant 0}} \mathcal{W}^{r,q,p}$$

$$F^k\left(\oplus \mathcal{D}^n_{\text{poly}}(\mathcal{M})\right) = \bigoplus_{\substack{p,q \in \mathbb{Z} \\ p+q=n \\ p \geqslant -1,\ q \geqslant k}} \mathscr{K}^{0,q,p} \qquad F^k\left(\oplus \mathscr{D}^n_{\text{poly}}(\mathcal{F})\right) = \bigoplus_{\substack{p,q,r \in \mathbb{Z} \\ p+q+r=n \\ p \geqslant -1,\ q \geqslant k,\ r \geqslant 0}} \mathscr{W}^{r,q,p}$$

Perturbing the filtered contractions of Proposition 5.3 homologically (see [20, Appendix] and references therein) by $\breve{\tau}_{\natural}(Q)$, we obtain

Proposition 5.5. *Given a finite-dimensional dg manifold $(\mathcal{M}, Q)$ and a torsion-free affine connection ∇ on $\mathcal{M}$, let $\mathcal{F} \to \mathcal{N}$ be the Fedosov dg Lie algebroid corresponding to the $\mathbb{Z}$-graded manifold $\mathcal{M}$ (as in Lemma 5.2) and let D^{∇} be the corresponding Fedosov homological vector field on $\mathcal{N}$ (as in Proposition 5.1). Then there are contractions*

(1) *at the level of polyvector fields:*

$$\left(\oplus \mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}), [Q, -]\right) \underset{\tilde{\sigma}_{\natural}}{\overset{\breve{\tau}_{\natural}}{\rightleftarrows}} \left(\oplus \mathscr{T}^{\bullet}_{\text{poly}}(\mathcal{F}), [D^{\nabla} + \breve{\tau}_{\natural}(Q), -]\right) \circlearrowleft \tilde{h}_{\natural}$$

(2) *at the level of polydifferential operators:*

$$\left(\oplus \mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M}), [\![m + Q, -]\!]\right) \underset{\tilde{\sigma}_{\natural}}{\overset{\breve{\tau}_{\natural}}{\rightleftarrows}} \left(\oplus \mathscr{D}^{\bullet}_{\text{poly}}(\mathcal{F}), [\![m + D^{\nabla} + \breve{\tau}_{\natural}(Q), -]\!]\right) \circlearrowleft \tilde{h}_{\natural}$$

(3) *and at the level of tensor fields of any type* (k,l)*:*

$$\Big(\big(\Gamma(\mathcal{M};T_{\mathcal{M}}^{\otimes k}\otimes(T_{\mathcal{M}}^{\vee})^{\otimes l})\big)^{\bullet},\mathcal{L}_Q\Big)\underset{\widetilde{\sigma}_\natural^{k,l}}{\overset{\breve{\tau}_\natural^{k,l}}{\rightleftarrows}}\Big(\big(\Gamma(\mathcal{N};\mathcal{F}^{\otimes k}\otimes(\mathcal{F}^{\vee})^{\otimes l})\big)^{\bullet},\mathcal{L}_{D^{\nabla}+\breve{\tau}_\natural(Q)}\Big)\circlearrowright \widetilde{h}_\natural^{k,l}$$

Remark. The maps $\breve{\tau}_\natural$ appearing in Proposition 5.3 and $\breve{\tau}_\natural$ appearing in Proposition 5.5 are actually identical. However, the homological perturbation modifies the maps $\sigma_\natural$ and $\breve{h}_\natural$ appearing in Proposition 5.3 and returns the new maps $\widetilde{\sigma}_\natural$ and $\widetilde{h}_\natural$ appearing in Proposition 5.5.

Remark. Writing $\big(\Gamma(\mathcal{N};\mathcal{F}^{\otimes k}\otimes(\mathcal{F}^{\vee})^{\otimes l})\big)^n$ in Proposition 5.5 is an abuse of notation. In fact, to ensure the convergence of the series involved in the homological perturbation, $\big(\Gamma(\mathcal{N};\mathcal{F}^{\otimes k}\otimes(\mathcal{F}^{\vee})^{\otimes l})\big)^n$ must be replaced by its subspace $\big(\bigoplus_{r\in\mathbb{Z}_{\geqslant 0}}\Gamma(\Lambda^r(T_{\mathcal{M}}^{\vee})\otimes\widehat{S}(T_{\mathcal{M}}^{\vee})\otimes T_{\mathcal{M}}^{\otimes k}\otimes(T_{\mathcal{M}}^{\vee})^{\otimes l})\big)^n$.

