C. R. Acad. Sci. Paris, Ser. I 355 (2017) 1168–1177

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Functional analysis

Common solutions to a system of variational inequalities over the set of common fixed points of demi-contractive operators

Solutions communes d'inégalités variationnelles sur l'ensemble des points fixes communs d'opérateurs semi-contractants

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A R T I C L E   I N F O

Article history:
Received 22 August 2017
Accepted after revision 30 October 2017
Available online 10 November 2017

Presented by the Editorial Board

Dedicated to Professor Ali Reza Medghalchi on the Occasion of His 65th Birthday

A B S T R A C T

In this paper, we introduce an explicit parallel algorithm for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators. Under suitable assumptions, we prove the strong convergence of this algorithm in the framework of a Hilbert space. The results obtained in this paper extend and improve the results of Tian and Jiang (2017), of Censor, Gibali and Reich (2012), and of many others.

R É S U M É

Dans cette Note, nous introduisons un algorithme parallèle explicite, trouvant les solutions communes d'un système d'inégalités variationnelles sur l'ensemble des points fixes communs à une famille finie d'opérateurs semi-contractants. Sous des hypothèses convenables, nous démontrons la convergence forte de cet algorithme dans le cadre des espaces de Hilbert. Les résultats obtenus étendent et améliorent ceux de Tian et Jiang (2017), de Censor, Gibali et Reich (2012), ainsi que de plusieurs autres auteurs.

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F:\mathcal{H}\to\mathcal{H}$ be a monotone operator. The classical variational inequality is formulated as the following problem:

$$\text{finding a point} \quad x^\star \in C \quad \text{such that} \quad \langle Fx^\star, y - x^\star\rangle \geq 0, \qquad \forall\, y \in C.$$

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https://doi.org/10.1016/j.crma.2017.10.018
1631-073X/

The set of solutions to this problem is denoted by $VI(F,C)$. In recent years, variational inequalities have been used to study a large variety of problems arising in structural analysis, economics, optimization, operations research, and engineering sciences (see, e.g., [20,28,19] and the references therein).

Observe that the feasible set C of the variational inequality problem can always be represented as the fixed point set of some operator, say, $C = \mathrm{Fix}(P_C)$ (P_C is the metric projection onto C). Following this idea, Yamada [26] considered the variational inequality problem $VI(F,\mathrm{Fix}(T))$, which calls for finding a point $x^\star \in \mathrm{Fix}(T)$ such that

$$\langle Fx^\star, y - x^\star\rangle \geq 0 \quad \text{for all} \quad y \in \mathrm{Fix}(T).$$

Yamada [26] considered the following hybrid steepest-descent iterative method:

$$x_{n+1} = (I - \mu\alpha_n F)Tx_n,$$

where F is a Lipschitzian continuous and strongly monotone operator and T is a nonexpansive operator. Under some appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique point in $VI(F,\mathrm{Fix}(T))$.

The literature on variational inequalities is vast, and the hybrid steepest-descent method has received great attention from many authors, who improved it in various ways; see, e.g., [27,7,6,29,5,15] and references therein.

Based on the hybrid steepest-descent method, recently (2017) Tian and Jiang [25] proved the following weak convergence theorem for zero points of an inverse strongly monotone operator and fixed points of a nonexpansive operator in a Hilbert space (Theorem 1.1).

Theorem 1.1. *Let $\mathcal{H}$ be a real Hilbert space and $T:\mathcal{H}\to\mathcal{H}$ be a nonexpansive operator with $\mathrm{Fix}(T)\neq\emptyset$. Let $F:\mathcal{H}\to\mathcal{H}$ be a k-inverse strongly monotone operator. Assume that $\mathrm{Fix}(T)\bigcap F^{-1}(0)\neq\emptyset$. Let $\{x_n\}$ and $\{y_n\}$, be sequences generated by $x_1\in\mathcal{H}$ and*

$$\begin{cases} y_n = (1-\lambda_n)x_n + \lambda_n T x_n, \\ x_{n+1} = (I - \mu\alpha_n F)y_n, \end{cases} \tag{1}$$

for each $n\in\mathbb{N}$, where $\{\lambda_n\}\in[a,b]$ for some $a,b\in(0,1)$ and $\mu\alpha_n\subset[c,d]$ for some $c,d\in(0,2k)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to a point $z\in\mathrm{Fix}(T)\bigcap F^{-1}(0)$, where $z=\lim_{n\to\infty}P_{\mathrm{Fix}(T)\bigcap F^{-1}(0)}x_n$. z is also a point in $VI(F,\mathrm{Fix}(T))$.

On the other hand, Censor, Gibali and Reich [11] (see also [12]), introduced the *Common Solutions to Variational Inequality Problem (CSVIP)*, which consists in finding common solutions to unrelated variational inequalities. The general form of the *CSVIP* is the following.

