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Combinatorics

Discriminantal arrangement, 3×3 minors of Plücker matrix and hypersurfaces in Grassmannian $Gr(3, n)$

Arrangement discriminant, mineurs 3×3 de la matrice de Plücker et hypersurfaces de la grassmannienne $Gr(3, n)$

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A R T I C L E   I N F O

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A B S T R A C T

We show that points in specific degree-2 hypersurfaces in the Grassmannian $Gr(3, n)$ correspond to generic arrangements of n hyperplanes in $\mathbb{C}^3$ with associated discriminantal arrangement having intersections of multiplicity three in codimension two.

R É S U M É

Nous montrons que les points d'hypersurfaces spécifiques de degré 2 de la grasmannienne $Gr(3, n)$ correspondent aux arrrangements génériques de n hyperplans dans $\mathbf{C}^3$, dont l'arrangement discriminant possède des intersections de triplets d'hyperplans de codimension deux.

1. Introduction

In 1989, Manin and Schechtman (cf. [10]) considered a family of arrangements of hyperplanes generalizing classical braid arrangements that they called the *discriminantal arrangements* (cf. [10] p. 209). Such an arrangement $\mathcal{B}(n, k), n, k \in \mathbf{N}$ for $k \geq 2$ depends on a choice $H_1^0, \ldots, H_n^0$ of collections of hyperplanes in general position in $\mathbb{C}^k$. It consists of parallel translates of $H_1^{t_1}, \ldots, H_n^{t_n}, (t_1, \ldots, t_n) \in \mathbb{C}^n$ that fail to form a generic arrangement in $\mathbb{C}^k$. $\mathcal{B}(n, k)$ can be viewed as a generalization of the pure braid group arrangement (cf. [12]) with which $\mathcal{B}(n, 1)$ coincides. These arrangements have several beautiful relations

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with diverse problems, including combinatorics (cf. [10], [1], [3] and also [4], which is an earlier appearance of discriminantal arrangements), the Zamolodchikov equation with its relation to higher category theory (cf. Kapranov–Voevodsky [7]), and vanishing of cohomology of bundles on toric varieties (cf. [13]). The paper [10] is concerned with arrangements $\mathcal{B}(n,k)$ whose combinatorics is constant on a Zariski open set $\mathcal{Z}$ in the space of generic arrangements $H_i^0, i = 1, \ldots, n$, but does not describe the set $\mathcal{Z}$ explicitly. In 1994 (see [5]), Falk showed that, contrary to what was frequently stated (see for instance [11], sect. 8, [12] or [8]), the combinatorial type of $\mathcal{B}(n,k)$ depends on the arrangement $\mathcal{A}$ of hyperplanes $H_i^0, i = 1, \ldots, n$ by providing an example of $\mathcal{A}$ for which the corresponding discriminantal arrangement has combinatorial type distinct from the one that occurs when $\mathcal{A}$ varies within the Zariski open set $\mathcal{Z}$. In 1997, Bayer and Brandt (cf. [3]) called the arrangements $\mathcal{A}$ in $\mathcal{Z}$ *very generic* and conjectured the full description of the intersection lattice of $\mathcal{B}(n,k)$ if $\mathcal{A} \in \mathcal{Z}$. In 1999, Athanasiadis proved their conjecture (cf. [1]). In particular, for the case of the arrangement $\mathcal{A}$ in $\mathbb{R}^k$, endowed with standard metric, he introduced a degree m polynomial $p_{\mathbb{T}}(a_{ij})$ (section 1 in [1] and subsection 2.3 in this paper) in the indeterminates (a_{ij}), where $\alpha_i = (a_{ij})$ is the normal vector to hyperplane H_i^0, $i \in L_h \in \mathbb{T}$, L_h is a subset of cardinality $k+1$ of $\{1, \ldots, n\}$ and $\mathbb{T}$ is a set of cardinality m. Since a null space of this polynomial corresponds to the intersection of hyperplanes in $\mathcal{B}(n,k)$, he provided, in the case of very generic arrangements, a full description of sets $\mathbb{T}$ such that $p_{\mathbb{T}}(a_{ij}) = 0$ (cf. Theorem 3.2 in [1]). In particular, all codimension-2 intersections of hyperplanes in $\mathcal{B}(n,k)$ have multiplicity 2 or $k+2$ if $\mathcal{A}$ is very generic.