6. Proof of main theorems

Recall Kontsevich's formality theorem for the vector space $\mathbb{k}^d$:

Theorem 6.1 *([19]). There exists a $GL(\mathbb{k}^d)$-equivariant L_∞ quasi-isomorphism*

$$\mathscr{U}:\mathcal{T}_{\mathrm{poly}}^{\bullet}(\mathbb{k}^d)\rightsquigarrow\mathcal{D}_{\mathrm{poly}}^{\bullet}(\mathbb{k}^d)$$

having the Hochschild–Kostant–Rosenberg map as first Taylor coefficient.

Kontsevich's original paper [19] lists additional properties satisfied by $\mathscr{U}$.

Kontsevich's formality L_∞ quasi-isomorphism $\mathscr{U}:\mathcal{T}_{\mathrm{poly}}^{\bullet}(V)\rightsquigarrow\mathcal{D}_{\mathrm{poly}}^{\bullet}(V)$ still makes sense for a finite-dimensional $\mathbb{Z}$-graded vector space V over the field $\mathbb{k}$ and is $GL(V)$-equivariant — see [7,32]. Applying the Taylor coefficients $\mathscr{U}_n:\Lambda^n\big(\mathcal{T}_{\mathrm{poly}}^{\bullet}(V)\big)\to\mathcal{D}_{\mathrm{poly}}^{\bullet}(V)[1-n]$ (with $n\in\mathbb{N}$) of Kontsevich's formality L_∞ morphism $\mathscr{U}$ essentially leafwise on $\mathcal{N}$ w.r.t. the $\mathcal{F}$-foliation, we obtain a sequence $(\mathscr{U}_n^{\mathrm{leaf}})_{n\in\mathbb{N}}$ of morphisms

$$\mathscr{U}_n^{\mathrm{leaf}}:\Lambda^n\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)\to{}_{\oplus}\mathscr{D}_{\mathrm{poly}}^{\bullet}(\mathcal{F})[1-n]$$

of left $\Gamma(\Lambda T_{\mathcal{M}}^{\vee})$-modules — the subalgebra $\Gamma(\Lambda T_{\mathcal{M}}^{\vee})$ of $C^\infty(\mathcal{N})$ consists of those functions on $\mathcal{N}$ that are constant along the $\mathcal{F}$-leaves. The vector field $\omega\in\mathfrak{X}(\mathcal{N})$ defined by

$$\omega=D^{\nabla}-d^{\nabla}+\breve{\tau}_\natural(Q)=-\delta+A^{\nabla}+\breve{\tau}_\natural(Q)$$

is tangent to the leaves of the $\mathcal{F}$-foliation of $\mathcal{N}$. Since $-\delta+A^{\nabla}\in\mathcal{W}^{1,0,0}$ and $\breve{\tau}_\natural(Q)\in\mathcal{W}^{0,1,0}$, we have $\omega\in{}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{0}(\mathcal{F})$. We do *not* claim that ω is a Maurer–Cartan element for any dgla structure on ${}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})$. Twisting nevertheless the sequence of maps $(\mathscr{U}_n^{\mathrm{leaf}})_{n\in\mathbb{N}}$ by ω, we define a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of morphisms of left $\Gamma(\Lambda T_{\mathcal{M}}^{\vee})$-modules

$$\Phi_n:\Lambda^n\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)\to{}_{\oplus}\mathscr{D}_{\mathrm{poly}}^{\bullet}(\mathcal{F})[1-n]$$

by the relations

$$\Phi_n(\gamma)=\sum_{j=0}^{\infty}\mathscr{U}_{n+j}^{\mathrm{leaf}}(\omega^j\wedge\gamma),\qquad\forall\gamma\in\Lambda^n\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big).$$

Now consider the dgla

$$\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)_Q=\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F}),[D^{\nabla}+\breve{\tau}_\natural(Q),-],[-,-]\big)$$

of "polyvector fields" on the dg Lie algebroid $\mathcal{F}_Q\to\mathcal{N}_Q$ and the dgla

$$\big({}_{\oplus}\mathscr{D}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)_Q=\big({}_{\oplus}\mathscr{D}_{\mathrm{poly}}^{\bullet}(\mathcal{F}),[\![m+D^{\nabla}+\breve{\tau}_\natural(Q),-]\!],[\![-,-]\!]\big)$$

of "polydifferential operators" on the dg Lie algebroid $\mathcal{F}_Q\to\mathcal{N}_Q$.