Let $\mathcal{H}$ be a Hilbert space. Let there be given, for each $i=1,2,\dots,m$, an operator $F_i:\mathcal{H}\to\mathcal{H}$ and a nonempty, closed and convex subset $C_i\subset\mathcal{H}$, with $\bigcap_{i=1}^m C_i\neq\emptyset$. The *CSVIP* (for single-valued operators) is to find a point $z\in\bigcap_{i=1}^m C_i$ such that, for each $i=1,2,\dots,m$,

$$\langle F_i z, x - z\rangle \geq 0, \qquad \forall x\in C_i, \quad 1\leq i\leq m. \tag{2}$$

We note that in *CSVIP*, if we choose all $F_i=0$, then the problem reduces to that of finding a point $z\in\bigcap_{i=1}^m C_i$ in the nonempty intersection of a finite family of closed and convex sets, which is the well-known *Convex Feasibility Problem (CFP)*.

Now, in this paper, we study the following problem.

Let $\mathcal{H}$ be a Hilbert space. Let there be given, for each $i=1,2,\dots,m$, an operator $F_i:\mathcal{H}\to\mathcal{H}$ and an operator $T_i:\mathcal{H}\to\mathcal{H}$ with $\bigcap_{i=1}^m\mathrm{Fix}(T_i)\neq\emptyset$. We intend to find a point $z\in\bigcap_{i=1}^m\mathrm{Fix}(T_i)$ such that, for each $i=1,2,\dots,m$,

$$\langle F_i z, x - z\rangle \geq 0, \qquad \forall x\in \mathrm{Fix}(T_i), \quad 1\leq i\leq m. \tag{3}$$

Recall that an operator $U:\mathcal{H}\to\mathcal{H}$ is said to be demicontractive [18] if there exists $\mu\in[0,1)$ such that

$$\|Ux-p\|^2\leq\|x-p\|^2+\mu\|x-Ux\|^2, \qquad \forall x\in\mathcal{H}, \quad \forall p\in\mathrm{Fix}(U). \tag{4}$$

In particular, if $\mu=0$ then U is called *quasi-nonexpansive* on $\mathcal{H}$. An operator satisfying (4) will be referred to as a μ-demicontractive operator. This class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization, see for example [23] and references therein.

In this paper, to solve (3), we introduce an explicit parallel algorithm based on the Halpern iterative method and the hybrid steepest-descent method for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators. We prove the strong convergence of the algorithm for a family of inverse strongly monotone operators in the framework of a Hilbert space. Finally, some applications of our main results have been obtained. Our results generalize and improve the results of Tian and Jiang [25], Censor, Gibali and Reich [11], and of many others.

2. Preliminaries

We use the following notation in the sequel:

- $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

Given a nonempty, closed convex set, $C \subset \mathcal{H}$, the mapping that assigns every point, $x \in \mathcal{H}$, to its unique nearest point in C is called the metric projection onto C and is denoted by P_C; i.e., $P_C(x) \in C$ and $\|x - P_C(x)\| = inf_{y\in C}\|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \le 0, \qquad \forall x \in \mathcal{H}, y \in C$$

(see for example, [[16], Section 3]).

We recall the following definitions concerning operator $F : \mathcal{H} \to \mathcal{H}$. The operator F is called:

- Lipschitz continuous with constant $L > 0$ if

$$\|F(x) - F(y)\| \le L\|x - y\|, \qquad \forall x, y \in \mathcal{H};$$

- monotone if

$$\langle F(x) - F(y), x - y\rangle \ge 0, \quad \forall x, y \in \mathcal{H};$$

- strongly monotone with constant $\beta > 0$, if

$$\langle F(x) - F(y), x - y\rangle \ge \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

- inverse strongly monotone with constant $\beta > 0$, $(\beta - ism)$ if

$$\langle F(x) - F(y), x - y\rangle \ge \beta\|F(x) - F(y)\|^2, \quad \forall x, y \in \mathcal{H}.$$

It is known that every β-inverse strongly monotone operator is monotone and Lipschitz continuous. We note that there exist some operators that are inverse strongly monotone, but not strongly monotone [25].

We also have the following definitions concerning $T : \mathcal{H} \to \mathcal{H}$. The operator T is called:

- nonexpansive, if

$$\|T(x) - T(y)\| \le \|x - y\|, \qquad \forall x, y \in \mathcal{H};$$

- β-strict pseudo-contractive [2], if there exists a constant $\beta \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \beta\|(x - Tx) - (y - Ty)\|^2, \qquad \forall x, y \in \mathcal{H};$$

- generalized nonexpansive [14] if there exists a constant $\mu \ge 0$ such that

$$\|Tx - Ty\| \le \|x - y\| + \mu\|x - Tx\|, \qquad \forall x, y \in \mathcal{H}.$$

We note that every generalized nonexpansive operator is quasi-nonexpansive. The class of demi-contractive operators contains the generalized nonexpansive operators and the strictly pseudo-contractive operators.