More recently, in 2016 (cf. [9]), Libgober and second author gave a sufficient geometric condition for an arrangement $\mathcal{A}$ not to be very generic. In particular, they gave a necessary and sufficient condition for multiplicity-3 codimension-2 intersections of hyperplanes in $\mathcal{B}(n,k)$ to appear (Theorem 3.8 [9] and Theorem 2.2 in this paper).

The purpose of this short note is double. From one side, it aims at rewriting the result obtained in [9] in terms of the polynomial $p_{\mathbb{T}}(a_{ij})$ introduced by Athanasiadis and at proving that, in case of non very generic arrangements, if $\mathbb{T}$ is a set of cardinality 3 such that $p_{\mathbb{T}}(a_{ij}) = 0$, then the polynomial $p_{\mathbb{T}}(a_{ij})$ has a simpler polynomial expression $\widetilde{p}_{\mathbb{T}}(a_{ij})$.

On the other side, the purpose is to show, by means of a more algebraic point of view, that non very generic arrangements $\mathcal{A}$ of cardinality n in $\mathbb{C}^3$ are points in a well-defined degree 2 hypersurface in the projective Grassmannian $Gr(3,n)$. Indeed, the space of generic arrangements of n lines in $(\mathbb{P}^2)^n$ is a Zariski open set U in the space of all arrangements of n lines in $(\mathbb{P}^2)^n$. On the other hand, in $Gr(3,n)$ there is an open set U' consisting of 3-spaces intersecting each coordinate hyperplane transversally (i.e. having dimension of intersection equal 2). One has also one set $\tilde{U}$ in $Hom(\mathbb{C}^3, \mathbb{C}^n)$ consisting of embeddings with image transversal to coordinate hyperplanes and $\tilde{U}/GL(3) = U'$ and $\tilde{U}/(\mathbb{C}^*)^n = U$. Hence, generic arrangements can be regarded as points in $Gr(3,n)$.

The content of paper is the following.

In section 2, we recall the definition of the discriminantal arrangement from [10], basic results in [9], the definition of $p_{\mathbb{T}}(a_{ij})$ in [1] and basic notions on the Grassmannian (cf. [6]). In section 3, we give a full description of the main example $\mathcal{B}(6,3)$ of 6 hyperplanes in $\mathbb{R}^3$. Section 4 contains the result stating the equivalence of polynomial $p_{\mathbb{T}}(a_{ij})$ with its reduced form $\widetilde{p}_{\mathbb{T}}(a_{ij})$ (cf. Theorem 4.4). The last section contains the last result of this paper (cf. Theorem 5.4), describing a family of hypersurfaces in the projective Grassmannian $Gr(3,n)$ in terms of non very generic arrangements $\mathcal{A}$ in $\mathbb{C}^3$. Notice that in Sections 3 and 4 $\mathcal{A}$ is an arrangement in $\mathbb{R}^k$, while in Section 5, $\mathcal{A}$ is an arrangement in $\mathbb{C}^k$.

2. Preliminaries

2.1. Discriminantal arrangement

Let $H_i^0, i = 1, \ldots, n$ be a generic arrangement in $\mathbb{C}^k, k < n$ i.e. a collection of hyperplanes such that $\dim \bigcap_{i \in K, |K|=k} H_i^0 = 0$. The space of parallel translates $\mathbb{S}(H_1^0, \ldots, H_n^0)$ (or simply $\mathbb{S}$ when dependence on H_i^0 is clear or not essential) is the space of n-tuples $H_1, \ldots, H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, \ldots, n$. One can identify $\mathbb{S}$ with the n-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, \ldots, H_n^0)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$ (see [9]).

We will use the compactification of $\mathbb{C}^k$ viewing it as $\mathbb{P}^k \setminus H_\infty$ endowed with a collection of hyperplanes $\bar{H}_i^0$ that are projective closures of affine hyperplanes H_i^0. The condition of genericity is equivalent to $\bigcup_i H_i^0$ being a normal crossing divisor in $\mathbb{P}^k$.

For a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, \ldots, n$, *the trace at infinity* (denoted by $\mathcal{A}_\infty$) is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$.

The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of n-tuples of hyperplanes in $\mathbb{P}^k$). For a t-tuple $H_{i_1}, \ldots, H_{i_t}$ $(t \geq 1)$ of hyperplanes in $\mathcal{A}$, recall that the arrangement that is obtained by intersections of hyperplanes $H \in \mathcal{A}, H \neq H_{i_s}, s = 1, \ldots, t$, with $H_{i_1} \cap \cdots \cap H_{i_t}$, is called the *restriction* of $\mathcal{A}$ to $H_{i_1} \cap \cdots \cap H_{i_t}$.