One can prove the following result:

Lemma 6.2. *The maps $(\Phi_n)_{n=1}^{\infty}$ are well defined and are the Taylor coefficients of an L_∞ morphism of dglas*

$$\Phi:\big({}_{\oplus}\mathscr{T}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)_Q\rightsquigarrow\big({}_{\oplus}\mathscr{D}_{\mathrm{poly}}^{\bullet}(\mathcal{F})\big)_Q.$$

We need to obtain a more explicit description of the first Taylor coefficient Φ_1 and to investigate the behavior of Φ_1 w.r.t. the associative algebra structures at hand.

Making use of the canonical inclusion $\Gamma(T_{\mathcal{M}}) \hookrightarrow \Omega^\bullet(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}) \cong \Gamma(\mathcal{N}; \mathcal{F})$, we define a "canonical" $\mathcal{F}$-connection ∇^{can} on $\mathcal{F}$ by the relation

$$\nabla^{\mathrm{can}}_a b = 0, \qquad \forall\, a, b \in \Gamma(T_{\mathcal{M}}).$$

The $\widehat{A}$ cocycle of the dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$ associated with the connection ∇^{can} is

$$\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q} = \mathrm{Ber}\left(\frac{\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}{e^{\frac{1}{2}\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}} - e^{-\frac{1}{2}\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}} \right).$$

We prove the following

Proposition 6.3.

(1) *The map $\Phi_1 : {}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}) \to {}_{\oplus}\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})$ is the modification*

$$\Phi_1 = \mathrm{hkr} \circ (\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}}$$

of the Hochschild–Kostant–Rosenberg map by (the square root of) the canonical $\widehat{A}$ cocycle.

(2) *Although the map Φ_1 does not intertwine the wedge product and the cup product, there exists a homotopy $H : {}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}) \times {}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}) \to {}_{\oplus}\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})$ satisfying*

$$\begin{aligned}\Phi_1(\alpha \wedge \beta) - \Phi_1(\alpha) \smile \Phi_1(\beta) = {} & [\![m + D^\nabla + \breve{\tau}_\natural(Q), H(\alpha, \beta)]\!] \\ & - H\big([D^\nabla + \breve{\tau}_\natural(Q), \alpha], \beta\big) + (-1)^a H\big(\alpha, [D^\nabla + \breve{\tau}_\natural(Q), \beta]\big)\end{aligned}$$

for all $\alpha \in {}_{\oplus}\mathscr{T}^a_{\mathrm{poly}}(\mathcal{F})$ and $\beta \in {}_{\oplus}\mathscr{T}^b_{\mathrm{poly}}(\mathcal{F})$.

The actions on ${}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})$, by contraction, of the square roots of the Todd cocycle $\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}$ and $\widehat{A}$ cocycle $\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q}$ are related to one another in a rather simple way, which we proceed to explain.

Lemma 6.4. *Let $\mathcal{A} \to \mathcal{M}$ be a dg Lie algebroid. Let $\mathcal{Q}$ denote the endomorphism of $\Gamma(\mathcal{A})$ encoding the dg structure and let $d_{\mathcal{A}}$ denote the Chevalley–Eilenberg differential. If $\xi \in \Gamma(\mathcal{A}^\vee)$ satisfies $d_{\mathcal{A}}(\xi) = 0$ and $\mathcal{Q}(\xi) = 0$, then the contraction with ξ is a derivation of the differential Gerstenhaber algebra $\big(\Gamma(\Lambda^\bullet \mathcal{A}), [-,-], \mathcal{Q}\big)$.*

The difference between

$$\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q} = \mathrm{Ber}\left(\frac{\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}{1 - e^{-\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}} \right) \qquad \text{and} \qquad \widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q} = \mathrm{Ber}\left(\frac{\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}{e^{\frac{1}{2}\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}} - e^{-\frac{1}{2}\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}}} \right)$$

is the factor $e^{\frac{1}{2}\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})}$. The section $\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})$ of the bundle $\mathcal{F}^\vee \to \mathcal{N}$ satisfies $d_{\mathcal{F}_Q}\big(\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})\big) = 0$ and $\mathcal{L}_{D^\nabla + \breve{\tau}_\natural(Q)}\big(\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})\big) = 0$. Therefore, applying Lemma 6.4 to the Fedosov dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$ and the section $\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})$ of its dual vector bundle and noting that $\Gamma(\Lambda^\bullet \mathcal{F}) \cong {}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})$ and $\mathcal{Q} = [D^\nabla + \breve{\tau}_\natural(Q), -]$, we obtain

Corollary 6.5.