An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be an averaged operator [1] if there exists some number $\alpha \in (0, 1)$ such that

$$T = (1 - \alpha) I + \alpha S, \tag{5}$$

where $I : \mathcal{H} \to \mathcal{H}$ is the identity operator and $S : \mathcal{H} \to \mathcal{H}$ is nonexpansive. More precisely, when (5) holds, we say that T is α-averaged. It is not difficult to see that the averaged operator T is also nonexpansive and $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.

Lemma 2.1. *[8] Let $\mathcal{H}$ be a real Hilbert space. Let $T : \mathcal{H} \to \mathcal{H}$ be an operator.*

(i) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-inverse strongly monotone.
(ii) If T is κ-inverse strongly monotone, then for $\gamma > 0$, γT is $\frac{\kappa}{\gamma}$-inverse strongly monotone.
(iii) For $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-inverse strongly monotone.

Definition 2.2. Let $U : C \to C$ be an operator, then $I - U$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Ux_n\| = 0$, imply $x = Ux$.

Lemma 2.3. *[13] Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and $U : C \to C$ be β-demicontractive operator. Then the fixed point set* $\mathrm{Fix}(U)$ *of U is closed and convex.*

Lemma 2.4. *[22] Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $U : C \to C$ be β-strict pseudo-contractive. Then $I - U$ is demiclosed at* 0.

Lemma 2.5. *[14] Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $U : C \to C$ be a generalized nonexpansive operator. Then $I - U$ is demiclosed at* 0.

Lemma 2.6. *For all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$, there holds the relation:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.7. *([17]) Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{n+1} \le (1-\lambda_n)s_n + \lambda_n\delta_n, & n \ge 0, \\ s_{n+1} \le s_n - \eta_n + \mu_n, & n \ge 0, \end{cases}$$

where (λ_n) is a sequence in $(0, 1)$, (η_n) is a sequence of nonnegative real numbers and (δ_n) and (μ_n) are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(ii) $\lim_{n\to\infty} \mu_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$, *implies* $\limsup_{k\to\infty} \delta_{n_k} \le 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n\to\infty} s_n = 0$.

3. An algorithm and its convergence analysis

In this section, we introduce an explicit parallel algorithm for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators.

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space. Let for each $i \in \{1, 2, ..., m\}$, $F_i : \mathcal{H} \to \mathcal{H}$ be a κ_i-inverse strongly monotone operator and $T_i : \mathcal{H} \to \mathcal{H}$ be a λ_i-demi-contractive operator such that $I - T_i$ is demiclosed at* 0*. Assume that $\mathcal{F} = \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^m F_i^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, \nu \in \mathcal{H}$ and by*

$$\begin{cases} y_n^{(i)} = (I - \mu^{(i)}\beta_n^{(i)}F_i)T_i^n x_n, & i = 1, 2, ..., m \\ x_{n+1} = \gamma_n^{(0)}\nu + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)}, & \forall n \ge 0, \end{cases} \tag{6}$$

where $T_i^n = \alpha_n^{(i)} I + (1-\alpha_n^{(i)})T_i$. Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\gamma_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1)$ *and* $\sum_{i=0}^m \gamma_n^{(i)} = 1$,
(ii) $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{n=1}^{\infty} \gamma_n^{(0)} = \infty$,
(iii) $\{\mu^{(i)}\beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\kappa_i)$,
(iv) $\lambda_i < \alpha_n^{(i)} \le e_i < 1$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}\nu \in \mathcal{F}$ which is also a point in $\bigcap_{i=1}^m VI(F_i, \mathrm{Fix}(T_i))$.

Proof. First we show that $\{x_n\}$ is bounded. Take $x^\star \in \mathcal{F}$. Since for each $i \in \{1, 2, ..., m\}$, T_i is λ_i-demicontractive, using Lemma 2.6 we arrive at

$$\begin{aligned} \|T_i^n x_n - x^\star\|^2 &= \|\alpha_n^{(i)} x_n + (1-\alpha_n^{(i)})T_i x_n - x^\star\|^2 \\ &= \alpha_n^{(i)}\|x_n - x^\star\|^2 + (1-\alpha_n^{(i)})\|T_i x_n - x^\star\|^2 - \alpha_n^{(i)}(1-\alpha_n^{(i)})\|T_i x_n - x_n\|^2 \\ &\le \alpha_n^{(i)}\|x_n - x^\star\|^2 + (1-\alpha_n^{(i)})(\|x_n - x^\star\|^2 + \lambda_i\|T_i x_n - x_n\|^2) \\ &- \alpha_n^{(i)}(1-\alpha_n^{(i)})\|T_i x_n - x_n\|^2 \\ &= \|x_n - x^\star\|^2 + (1-\alpha_n^{(i)})(\lambda_i - \alpha_n^{(i)})\|T_i x_n - x_n\|^2. \end{aligned} \tag{7}$$