For a generic arrangement $\mathcal{A}_\infty$, consider the closed subset of $\mathbb{S}$ formed by those collections that fail to form a generic arrangement. This subset is a union of hyperplanes with each hyperplane D_L corresponding to a subset $L = \{i_1, \ldots, i_{k+1}\} \subset [n] := \{1, \ldots, n\}$ and consisting of n-tuples of translates of hyperplanes $H_1^0, \ldots, H_n^0$ in which translates of $H_{i_1}^0, \ldots, H_{i_{k+1}}^0$ fail to form a generic arrangement. The arrangement $\mathcal{B}(n,k,\mathcal{A}_\infty)$ of hyperplanes D_L is called *the discriminantal arrangement* and has been introduced by Manin and Schechtman (see [10]). Notice that since the combinatorics of discriminantal arrangement depends on the arrangement $\mathcal{A}_\infty$ rather than $\mathcal{A}$, we denote it by $\mathcal{B}(n,k,\mathcal{A}_\infty)$ following the notation in [9].

2.2. Good 3s-partitions

Given $s \geq 2$ and $n \geq 3s$, consider the set $\mathbb{T} = \{L_1, L_2, L_3\}$, with L_i subsets of $[n]$ such that $|L_i| = 2s$, $|L_i \cap L_j| = s$ $(i \neq j)$, $L_1 \cap L_2 \cap L_3 = \emptyset$ (in particular $|\bigcup L_i| = 3s$) with a choice $L_1 = \{i_1, \dots, i_{2s}\}, L_2 = \{i_{s+1}, \dots, i_{3s}\}, L_3 = \{i_1, \dots, i_s, i_{2s+1}, \dots, i_{3s}\}$. We call the set $\mathbb{T} = \{L_1, L_2, L_3\}$ *a good* 3*s-partition*.

Given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$, subsets L_i define hyperplanes D_{L_i} in the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}_\infty)$. In the rest of the paper, we will always use D_L to denote hyperplanes in discriminantal arrangement. With the above notations, the following lemma holds.

Lemma 2.1. *(Lemma 3.1 [9]) Let $s \geq 2$, $n = 3s$, $k = 2s - 1$ and $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{C}^k$. Given a good 3s-partition $\mathbb{T} = \{L_1, L_2, L_3\}$ of $[n] = [3s]$, consider the triple of codimension-s subspaces $H_{\infty,i,j} = \bigcap_{t \in L_i \cap L_j} H_{\infty,t}$ of the hyperplane at infinity H_∞. Then $H_{\infty,i,j}$ span a proper subspace in H_∞ if and only if the codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2.*

In [9], the authors define a notion of dependency for a generic arrangement $\mathcal{A}_\infty = \{W_{\infty,1}, \dots, W_{\infty,3s}\}$ in $\mathbb{P}^{2s-2}$, $s \geq 2$ based on Lemma 2.1 as follows. If there exists a partition I_1, I_2 and I_3 of $[3s]$ such that $P_i = \bigcap_{t \in I_i} W_{\infty,t}$ span a proper subspace in $\mathbb{P}^{2s-2}$, then $\mathcal{A}_\infty$ is called *dependent*. Remark that if $\{L_1, L_2, L_3\}$ is a good 3s-partition and if we set $I_1 = L_1 \cap L_2$, $I_2 = L_1 \cap L_3$, $I_3 = L_2 \cap L_3$, then the assumption of Lemma 2.1 is that the trace at infinity $\mathcal{A}_\infty$ of $\mathcal{A}$ is dependent and the following theorem holds.

Theorem 2.2. *(Theorem 3.8 [9]) Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{C}^k$ and $\mathcal{A}_\infty$ the trace at infinity of $\mathcal{A}$.*

1. The arrangement $\mathcal{B}(n, k, \mathcal{A}_\infty)$ has $\binom{n}{k+2}$ codimension-2 strata of multiplicity $k + 2$.

2. There is a one-to-one correspondence between

(a) restriction arrangements of $\mathcal{A}_\infty$ that are dependent, and

(b) triples of hyperplanes in $\mathcal{B}(n, k, \mathcal{A}_\infty)$ for which the codimension of their intersection is equal to 2.

3. There are no codimension-2 strata having multiplicity 4 unless $k = 3$. All codimension-2 strata of $\mathcal{B}(n, k, \mathcal{A}_\infty)$ not mentioned in part 1 have multiplicity either 2 or 3.