(1) *The contraction by $\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})$ is a derivation of the differential Gerstenhaber algebra $\big({}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}), [D^\nabla + \breve{\tau}_\natural(Q), -], \wedge, [-,-]\big)$.*

(2) *The contraction by $e^{\frac{1}{2}\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})}$ is an automorphism of the differential Gerstenhaber algebra $\big({}_{\oplus}\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}), [D^\nabla + \breve{\tau}_\natural(Q), -], \wedge, [-,-]\big)$, which satisfies the equation*

$$(\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}} \circ e^{\frac{1}{2}\mathrm{str}(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q})} = (\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}}.$$

One significant technical difficulty is the absence of a simple direct relation between $\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}$ and the Todd cocycle $\mathrm{td}^{\nabla}_{(\mathcal{M},Q)}$ of the dg manifold $(\mathcal{M}, Q)$. Overcoming it requires the introduction of an intermediate $\mathcal{F}$-connection $\breve{\nabla}$ on $\mathcal{F}$, which is obtained by pushing forward the connection ∇ through the map $\breve{\tau}_\natural$:

$$\breve{\tau}_\natural(\nabla_X Y) = \breve{\nabla}_{\breve{\tau}_\natural(X)} \breve{\tau}_\natural(Y), \qquad \forall\, X, Y \in \mathfrak{X}(\mathcal{M}).$$

The Todd cocycle of the Fedosov dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$ associated with this intermediate connection $\breve{\nabla}$ is denoted $\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}$.

The following lemma can be easily checked.

Lemma 6.6. *The following identities hold.*

(1) $\breve{\tau}_\natural^{1,2}(\mathrm{at}^{\nabla}_{(\mathcal{M},Q)}) = \mathrm{at}^{\breve{\nabla}}_{\mathcal{F}_Q}$
(2) $\breve{\tau}_\natural(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)}) = \mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}$

The square roots of the Todd cocycles $\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}$ and $\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}$ act on $_{\oplus}\mathscr{T}^{\bullet}_{\mathrm{poly}}(\mathcal{F})$ by contraction.

Lemma 6.7. *The chain map $\big(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}}$ from the cochain complex $\big({}_{\oplus}\mathscr{T}^{\bullet}_{\mathrm{poly}}(\mathcal{F}), [D^{\nabla} + \breve{\tau}_\natural(Q), -]\big)$ to itself is chain homotopic to the identity map.*

Proof. Since the Atiyah class of a dg vector bundle is independent of the connection used to compute it, we have $\mathrm{at}^{\breve{\nabla}}_{\mathcal{F}_Q} = \mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q} + \mathcal{L}_Q\beta$, for some β in $\big(\Gamma(\mathcal{F}^\vee \otimes \mathrm{End}\,\mathcal{F})\big)^0$. It follows that $(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}} = (\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}} + \mathcal{L}_Q\zeta$, for some ζ in $\prod_{k\geqslant 0}\big(\Gamma(\Lambda^k\mathcal{F}^\vee)\big)^{k-1}$, then $(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \wedge (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}} = 1 + (\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \wedge \mathcal{L}_Q\zeta = 1 + \mathcal{L}_Q(\xi)$, for some ξ in $\prod_{k\geqslant 0}\big(\Gamma(\Lambda^k\mathcal{F}^\vee)\big)^{k-1}$, and finally $(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \circ (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}} = \mathrm{id} + i_{\mathcal{L}_Q(\xi)}$. Since $i_{\mathcal{L}_Q(\xi)} = \mathcal{L}_Q \circ i_\xi + i_\xi \circ \mathcal{L}_Q$, the chain map $(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \circ (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}}$ is chain homotopic to the identity:

$$(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \circ (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}} - \mathrm{id} = \mathcal{L}_Q \circ i_\xi + i_\xi \circ \mathcal{L}_Q. \qquad \square$$

The following proposition is a consequence of the "homotopy transfer theorem" for L_∞ algebra structures [1, Theorem 1.9 and Lemma 1.11] (see also [17, Theorem 4.1 and Proposition 4.2]).

Proposition 6.8. *If $V^\bullet$ is an L_∞ algebra and $f: V^\bullet \to V^\bullet$ is a chain map homotopic to the identity, then there exists an L_∞ morphism $F: V^\bullet \rightsquigarrow V^\bullet$ having f as first Taylor coefficient. Moreover, F admits an explicit algebraic expression in terms of f and the chain homotopy.*