Put $w_n^{(i)} = T_i^n x_n$. From condition (iii) and by the assumption that F_i is a κ_i-inverse strongly monotone, we get that

$$\begin{aligned} \|y_n^{(i)} - x^\star\|^2 &= \|(I - \mu^{(i)}\beta_n^{(i)}F_i)w_n^{(i)} - (I - \mu^{(i)}\beta_n^{(i)}F_i)x^\star\|^2 \\ &\le \|w_n^{(i)} - x^\star\|^2 + (\mu^{(i)}\beta_n^{(i)})^2\|F_i w_n^{(i)} - F_i x^\star\|^2 - 2\mu^{(i)}\beta_n^{(i)}\langle F_i w_n^{(i)} - F_i x^\star, w_n^{(i)} - x^\star\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq \|w_n^{(i)} - x^\star\|^2 + (\mu^{(i)}\beta_n^{(i)})^2\|F_i w_n^{(i)} - F_i x^\star\|^2 - 2\mu^{(i)}\beta_n^{(i)}\kappa_i\|F_i w_n^{(i)} - F_i x^\star\|^2 \\
&\leq \|w_n^{(i)} - x^\star\|^2 + \mu^{(i)}\beta_n^{(i)}(\mu^{(i)}\beta_n^{(i)} - 2\kappa_i)\|F_i w_n^{(i)} - F_i x^\star\|^2 \\
&\leq \|w_n^{(i)} - x^\star\|^2 \\
&\leq \|x_n - x^\star\|^2.
\end{aligned}
$$

By the convexity of $\|.\|^2$, we have:

$$
\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\gamma_n^{(0)}\nu + \sum_{i=1}^{m}\gamma_n^{(i)} y_n^{(i)} - x^\star\|^2 \\
&\leq \gamma_n^{(0)}\|\nu - x^\star\|^2 + \sum_{i=1}^{m}\gamma_n^{(i)}\|y_n^{(i)} - x^\star\|^2 \\
&\leq \gamma_n^{(0)}\|\nu - x^\star\|^2 + (1-\gamma_n^{(0)})\|x_n - x^\star\|^2 \\
&\leq \max\{\|\nu - x^\star\|^2, \|x_n - x^\star\|^2\} \\
&\leq \ldots \leq \max\{\|\nu - x^\star\|^2, \|x_0 - x^\star\|^2\}.
\end{aligned}
\tag{8}
$$

This yields that the sequence $\{x_n\}$ is bounded. Furthermore, the sequence $\{y_n^{(i)}\}$ is bounded. Next we prove that the sequences $\{x_n\}$ converge strongly to $\nu^\star = P_{\mathcal{F}}\nu$. From the inequality, $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle (\forall x, y \in \mathcal{H})$, we find that

$$
\begin{aligned}
\|x_{n+1} - \nu^\star\|^2 &\leq \|\sum_{i=1}^{m}\gamma_n^{(0)} y_n^{(i)} - (1-\gamma_n^{(0)})\nu^\star\|^2 \\
&\quad + 2\gamma_n^{(0)}\langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle \\
&= \|(1-\gamma_n^{(0)})(\sum_{i=1}^{m}\frac{\gamma_n^{(i)}}{1-\gamma_n^{(0)}} y_n^{(i)} - \nu^\star)\|^2 \\
&\quad + 2\gamma_n^{(0)}\langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle \\
&= (1-\gamma_n^{(0)})^2\sum_{i=1}^{m}\frac{\gamma_n^{(i)}}{1-\gamma_n^{(0)}}\|y_n^{(i)} - \nu^\star\|^2 \\
&\quad + 2\gamma_n^{(0)}\langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle \\
&\leq (1-\gamma_n^{(0)})\sum_{i=1}^{m}\gamma_n^{(i)}\|y_n^{(i)} - \nu^\star\|^2 \\
&\quad + 2\gamma_n^{(0)}\langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle \\
&\leq (1-\gamma_n^{(0)})^2\|x_n - \nu^\star\|^2 \\
&\quad + 2\gamma_n^{(0)}\langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle.
\end{aligned}
$$