4. The combinatorial type of $\mathcal{B}(n, 2, \mathcal{A}_\infty)$ is independent of $\mathcal{A}$.

2.3. Matrices $A(\mathcal{A}_\infty)$ and $A_{\mathbb{T}}(\mathcal{A}_\infty)$

Let $\alpha_i = (a_{i1}, \dots, a_{ik})$ be the normal vectors of hyperplanes H_i^0, $1 \leq i \leq n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the usual dot product

$$(a_1, \dots, a_k) \cdot (v_1, \dots, v_k) = \sum_i a_i v_i \quad .$$

Then the normal vectors to hyperplanes D_L, $L = \{s_1 < \cdots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \simeq \mathbb{C}^n$ are nonzero vectors of the form

$$\alpha_L = \sum_{i=1}^{k+1} (-1)^i \det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}}) e_{s_i}, \tag{1}$$

where $\{e_j\}_{1 \leq j \leq n}$ is the standard basis of $\mathbb{C}^n$ (cf. [1]).

Let $\mathcal{P}_{k+1}([n]) = \{L \subset [n] \mid |L| = k + 1\}$ be the set of cardinality $k + 1$ subsets of $[n]$, we denote by

$$A(\mathcal{A}_\infty) = (\alpha_L)_{L \in \mathcal{P}_{k+1}([n])} \tag{2}$$

the matrix having in each row the entries of vectors α_L normal to hyperplanes D_L and by $A_{\mathbb{T}}(\mathcal{A}_\infty)$ the submatrix of $A(\mathcal{A}_\infty)$ with rows α_L, $L \in \mathbb{T}$, $\mathbb{T} \subset \mathcal{P}_{k+1}([n])$ of cardinality m.

2.4. Polynomial $p_{\mathbb{T}}(a_{ij})$

The construction in Subsection 2.3 naturally holds also in the real case, i.e. $\mathcal{A}$ arrangement in $\mathbb{R}^k$. In this case, Athanasiadis (see [1]) defined the polynomial

$$p_{\mathbb{T}}(a_{ij}) = \sum_{\substack{J \subset [n] \\ |J| = m}} \det[A_{\mathbb{T},J}(\mathcal{A}_\infty)]^2 \tag{3}$$

in the variable a_{ij} given by the sum of the squares of determinants of the $m \times m$ submatrices $A_{\mathbb{T},J}$ of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ obtained considering the columns $j \in J$. Notice that if $\mathcal{A}$ is a generic arrangement in $\mathbb{R}^k$, if $\mathbb{T} = \{L_1, L_2, L_3\}$ is a good 3s-partition, then the condition in Lemma 2.1 is equivalent to $p_{\mathbb{T}}(a_{ij}) = 0$.

2.5. Grassmannian $Gr(k,n)$

Let $Gr(k,n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ and

$$\begin{aligned}\gamma : Gr(k,n) &\to \mathbb{P}(\bigwedge^k \mathbb{C}^n)\\ < v_1,\dots,v_k > &\mapsto [v_1\wedge\cdots\wedge v_k],\end{aligned} \tag{4}$$

the Plücker embedding. Then $[x]\in\mathbb{P}(\bigwedge^k\mathbb{C}^n)$ is in $\gamma(Gr(k,n))$ if and only if the map

$$\begin{aligned}\varphi_x : \mathbb{C}^n &\to \bigwedge^{k+1}\mathbb{C}^n\\ v &\mapsto v\wedge x\end{aligned} \tag{5}$$

has a kernel of dimension k, i.e. $\ker\varphi_x =< v_1,\dots,v_k >$. If $e_1,\dots,e_n$ is a basis of $\mathbb{C}^n$, then $e_I = e_{i_1}\wedge\dots\wedge e_{i_k}$, $I=\{i_1,\dots,i_k\}\subset[n]$, $i_1<\cdots<i_k$, is a basis for $\bigwedge^k\mathbb{C}^n$, and $x\in\bigwedge^k\mathbb{C}^n$ can be written uniquely as

$$x=\sum_{\substack{I\subseteq[n]\\|I|=k}}\beta_I\, e_I = \sum_{1\le i_1<\cdots<i_k\le n}\beta_{i_1\dots i_k}(e_{i_1}\wedge\cdots\wedge e_{i_k}) \tag{6}$$