Proof. Let $K: V^\bullet \to V^\bullet$ be the chain homotopy between id_V and f, i.e. $f - \mathrm{id}_V = d_V K + K d_V$. Consider the L_∞ algebra $V^\bullet \otimes \Bbbk[t, \mathrm{d}t]$ obtained by tensoring the L_∞ algebra $V^\bullet$ with the 2-term cdga $\Bbbk[t, \mathrm{d}t]$ (where the scalars have degree 0, the variable t has degree 0, and its image dt under the differential has degree 1). The evaluation maps $\mathrm{ev}_0: V^\bullet \otimes \Bbbk[t, \mathrm{d}t] \to V^\bullet$ and $\mathrm{ev}_1: V^\bullet \otimes \Bbbk[t, \mathrm{d}t] \to V^\bullet$ defined by

$$\mathrm{ev}_0\Big(\sum_{p\geq 0} t^p a_p + \mathrm{d}t \sum_{q\geq 0} t^q b_q\Big) = a_0$$
$$\mathrm{ev}_1\Big(\sum_{p\geq 0} t^p a_p + \mathrm{d}t \sum_{q\geq 0} t^q b_q\Big) = \sum_{p\geq 0} a_p$$

preserve the L_∞ multi-brackets. Therefore, they are the respective first Taylor coefficients of a pair of L_∞ morphisms $\mathrm{ev}_0: V^\bullet \otimes \Bbbk[t, \mathrm{d}t] \rightsquigarrow V^\bullet$ and $\mathrm{ev}_1: V^\bullet \otimes \Bbbk[t, \mathrm{d}t] \rightsquigarrow V^\bullet$ whose higher Taylor coefficients are all equal to zero.

Consider the contraction

$$\big(V^\bullet, d_V\big) \underset{\mathrm{ev}_0}{\overset{j}{\rightleftarrows}} \big(V^\bullet \otimes \Bbbk[t, \mathrm{d}t], d\big) \circlearrowleft H \tag{5}$$

defined by

$$j(x) = (1-t)x + tf(x) + \mathrm{d}t\, K(x),$$
$$H\Big(\sum_{p\geq 0} t^p a_p + \mathrm{d}t \sum_{q\geq 0} t^q b_q\Big) = tK(a_0) - \sum_{q\geq 0} \frac{1}{q+1} t^{q+1} b_q.$$

According to Lemma 1.11 in [1], the L_∞ structure on $V^\bullet$ obtained by homotopy transfer [1, Theorem 1.9] of the L_∞ structure on $V^\bullet \otimes \Bbbk[t, \mathrm{d}t]$ through the contraction (5) coincides with the original L_∞ structure on $V^\bullet$. Furthermore, there exists, according to Theorem 1.9 in [1], an L_∞ morphism $J : V^\bullet \rightsquigarrow V^\bullet \otimes \Bbbk[t, dt]$ having j as its first Taylor coefficient. The desired L_∞ morphism $F : V^\bullet \rightsquigarrow V^\bullet$ is the composition $F = \mathrm{ev}_1 \circ J$. Its first Taylor coefficient is $\mathrm{ev}_1 \circ j = f$. □

Applying Proposition 6.8 to the dgla $V^\bullet = \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ and the chain map $f = \big(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}}$, we obtain the following

Lemma 6.9. *There exists an L_∞ automorphism $\Xi^{\breve{\nabla}}$ of the dgla $\big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ having the chain map $\big(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}}$ as first Taylor coefficient $\Xi^{\breve{\nabla}}_1$.*

Proof. Since the chain map $(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{-\frac{1}{2}} \circ (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}}$ is chain homotopic to the identity (Lemma 6.7), there exists, according to Proposition 6.8, an L_∞ morphism $\Xi^{\breve{\nabla}} : \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q \rightsquigarrow \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ having $\big(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}}$ as its first Taylor coefficient. Furthermore, since the chain map $\big(\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q}\big)^{-\frac{1}{2}} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}}$ is an automorphism of the cochain complex $\big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}), [D^\nabla + \breve{\tau}_\natural(Q), -]\big)$, the lifted L_∞ morphism $\Xi^{\breve{\nabla}}$ is actually an automorphism of the L_∞ algebra $\big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ – see [24, Corollary 12.5.5]. □

We are thus led to consider the L_∞ morphism

$$\Psi : \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q \rightsquigarrow \big({}_\oplus\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q,$$

from the dgla of "polyvector fields" on the dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$ to the dgla of "polydifferential operators" on the dg Lie algebroid $\mathcal{F}_Q \to \mathcal{N}_Q$, defined as the composition

$$\Psi = \Phi \circ e^{\frac{1}{2}\operatorname{str}\left(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}\right)} \circ \Xi^{\breve{\nabla}}.$$

Proposition 6.10. *The first Taylor coefficient $\Psi_1 : {}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}) \to {}_\oplus\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})$ of the L_∞ morphism $\Psi : \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q \rightsquigarrow \big({}_\oplus\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ satisfies the following two properties:*