It immediately follows that

$$
\begin{aligned}
\Gamma_{n+1} &\leq (1-\gamma_n^{(0)})^2\Gamma_n + 2\gamma_n^{(0)}\eta_n \\
&= (1-2\gamma_n^{(0)})\Gamma_n + (\gamma_n^{(0)})^2\Gamma_n + 2\gamma_n^{(0)}\eta_n \\
&\leq (1-2\gamma_n^{(0)})\Gamma_n + 2\gamma_n^{(0)}\{\frac{\gamma_n^{(0)}N}{2} + \eta_n) \\
&\leq (1-\rho_n)\Gamma_n + \rho_n\delta_n,
\end{aligned}
\tag{9}
$$

where $\Gamma_n = \|x_n - \vartheta^\star\|^2$, $\eta_n = \langle \nu - \nu^\star, x_{n+1} - \nu^\star\rangle$, $N = \sup\{\|x_n - \nu^\star\|^2 : n \geq 0\}$, $\rho_n = 2\gamma_n^{(0)}$ and $\delta_n = \frac{\gamma_n^{(0)}N}{2} + \eta_n$. We observe that $\rho_n \to 0$, $\sum_{n=1}^{\infty}\rho_n = \infty$.

Since F_i is κ_i-inverse strongly monotone, we can rewrite $y_n^{(i)}$ as

$$
y_n^{(i)} = (1-\xi_n^{(i)})w_n^{(i)} + \xi_n^{(i)} S_n^{(i)} w_n^{(i)},
$$

by using Lemma 2.1, where $\xi_n^{(i)} = \frac{\mu^{(i)}\beta_n^{(i)}}{2\kappa_i}$ and $S_n^{(i)}$ are nonexpansive operators of $\mathcal{H}$ into $\mathcal{H}$. Utilizing Lemma 2.6, we have:

$$\begin{aligned}\|y_n^{(i)} - v^\star\|^2 &= \|(1-\xi_n^{(i)})w_n^{(i)} + \xi_n^{(i)} S_n^{(i)} w_n^{(i)} - v^\star\|^2 \\ &\le (1-\xi_n^{(i)})\|w_n^{(i)} - v^\star\|^2 + \xi_n^{(i)}\|S_n^{(i)} w_n^{(i)} - v^\star\|^2 - \xi_n^{(i)}(1-\xi_n^{(i)})\|S_n^{(i)} w_n^{(i)} - w_n^{(i)}\|^2 \\ &\le \|w_n^{(i)} - v^\star\|^2 - \xi_n^{(i)}(1-\xi_n^{(i)})\|S_n^{(i)} w_n^{(i)} - w_n^{(i)}\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - v^\star\|^2 &= \|\gamma_n^{(0)} v + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)} - v^\star\|^2 \\ &\le \gamma_n^{(0)}\|v - v^\star\|^2 + \sum_{i=1}^m \gamma_n^{(i)}\|y_n^{(i)} - v^\star\|^2 \\ &\le \gamma_n^{(0)}\|v - v^\star\|^2 + (1-\gamma_n^{(0)})\|x_n^{(i)} - v^\star\|^2 \\ &\quad - \sum_{i=1}^m \gamma_n^{(i)}\xi_n^{(i)}(1-\xi_n^{(i)})\|S_n^{(i)} w_n^{(i)} - w_n^{(i)}\|^2 \\ &\quad - \sum_{i=1}^m \gamma_n^{(i)}(1-\alpha_n^{(i)})(\alpha_n^{(i)} - \lambda_i)\|T_i x_n - x_n\|^2.\end{aligned} \tag{10}$$

Now by setting

$$\begin{aligned}\zeta_n &= \sum_{i=1}^m \gamma_n^{(i)}\xi_n^{(i)}(1-\xi_n^{(i)})\|S_n^{(i)} w_n^{(i)} - w_n^{(i)}\|^2 \\ &\quad + \sum_{i=1}^m \gamma_n^{(i)}(1-\alpha_n^{(i)})(\alpha_n^{(i)} - \lambda_i)\|T_i x_n - x_n\|^2,\end{aligned} \tag{11}$$

and

$$\varrho_n = \gamma_n^{(0)}\|v - v^\star\|^2, \tag{12}$$

the inequality (10) can be rewritten in the following form:

$$\Gamma_{n+1} \le \Gamma_n - \zeta_n + \varrho_n. \tag{13}$$

To use Lemma 2.7 (considering inequalities (9) and (13)), it suffices to verify that, for all subsequences $\{n_k\} \subset \{n\}$, $\lim_{k\to\infty} \zeta_{n_k} = 0$ implies

$$\limsup_{k\to\infty} \delta_{n_k} \le 0.$$

We assume that $\lim_{k\to\infty} \zeta_{n_k} = 0$. From (11) and by our assumptions on $\{\gamma_n^{(i)}\}$, $\{\alpha_n^{(i)}\}$, and $\{\xi_n^{(i)}\}$, we have:

$$\lim_{k\to\infty} \|T_i x_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|S_{n_k}^{(i)} w_{n_k}^{(i)} - w_{n_k}^{(i)}\| = 0. \tag{14}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \widehat{x}$. Since $\lim_{n\to\infty} \|x_n - w_n^{(i)}\| = 0$, we have $w_{n_k}^{(i)} \rightharpoonup \widehat{x}$. Since $\{\beta_{n_k}^{(i)}\}$ is bounded, we can find a subsequence $\{\beta_{n_{k_j}}^{(i)}\}$ converging to $\beta^{(i)}$ such that $\mu^{(i)}\beta^{(i)} \subset [c_i, d_i]$. Since $\{w_{n_{k_j}}^{(i)}\}$ is bounded and F_i is inverse strongly monotone, we know that $\{F_i w_{n_{k_j}}^{(i)}\}$ is bounded. Hence, we have:

$$\|(I - \mu^{(i)}\beta_{n_{k_j}}^{(i)} F_i) w_{n_{k_j}} - (I - \mu^{(i)}\beta^{(i)} F_i) w_{n_{k_j}}\| \le |\mu^{(i)}\beta_{n_{k_j}}^{(i)} - \mu^{(i)}\beta^{(i)}|\|F_i w_{n_{k_j}}^{(i)}\| \to 0.$$

From (14) we have $\lim_{n\to\infty} \|y_n^{(i)} - w_n^{(i)}\| = 0$, hence

$$\|(I - \mu^{(i)}\beta_{n_{k_j}}^{(i)} F_i) w_{n_{k_j}}^{(i)} - w_{n_{k_j}}^{(i)}\| \to 0.$$

Therefore, we get

$$\begin{aligned}\|(I - \mu^{(i)}\beta^{(i)} F_i) w_{n_{k_j}}^{(i)} - w_{n_{k_j}}^{(i)}\| &\le \|(I - \mu^{(i)}\beta_{n_{k_j}}^{(i)} F_i) w_{n_{k_j}}^{(i)} - (I - \mu^{(i)}\beta^{(i)} F_i) w_{n_{k_j}}^{(i)}\| \\ &\quad + \|(I - \mu^{(i)}\beta_{n_{k_j}}^{(i)} F) w_{n_{k_j}}^{(i)} - w_{n_{k_j}}^{(i)}\| \to 0.\end{aligned}$$

From the demiclosedness of $I - \mu^{(i)}\beta^{(i)}F_i$, we obtain that

$$\widehat{x} \in \mathrm{Fix}(I - \mu^{(i)}\beta^{(i)}F_i) = F_i^{-1}(0), \quad i \in \{1, 2, \ldots, m\}.$$

From the demiclosedness of $I - T_i$ and using (14), we get that $\widehat{x} \in \bigcap_{i=1}^m \mathrm{Fix}(T_i)$. Thus $\widehat{x} \in \mathcal{F}$. Now we show that

$$\limsup_{k\to\infty} \delta_{n_k} = \limsup_{k\to\infty} \langle \vartheta - \vartheta^\star, x_{n_k} - \vartheta^\star \rangle \le 0. \tag{15}$$

To show this inequality, we choose a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty} \langle \nu - \nu^\star, x_{n_{k_j}} - \nu^\star \rangle = \limsup_{k\to\infty} \langle \nu - \nu^\star, x_{n_k} - \nu^\star \rangle.$$

Since $\{x_{n_{k_j}}\}$ converges weakly to $\widehat{x}$, it follows that

$$\limsup_{k\to\infty} \langle \nu - \nu^\star, x_{n_k} - \nu^\star \rangle = \lim_{j\to\infty} \langle \nu - \nu^\star, x_{n_{k_j}} - \nu^\star \rangle = \langle \nu - P_{\mathcal{F}}(\nu), \widehat{x} - P_{\mathcal{F}}(\nu) \rangle \le 0. \tag{16}$$

Hence, all conditions of Lemma 2.7 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \|x_n - \nu^\star\| = 0$, that is $\{x_n\}$ converges strongly to $\nu^\star = P_{\mathcal{F}}(\nu)$, which completes the proof. $\square$

Theorem 3.2. *Let $\mathcal{H}$ be a real Hilbert space. Let, for each $i \in \{1, 2, \ldots, m\}$, $F_i : \mathcal{H} \to \mathcal{H}$ be a κ_i-inverse strongly monotone operator and $T_i : \mathcal{H} \to \mathcal{H}$ be a λ_i-strict pseudo-contractive operator. Assume that $\mathcal{F} = \bigcap_{i=1}^m \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^m F_i^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 \in \mathcal{H}$ and*

$$x_{n+1} = \sum_{i=1}^m \gamma_n^{(i)} (I - \mu^{(i)}\beta_n^{(i)}F_i) T_i^n x_n, \qquad \forall n \ge 0, \tag{17}$$

where $T_i^n = \alpha_n^{(i)} I + (1 - \alpha_n^{(i)}) T_i$. Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\gamma_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1)$,
(ii) $\sum_{i=1}^m \gamma_n^{(i)} = 1 - \gamma_n^{(0)}$ *where* $\gamma_n^{(0)} \in (0, 1)$, $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{n=1}^\infty \gamma_n^{(0)} = \infty$,
(iii) $\{\mu^{(i)}\beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\kappa_i)$,
(iv) $\lambda_i < \alpha_n^{(i)} \le e_i < 1$.