where homogeneous coordinates β_I are the Plücker coordinates on $\mathbb{P}(\bigwedge^k\mathbb{C}^n)=\mathbb{P}^{\binom{n}{k}-1}$ associated with the ordered basis $e_1,\dots,e_n$ of $\mathbb{C}^n$. With this choice of basis for $\mathbb{C}^n$, the matrix $M_x=(b_{ij})$ associated with φ_x is the $\binom{n}{k+1}\times n$ matrix with rows indexed by ordered subsets $I\subseteq[n]$, $|I|=k$, and entries $b_{ij}=(-1)^i\beta_{I\cup\{j\}\setminus\{i\}}$ if $i\in I$, $b_{ij}=0$ otherwise. The Plücker relations, i.e. conditions for $\dim(\ker\varphi_x)=k$, are vanishing conditions of all $(n-k+1)\times(n-k+1)$ minors of M_x. It is well known (see for instance [6]) that Plücker relations are degree-2 relations and that they can also be written as

$$\sum_{l=0}^{k}(-1)^l\beta_{i_1\dots i_{k-1}j_l}\beta_{j_0\dots\hat{j}_l\dots j_k}=0 \tag{7}$$

for any $2k$-tuple $(i_1,\dots,i_{k-1},j_0,\dots,j_k)$.

Remark 2.3. Notice that vectors α_L in equation (1) normal to hyperplanes D_L correspond to rows $I=L$ in the Plücker matrix M_x, that is

$$A(\mathcal{A}_\infty)=M_x \quad .$$

For this reason, in the rest of the paper, we will call $A(\mathcal{A}_\infty)$ Plücker coordinate matrix. Notice that, in particular, $\det(\alpha_{s_1},\dots,\hat{\alpha_{s_i}},\dots,\alpha_{s_{k+1}})$ is the Plücker coordinate β_I, $I=\{s_1,s_2,\dots,s_{k+1}\}\setminus\{s_i\}$.

In the following section, we give an example to illustrate the general Theorem in section 4. This example appears also in [5], [9] and, in the context of oriented matroids, in [2].

3. Example $\mathcal{B}(6,3,\mathcal{A}_\infty)$ in a real case

Consider $\mathcal{A}=\{H_1^0,H_2^0,\dots,H_6^0\}$ to be a generic arrangement of hyperplanes in $\mathbb{R}^3$ with normal vectors $\alpha_i=(a_{i1},a_{i2},a_{i3})$, $1\le i\le 6$ and $H_i^{t_i}$ to be a hyperplane obtained by translating H_i^0 along the direction α_i, i.e. $H_i^{t_i}=H_i^0+t_i\alpha_i$, $t_i\in\mathbb{R}$. Let $\mathbb{T}=\{L_1,L_2,L_3\}$ be the good 6-partition with $L_1=\{1,2,3,4\}$, $L_2=\{1,2,5,6\}$ and $L_3=\{3,4,5,6\}$, then

$$A_{\mathbb{T}}(\mathcal{A}_\infty)=\begin{pmatrix}\alpha_{L_1}\\ \alpha_{L_2}\\ \alpha_{L_3}\end{pmatrix}=\begin{pmatrix}-\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0\\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125}\\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345}\end{pmatrix},\quad \beta_{ijk}=\det\begin{pmatrix}a_{i1} & a_{j1} & a_{k1}\\ a_{i2} & a_{j2} & a_{k2}\\ a_{i3} & a_{j3} & a_{k3}\end{pmatrix}$$

is a submatrix of the Plücker coordinate matrix $A(\mathcal{A}_\infty)$.

Let $\alpha_i\times\alpha_j$ be the cross product of α_i,α_j corresponding to the direction orthogonal to both α_i and α_j, and denote by $(\alpha_i\times\alpha_{i+1})$ the matrix $\begin{pmatrix}\alpha_1\times\alpha_2\\ \alpha_3\times\alpha_4\\ \alpha_5\times\alpha_6\end{pmatrix}$. Then $\alpha_i\times\alpha_j$ is the direction of the line $H_i\cap H_j$, since α_i and α_j are, respectively, directions orthogonal to H_i and H_j and $\operatorname{rank} A_{\mathbb{T}}(\mathcal{A}_\infty)=2$ if and only $\operatorname{rank}(\alpha_i\times\alpha_{i+1})=2$. Indeed $\operatorname{rank}(A_{\mathbb{T}}(\mathcal{A}_\infty))=2$ is equivalent to $\operatorname{codim}(D_{L_1}\cap D_{L_2}\cap D_{L_3})=2$; hence, by Lemma 2.1, the points $\bigcap_{i\in L_1\cap L_2}\bar{H}_i^{t_i}\cap H_\infty=\bar{H}_3^{t_3}\cap\bar{H}_4^{t_4}\cap H_\infty$, $\bigcap_{i\in L_1\cap L_3}\bar{H}_i^{t_i}\cap$