(1) *Ψ_1 preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;*
(2) *Ψ_1 is the composition $\mathrm{hkr} \circ (\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}}$ of the action of the square root of the Todd cocycle $\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}$ on ${}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})$, by contraction, with the Hochschild–Kostant–Rosenberg map $\mathrm{hkr} : {}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F}) \to {}_\oplus\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F})$.*

Sketch of proof.

$$\begin{aligned}
\Psi_1 &= \Phi_1 \circ e^{\frac{1}{2}\operatorname{str}\left(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}\right)} \circ \Xi^{\breve{\nabla}}_1 \\
&= \mathrm{hkr} \circ (\widetilde{\mathrm{td}}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}} \circ e^{\frac{1}{2}\operatorname{str}\left(\mathrm{at}^{\mathrm{can}}_{\mathcal{F}_Q}\right)} \circ \Xi^{\breve{\nabla}}_1 && \text{(by Proposition 6.3)} \\
&= \mathrm{hkr} \circ (\mathrm{td}^{\mathrm{can}}_{\mathcal{F}_Q})^{\frac{1}{2}} \circ \Xi^{\breve{\nabla}}_1 && \text{(by Corollary 6.5)} \\
&= \mathrm{hkr} \circ \big(\mathrm{td}^{\breve{\nabla}}_{\mathcal{F}_Q}\big)^{\frac{1}{2}} && \text{(by Lemma 6.9)} \quad \square
\end{aligned}$$

One proves that the map $\breve{\tau}_\natural$ on polyvector fields of Proposition 5.5 is a morphism of Gerstenhaber algebras:

Proposition 6.11.

(1) *The map $\breve{\tau}_\natural : {}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}) \to {}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})$ is a morphism of associative algebras as it preserves the wedge product of polyvector fields.*
(2) *The map $\breve{\tau}_\natural : \big({}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M})\big)_Q \to \big({}_\oplus\mathscr{T}^\bullet_{\mathrm{poly}}(\mathcal{F})\big)_Q$ is a morphism of differential graded Lie algebras as it intertwines the differentials $[Q, -]$ and $[D^\nabla + \breve{\tau}_\natural(Q), -]$ and preserves the Schouten bracket of polyvector fields.*

Likewise, for polydifferential operators, the quasi-isomorphism

$$\Big({}_\oplus\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M}), [\![m + Q, -]\!]\Big) \xrightarrow{\breve{\tau}_\natural} \Big({}_\oplus\mathscr{D}^\bullet_{\mathrm{poly}}(\mathcal{F}), [\![m + D^\nabla + \breve{\tau}_\natural(Q), -]\!]\Big)$$

of Proposition 5.5 preserves the cup product of polydifferential operators. Therefore the map on cohomology induced by $\check{\tau}_\natural$ is an isomorphism of associative algebras. Since the maps $\check{\tau}_\natural$ and $\widetilde{\sigma}_\natural$ constitute a contraction, the pair of maps they induce in cohomology are mutual inverses. Hence, the map on cohomology induced by $\widetilde{\sigma}_\natural$ is an isomorphism of associative algebras as well.

Proposition 6.12. *The quasi-isomorphism $\widetilde{\sigma}_\natural$ of Proposition 5.5 induces an isomorphism*

$$\widetilde{\sigma}_\natural : \mathbb{H}^\bullet\big({}_\oplus\mathscr{D}_{\text{poly}}(\mathcal{F}), [\![m + D^\nabla + \check{\tau}_\natural(Q), -]\!]\big) \to \mathbb{H}^\bullet\big({}_\oplus\mathcal{D}_{\text{poly}}(\mathcal{M}), [\![m + Q, -]\!]\big)$$

of associative algebras *on the cohomology level.*

Let $\mathrm{T} : \big({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q \rightsquigarrow \big({}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q$ denote the L_∞ quasi-isomorphism having the quasi-isomorphism of cochain complexes $\check{\tau}_\natural : \big({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}), [Q, -]\big) \to \big({}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F}), [D^\nabla + \check{\tau}_\natural(Q), -]\big)$ as first Taylor coefficient and all higher Taylor coefficients equal to zero.

Since the quasi-isomorphism of cochain complexes $\widetilde{\sigma}_\natural : \big({}_\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F}), [\![m + D^\nabla + \check{\tau}_\natural(Q), -]\!]\big) \to \big({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M}), [\![m + Q, -]\!]\big)$ is part of a contraction (Proposition 5.5), standard results about L_∞ morphisms (see [24, Chapter 13] and [1]) assert the existence of an L_∞ quasi-isomorphism $\Sigma : \big({}_\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q \rightsquigarrow \big({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q$ having the quasi-isomorphism $\widetilde{\sigma}_\natural$ as its first Taylor coefficient.