Then, the sequence $\{x_n\}$ converges strongly to $x^\star \in \mathcal{F}$, which satisfies $\|x^\star\| = \min\{\|x\| : x \in \mathcal{F}\}$.

Proof. We note that every strict pseudo-contractive mapping is demi-contractive. Also, from Lemma 2.4 we know that $I - T_i$ are demiclosed at 0. Now setting $\nu = 0$ in Theorem 3.1 we obtain the desired result. $\square$

Now we consider an algorithm similar to algorithm (6) for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of strongly quasi-nonexpansive operators. Recall that an operator $U : \mathcal{H} \to \mathcal{H}$ is said to be ρ-strongly quasi-nonexpansive, where $\rho \ge 0$, if

$$\|Ux - p\|^2 \le \|x - p\|^2 - \rho\|x - Ux\|^2, \qquad \forall x \in \mathcal{H}, \quad \forall p \in \mathrm{Fix}(U). \tag{18}$$

More information on strongly quasi-nonexpansive operators can be found in Section 2.2 of [21].

Theorem 3.3. *Let $\mathcal{H}$ be a real Hilbert space. Let for each $i \in \{1, 2, \ldots, m\}$, $F_i : \mathcal{H} \to \mathcal{H}$ be a κ_i-inverse strongly monotone operator and $U_i : \mathcal{H} \to \mathcal{H}$ be a ρ_i-strongly quasi-nonexpansive operator where $\rho_i > 0$ and such that $I - U_i$ is demiclosed at 0. Assume that $\mathcal{F} = (\bigcap_{i=1}^m \mathrm{Fix}(U_i)) \bigcap (\bigcap_{i=1}^m F_i^{-1}(0)) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, \nu \in \mathcal{H}$ and by*

$$\begin{cases} y_n^{(i)} = (I - \mu^{(i)}\beta_n^{(i)}F_i) U_i x_n, \quad i = 1, 2, \ldots, m \\ x_{n+1} = \gamma_n^{(0)}\nu + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)}, \qquad \forall n \ge 0. \end{cases} \tag{19}$$

Let the sequences $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\gamma_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1)$ *and* $\sum_{i=0}^m \gamma_n^{(i)} = 1$,
(ii) $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{n=1}^\infty \gamma_n^{(0)} = \infty$,
(iii) $\{\mu^{(i)}\beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\kappa_i)$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}\nu \in \mathcal{F}$ which is also a point in $\bigcap_{i=1}^m VI(F_i, \mathrm{Fix}(U_i))$.

Proof. Since U_i is ρ_i-strongly quasi-nonexpansive operator with $\rho_i > 0$, for each $x^\star \in \mathcal{F}$, we have:

$$\|U_i x_n - x^\star\|^2 \le \|x_n - x^\star\|^2 - \rho_i \|x_n - U_i x_n\|^2. \tag{20}$$

On substituting inequality (20) into inequality (7) in Theorem 3.1 and by similar arguments, we obtain the desired result. □

Remark 3.4. In [25], Tian and Jiang proved a weak convergence theorem (see Theorem 1.1) for finding zero points of an inverse strongly monotone operator and fixed points of a nonexpansive operator in a Hilbert space. In this paper, we generalized the result for finding common fixed points of a finite family of demi-contractive operators (as a general class of operators) and of zero points of a family of inverse strongly monotone operators. We also proved a strong convergence theorem, which is more desirable than weak convergence.

4. Applications

In this section, we present some application of our main result.

4.1. The multiple-set split feasibility problem

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and let $\{C_i\}_{i=1}^p$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$. The multiple-set split feasibility problem (MSSFP) was introduced by Censor et al. (2005) [10], and is formulated as finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad Ax^\star \in \bigcap_{i=1}^{r} Q_i.$$

Masad and Riech [24] studied the constrained multiple-set split convex feasibility problem (CMSSCFP). Let $A_i : \mathcal{H} \to \mathcal{K}$, $i = 1, 2, \ldots, r$, be r bounded linear operators and let Ω be another closed and convex subset of $\mathcal{H}$. The CMSSCFP is formulated as follows:

$$\text{find a point} \quad x^\star \in \Omega \quad \text{such that} \quad x^\star \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad A_i(x^\star) \in Q_i \quad \text{for each } i = 1, 2, \ldots, r.$$

The multiple-set split feasibility problem with $p = r = 1$ is known as the split feasibility problem (SFP), which is formulated as finding a point $x^\star$ with the property:

$$x^\star \in C \quad \text{and} \quad Ax^\star \in Q,$$

where C and Q are nonempty closed convex subsets of $\mathcal{H}$ and $\mathcal{K}$, respectively. The split feasibility problem was introduced by Censor and Elfving (1994) ([9]). It has attracted many authors attention due to its application in optimization problem and signal processing. To solve the SFP, Byrne [3,4] proposed his CQ algorithm, which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n$$

where $\lambda \in (0, \frac{2}{\|A\|^2})$, A^* is the adjoint of A.

