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Homological algebra

Derived invariance of the cap product in Hochschild theory

Invariance dérivée du cap produit en théorie de Hochschild

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ABSTRACT

We prove the derived invariance of the cap product for associative algebras projective over a commutative ring.

RÉSUMÉ

Nous démontrons l'invariance dérivée du cap produit pour les algèbres associatives projectives sur un anneau commutatif.

1. Introduction

It has been known since Rickard's work [3] that Hochschild cohomology is preserved under derived equivalence as a graded algebra with the cup product. Using the methods of [3], one can also show that Hochschild homology is preserved as a graded space, see for example [5]. Nevertheless, the derived invariance of the cap product – which provides an action of the Hochschild cohomology algebra on the Hochschild homology – has not been considered. In this note, we prove that derived invariance holds as well for the cap product.

This paper is part of the Ph. D. thesis of the first author, whose advisors are Claude Cibils and José Antonio de la Peña, to whom he is very grateful. It enters into the first author's project of showing the derived invariance of the Tamarkin–Tsygan calculus associated with a k-projective algebra.

2. Derived invariance

Let k be a commutative ring and A an associative k-algebra, projective as a k-module. We write A^e for the envelopping algebra $A \otimes_k A^{op}$. We denote by $D(A)$ the unbounded derived category of the category of right A-modules. For a bimod-

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ule M, we denote by $HH^{\bullet}(A,M)$ the Hochschild cohomology with coefficients in M and by $HH_{\bullet}(A,M)$ the Hochschild homology with coefficients in M, see for example [1] or [4]. We have canonical isomorphisms

$$HH^n(A,M) \xrightarrow{\sim} H^n(\mathrm{RHom}_{A^e}(A,M)) = \mathrm{Hom}_{D(A^e)}(A, M[n])$$

and

$$HH_n(A,M) \xrightarrow{\sim} H_n(A \overset{\mathbf{L}}{\otimes}_{A^e} M).$$

Let $f \in HH^m(A,A)$. The cap product by f is a map

$$f \cap ? : HH_n(A,M) \to HH_{n-m}(A,M).$$

The following lemma gives an interpretation of the cap product in terms of the derived category.

Lemma 2.1. *The following square commutes, where the vertical arrows are the canonical identifications.*

$$\begin{tikzcd}
HH_m(A,M) \arrow[r, "f\cap ?"] \arrow[d] & HH_{m-n}(A,M) \arrow[d] \\
H_0(M \overset{\mathbf{L}}{\otimes}_{A^e} A[-m]) \arrow[r, "H_0(\mathrm{id}\otimes f)"'] & H_0(M \overset{\mathbf{L}}{\otimes}_{A^e} A[n-m]).
\end{tikzcd}$$

Proof. Let $Bar(A)$ be the bar resolution of A, we get

$$M \overset{\mathbf{L}}{\otimes}_{A^e} A = Tot(M \otimes_{A^e} Bar(A)) = M \otimes_{A^e} Bar(A).$$

Let $x \in M$ and $y \in Bar(A)$, then

$$H_0(\mathrm{id} \otimes f)([x \otimes y]) = [x \otimes f(y)] = f \cap [x \otimes y]. \quad \square$$

Now suppose that A is derived equivalent to a k-projective algebra B. By Rickard's Morita theory for derived categories [2] [3], this implies that there exist bimodule complexes $X \in D(A^{op} \otimes_k B)$ and $X^{\vee} \in D(B^{op} \otimes_k A)$ such that there are isomorphisms $\eta : A \xrightarrow{\sim} X \overset{\mathbf{L}}{\otimes}_B X^{\vee}$ and $\varepsilon : X^{\vee} \overset{\mathbf{L}}{\otimes}_A X \xrightarrow{\sim} B$ in $D(A^e)$, respectively $D(B^e)$. We may and will suppose that these isomorphisms make the following triangles commutative:

$$\begin{tikzcd}
X \arrow[r, "\eta\otimes X"] \arrow[dr, "="'] & X \overset{\mathbf{L}}{\otimes}_B X^{\vee} \overset{\mathbf{L}}{\otimes}_A X \arrow[d, "X\otimes\varepsilon"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{\vee} \arrow[r, "X^{\vee}\otimes\eta"] \arrow[dr, "="'] & X^{\vee} \overset{\mathbf{L}}{\otimes}_A X \overset{\mathbf{L}}{\otimes}_B X^{\vee} \arrow[d, "\varepsilon\otimes X^{\vee}"] \\
& X^{\vee}.
\end{tikzcd}$$