Consider the L_∞ morphism

$$\mathcal{I} : \big({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q \rightsquigarrow \big({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q$$

obtained as the composition

$$\mathcal{I} = \Sigma \circ \Psi \circ \mathrm{T}$$

of the L_∞ morphism $\Psi : \big({}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q \rightsquigarrow \big({}_\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q$ of Proposition 6.10 with the L_∞ quasi-isomorphisms T and Σ:

$$\begin{array}{ccc} \big({}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q & \overset{\Psi}{\rightsquigarrow} & \big({}_\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F})\big)_Q \\ \mathrm{T}\uparrow & & \downarrow\Sigma \\ \big({}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q & \underset{\mathcal{I}}{\rightsquigarrow} & \big({}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M})\big)_Q \end{array} \tag{6}$$

The following result follows immediately from Lemma 6.6:

Lemma 6.13. *The diagram*

$$\begin{array}{ccc} {}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F}) & \xrightarrow{(\mathrm{td}^{\check{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}}} & {}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F}) \\ \check{\tau}_\natural\uparrow & & \uparrow\check{\tau}_\natural \\ {}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}) & \xrightarrow[(\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}}]{} & {}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}) \end{array}$$

commutes.

The following result is also immediate:

Lemma 6.14. *The diagram*

$$\begin{array}{ccc} {}_\oplus\mathscr{T}^\bullet_{\text{poly}}(\mathcal{F}) & \xrightarrow{\mathrm{hkr}} & {}_\oplus\mathscr{D}^\bullet_{\text{poly}}(\mathcal{F}) \\ \check{\tau}_\natural\uparrow & & \uparrow\check{\tau}_\natural \\ {}_\oplus\mathcal{T}^\bullet_{\text{poly}}(\mathcal{M}) & \xrightarrow[\mathrm{hkr}]{} & {}_\oplus\mathcal{D}^\bullet_{\text{poly}}(\mathcal{M}) \end{array}$$

commutes.

Therefore, the first Taylor coefficient of the L_∞ morphism $\mathcal{I}$ is

$$\begin{aligned}
\mathcal{I}_1 &= \Sigma_1 \circ \Psi_1 \circ \mathrm{T}_1 \\
&= \widetilde{\sigma}_\natural \circ \mathrm{hkr} \circ (\mathrm{td}^{\check{\nabla}}_{\mathcal{F}_Q})^{\frac{1}{2}} \circ \check{\tau}_\natural && \text{(by Proposition 6.10)} \\
&= \widetilde{\sigma}_\natural \circ \mathrm{hkr} \circ \check{\tau}_\natural \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} && \text{(by Lemma 6.13)} \\
&= \widetilde{\sigma}_\natural \circ \check{\tau}_\natural \circ \mathrm{hkr} \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} && \text{(by Lemma 6.14)} \\
&= \mathrm{hkr} \circ (\mathrm{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}} && \text{(by Proposition 5.5).}
\end{aligned}$$

Finally, since hkr is a quasi-isomorphism of cochain complexes (Proposition 4.1), so is $\mathcal{I}_1$. Hence $\mathcal{I}$ is an L_∞ quasi-isomorphism.

According to Proposition 6.11; Proposition 6.10; and Proposition 6.12 respectively, the morphism $\mathrm{T}_1 = \check{\tau}_\natural$; Ψ_1; and $\Sigma_1 = \widetilde{\sigma}_\natural$ preserve the associative algebra structures on the cohomology level. Therefore, so does $\mathcal{I}_1$.

The proof of Theorem 4.2 is thus complete.

7. Applications

Theorem 4.2 can be applied to various interesting examples of dg manifolds to the study of deformation quantization problem of (0-shifted) derived Poisson manifolds or P_∞-manifolds [2,7]. As another application, Theorem 4.3 can be applied to various geometric situations so as to obtain Duflo-type theorems. In particular, we can recover the Kontsevich–Duflo theorem for Lie algebras [14,19] and the Kontsevich theorem for complex manifolds [19] and unify them in a common framework by considering two special classes of dg manifolds.