Now we present an algorithm for solving the multiple-set split feasibility problem when C_i are the fixed point set of nonlinear operators.

Theorem 4.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces. Let for each $i \in \{1, 2, \ldots, m\}$, $A_i : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and $T_i : \mathcal{H} \to \mathcal{H}$ be a generalized nonexpansive mapping. Let $\{Q_i\}_{i=1}^m$ be a family of nonempty closed convex subsets in $\mathcal{K}$. Assume that $\mathcal{F} = \{x^\star \in \bigcap_{i=1}^m \mathrm{Fix}(T_i) : A_i(x^\star) \in Q_i, \quad i = 1, 2, \ldots, m\} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, v \in \mathcal{H}$ and by*

$$\begin{cases} y_n^{(i)} = (I - \mu^{(i)} \beta_n^{(i)} A_i{}^*(I - P_{Q_i})A_i)T_i^n x_n, \quad i = 1, 2, \ldots, m \\ x_{n+1} = \gamma_n^{(0)} v + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)}, \qquad \forall n \ge 0, \end{cases} \tag{21}$$

where $T_i^n = \alpha_n^{(i)} I + (1 - \alpha_n^{(i)})T_i$. Let the sequences $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\gamma_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1)$ *and* $\sum_{i=0}^m \gamma_n^{(i)} = 1$,
(ii) $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{n=1}^\infty \gamma_n^{(0)} = \infty$,
(iii) $\{\mu^{(i)} \beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, \frac{2}{\|A_i\|^2})$,
(iv) $\{\alpha_n^{(i)}\} \subset [e_i, l_i] \subset (0, 1)$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} v \in \mathcal{F}$.

Proof. Notice that $A_i x^\star \in Q_i$ if and only if $x^\star \in (A_i{}^*(I - P_{Q_i})A_i)^{-1}(0)$. Putting $F_i = A_i{}^*(I - P_{Q_i})A_i$ we see that F_i is $\frac{1}{\|A_i\|^2}$-inverse strongly monotone operator (see [25] for details). We note that every generalized nonexpansive operator is 0-demi-contractive. Also, from Lemma 2.5 we know that $I - T_i$ are demiclosed at 0. Now, utilizing Theorem 3.1, we obtain the desired result. □

4.2. Common solutions to a system of variational inequalities

Now, we present a strong convergence theorem for finding common solutions to a system of variational inequalities that generalizes the result of Censor, Gibali, and Reich [11].

Theorem 4.2. *Let $\mathcal{H}$ be a real Hilbert space. Let for each $i \in \{1, 2, ..., m\}$, C_i be a nonempty, closed convex subset of $\mathcal{H}$ and $F_i : \mathcal{H} \to \mathcal{H}$ be a κ_i-inverse strongly monotone operator. Assume that $\mathcal{F} = \bigcap_{i=1}^m VI(F_i, C_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, \nu \in \mathcal{H}$ and by*

$$\begin{cases} y_n^{(i)} = (I - \mu^{(i)}\beta_n^{(i)} F_i) P_{C_i} x_n, \quad i = 1, 2, ..., m \\ x_{n+1} = \gamma_n^{(0)} \nu + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)}, \qquad \forall n \geq 0. \end{cases} \tag{22}$$

Let the sequences $\{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:

(i) $\{\gamma_n^{(i)}\} \subset [a_i, b_i] \subset (0, 1)$ *and* $\sum_{i=0}^m \gamma_n^{(i)} = 1$,
(ii) $\lim_{n\to\infty} \gamma_n^{(0)} = 0$ *and* $\sum_{n=1}^\infty \gamma_n^{(0)} = \infty$,
(iii) $\{\mu^{(i)}\beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, 2\kappa_i)$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}}\nu \in \bigcap_{i=1}^m VI(F_i, C_i)$.

Proof. We note that the metric projection P_C is a 1-strongly quasi-nonexpansive operator. Now utilizing Theorem 3.3, we obtain the desired result. □

Acknowledgements

The author would like to thank the referees for their comments and suggestions, which have allowed him to improve an earlier version of this paper.

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