As a consequence, the functor

$$F = ? \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^{\vee}) : D(A^e) \to D(B^e)$$

is an equivalence with quasi-inverse $G = ? \overset{\mathbf{L}}{\otimes}_{B^e} (X \overset{\mathbf{L}}{\otimes}_k X^{\vee})$. We have canonical isomorphisms

$$FA = A \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^{\vee}) = X^{\vee} \overset{\mathbf{L}}{\otimes}_A A \overset{\mathbf{L}}{\otimes}_A X = X^{\vee} \overset{\mathbf{L}}{\otimes}_A X \xrightarrow{\sim} B$$

and

$$GB = B \overset{\mathbf{L}}{\otimes}_{B^e} (X^{\vee} \overset{\mathbf{L}}{\otimes}_k X) = X \overset{\mathbf{L}}{\otimes}_B B \overset{\mathbf{L}}{\otimes}_B X^{\vee} = X \overset{\mathbf{L}}{\otimes}_B X^{\vee} \xrightarrow{\sim} A.$$

We obtain a canonical isomorphism

$$\begin{aligned}
HH^n(A,A) = \mathrm{Hom}_{D(A^e)}(A, A[n]) &\xrightarrow{\sim} \mathrm{Hom}_{D(B^e)}(X^{\vee} \overset{\mathbf{L}}{\otimes}_A X, X^{\vee} \overset{\mathbf{L}}{\otimes}_A X[n]) \\
&\xrightarrow{\sim} \mathrm{Hom}_{D(B^e)}(B, B[n]) = HH^n(B,B).
\end{aligned}$$

By abuse of notation, we will still denote it by $f \mapsto Ff$. Let us suppose that M is an A-bimodule such that $N = FM$ is concentrated in degree 0. For example, if $M = A$, then $N = B$.

Theorem 2.2. *There is a canonical isomorphism*

$$HH_\bullet(A, M) \xrightarrow{\sim} HH_\bullet(B, N)$$

such that, for each $f \in HH^m(A, A)$*, the following square commutes:*

$$\begin{tikzcd} HH_n(A, M) \arrow[r, "f\cap ?"] \arrow[d, "\cong"'] & HH_{n-m}(A, M) \arrow[d, "\cong"] \\ HH_n(B, N) \arrow[r, "Ff\cap ?"'] & HH_{n-m}(B, N). \end{tikzcd}$$

Proof. We define the isomorphism

$$HH_\bullet(A, M) \xrightarrow{\sim} HH_\bullet(B, N)$$

to be induced by the canonical chain of isomorphisms in $D(k)$

$$M \overset{\mathbf{L}}{\otimes}_{A^e} A \xrightarrow{\sim} M \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_B X^\vee) = M \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} B = FM \overset{\mathbf{L}}{\otimes}_{B^e} B = N \overset{\mathbf{L}}{\otimes}_{B^e} B.$$

Let $f \in HH^m(A, A)$. It suffices to show that the following square is commutative:

$$\begin{tikzcd} M \overset{\mathbf{L}}{\otimes}_{A^e} A \arrow[r] \arrow[d, "M\otimes f"'] & M \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} B \arrow[d, "M\otimes X\otimes X^\vee\otimes Ff"] \\ M \overset{\mathbf{L}}{\otimes}_{A^e} A[m] \arrow[r] & M \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} B[m]. \end{tikzcd}$$

This is implied by the commutativity of the square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes}_k X) \arrow[d, "(Ff)\otimes X^\vee\otimes X"] \\ A[m] \arrow[r] & B[m] \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes}_k X). \end{tikzcd}$$

In turn, this will follow from the commutativity of the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes}_k X) \arrow[d, "f\otimes X\otimes X^\vee\otimes X^\vee\otimes X"] \\ A[m] \arrow[r] & A[m] \overset{\mathbf{L}}{\otimes}_{A^e} (X \overset{\mathbf{L}}{\otimes}_k X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \overset{\mathbf{L}}{\otimes}_k X). \end{tikzcd}$$

This last commutativity follows from the naturality of the adjunction morphism $A \xrightarrow{\sim} GFA$. □

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