7.1. Kontsevich–Duflo theorem

We apply Theorem 4.3 to the dg manifold $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ arising from a finite-dimensional Lie algebra $\mathfrak{g}$. The cohomology of polyvector fields is

$$\mathbb{H}^\bullet({}_\oplus\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), \mathcal{Q}) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})), \tag{7}$$

and the cohomology of polydifferential operators is

$$\mathbb{H}^\bullet({}_\oplus\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), \mathcal{Q}) \cong HH^\bullet(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})). \tag{8}$$

The symbols $\cong$ in Equations (7) and (8) denote isomorphisms of associative algebra structures.[1] Since we shift the degree of $\mathfrak{g}$ by one, the map hkr becomes the well known symmetrization map $\mathrm{pbw} : H^\bullet_{\mathrm{CE}}\big(\mathfrak{g}, S(\mathfrak{g})\big) \to H^\bullet_{\mathrm{CE}}\big(\mathfrak{g}, \mathcal{U}(\mathfrak{g})\big)$. Moreover, the Todd class of the dg manifold $(\mathfrak{g}[1], d_{\mathrm{CE}})$ is essentially the Duflo element $J \in \big(\widehat{S}(\mathfrak{g}^\vee)\big)^{\mathfrak{g}}$ — see [28]. Theorem 4.3 implies

Theorem 7.1 *([19,29]). Let $\mathfrak{g}$ be a Lie algebra. The map*

$$\mathrm{pbw} \circ J^{\frac{1}{2}} : H^\bullet_{\mathrm{CE}}\big(\mathfrak{g}, S(\mathfrak{g})\big) \xrightarrow{\cong} H^\bullet_{\mathrm{CE}}\big(\mathfrak{g}, \mathcal{U}(\mathfrak{g})\big)$$

is an isomorphism of associative algebras.

Restriction of this isomorphism to the subalgebras consisting solely of the cohomology groups of degree 0 yields the classical Duflo theorem [14].

7.2. Kontsevich theorem for complex manifolds

The following result was proved in [10].

Proposition 7.2 *([10]). Let $(\mathcal{M}, Q) = (T^{0,1}_X[1], \bar{\partial})$ be the dg manifold arising from a complex manifold X. Then there exists a canonical isomorphism*

$$\Phi^{k,l} : \mathbb{H}^\bullet\big(\Gamma(T^{\otimes k}_{\mathcal{M}} \otimes (T^\vee_{\mathcal{M}})^{\otimes l}), \mathcal{Q}\big) \xrightarrow{\cong} H^\bullet_{\mathrm{sheaf}}\big(X, T^{\otimes k}_X \otimes (T^\vee_X)^{\otimes l}\big) \tag{9}$$

such that

[1] The right hand sides of Equations (7) and (8) actually admit Gerstenhaber algebra structures. However, unlike their associative algebra structures, their Lie algebra structures are not so obvious.

(1) $\Phi^{\bullet,0} : \mathbb{H}^\bullet({}_{\oplus}\mathcal{T}_{\mathrm{poly}}(\mathcal{M}), \mathcal{Q}) \xrightarrow{\cong} \mathbb{H}^\bullet_{\mathrm{sheaf}}(X, \Lambda^\bullet T_X)$ *is an isomorphism of Gerstenhaber algebras.*
(2) $\Phi^{0,\bullet}(\mathrm{Td}_{(\mathcal{M},Q)}) = \mathrm{Td}_X$

We can also prove the following

Proposition 7.3. *Let* $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar\partial)$ *be the dg manifold arising from a complex manifold* X*. Then there exists a canonical isomorphism of Gerstenhaber algebras*

$$\mathbb{H}^\bullet({}_{\oplus}\mathcal{D}_{\mathrm{poly}}(\mathcal{M}), \mathcal{Q}) \xrightarrow{\cong} HH^\bullet(X).$$

Combining Theorem 4.3 with Proposition 7.2 and Proposition 7.3, we recover Kontsevich's theorem:

Theorem 7.4 *(Kontsevich theorem for complex manifolds [19,21,6]). For every complex manifold* X*, the composition*

$$\mathrm{hkr} \circ (\mathrm{Td}_X)^{\frac{1}{2}} : \mathbb{H}^\bullet_{\mathrm{sheaf}}(X, \Lambda^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$$

is an isomorphism of Gerstenhaber algebras. It is understood that the square root of the Todd class

$$\mathrm{Td}_X \in \bigoplus_{k=0} H^{k,k}(X) \cong \bigoplus_{k=0} H^k_{\mathrm{sheaf}}(X, \Omega_X^k)$$

acts on $\mathbb{H}^\bullet_{\mathrm{sheaf}}(X, \Lambda^\bullet T_X)$ *by contraction.*

The Kontsevich theorem for complex manifolds is due to Kontsevich [19] (for associative algebra structures). A detailed proof appeared in [6], where the additional Gerstenhaber algebra structures were also addressed. See also [21] for another proof using formality for Lie pairs.

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