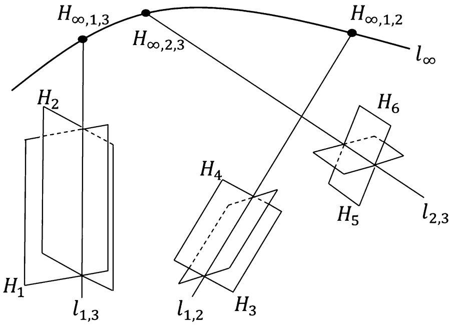

Fig. 1. Picture of case $\mathcal{B}(6,3,\mathcal{A}^0_\infty)$.

$H_\infty = \bar{H}_1^{t_1} \cap \bar{H}_2^{t_2} \cap H_\infty$, and $\bigcap_{i\in L_2\cap L_3} \bar{H}_i^{t_i} \cap H_\infty = \bar{H}_5^{t_5} \cap \bar{H}_6^{t_6} \cap H_\infty$ are collinear, which means that the directions of $H_i^{t_i} \cap H_{i+1}^{t_{i+1}}$ are dependent, and hence that $\mathrm{rank}(\alpha_i \times \alpha_{i+1}) = 2$ (see Fig. 1).

The rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if β_{ijk} are solutions to the system:

$$(I)\begin{cases} -\beta_{456}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ \beta_{356}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ -\beta_{346}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ \beta_{345}(\beta_{134}\beta_{256}-\beta_{234}\beta_{156})=0\\ -\beta_{256}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ \beta_{156}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ -\beta_{126}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ \beta_{125}(\beta_{124}\beta_{356}-\beta_{123}\beta_{456})=0\\ -\beta_{234}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ \beta_{134}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ -\beta_{124}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0\\ \beta_{123}(\beta_{125}\beta_{346}-\beta_{126}\beta_{345})=0 \end{cases} \quad \text{and} \quad (II)\begin{cases} \beta_{234}\beta_{126}\beta_{456}+\beta_{124}\beta_{256}\beta_{346}=0\\ -(\beta_{234}\beta_{125}\beta_{456}+\beta_{124}\beta_{256}\beta_{345})=0\\ -(\beta_{234}\beta_{126}\beta_{356}+\beta_{123}\beta_{256}\beta_{346})=0\\ \beta_{234}\beta_{125}\beta_{356}+\beta_{123}\beta_{256}\beta_{345}=0\\ -(\beta_{134}\beta_{126}\beta_{456}+\beta_{124}\beta_{156}\beta_{346})=0\\ \beta_{134}\beta_{125}\beta_{456}+\beta_{124}\beta_{156}\beta_{345}=0\\ \beta_{134}\beta_{126}\beta_{356}+\beta_{123}\beta_{156}\beta_{346}=0\\ -(\beta_{134}\beta_{125}\beta_{356}+\beta_{123}\beta_{156}\beta_{345})=0 \end{cases} \tag{8}$$

and the polynomial $p_{\mathbb{T}}(a_{ij})$ is

$$\begin{aligned} p_{\mathbb{T}}(a_{ij}) = \sum_{\substack{J\subset[6]\\|J|=3}} \det(A_{\mathbb{T},J})^2 &= (\beta_{134}\beta_{256}-\beta_{234}\beta_{156})^2\Big(\sum_{\substack{I_1\subset\{3,4,5,6\}\\|I_1|=3}} \beta_{I_1}^2\Big) + (\beta_{124}\beta_{356}-\beta_{123}\beta_{456})^2\Big(\sum_{\substack{I_2\subset\{1,2,5,6\}\\|I_2|=3}} \beta_{I_2}^2\Big)\\ &+ (\beta_{125}\beta_{346}-\beta_{126}\beta_{345})^2\Big(\sum_{\substack{I_3\subset\{1,2,3,4\}\\|I_3|=3}} \beta_{I_3}^2\Big) + \sum_{\substack{i=5,6\\j=3,4}} (\beta_{234}\beta_{12i}\beta_{j56}+\beta_{12j}\beta_{256}\beta_{34i})^2\\ &+ \sum_{\substack{i=5,6\\j=3,4}} (\beta_{134}\beta_{12i}\beta_{j56}+\beta_{12j}\beta_{156}\beta_{34i})^2. \end{aligned}$$

On the other hand, the condition $\mathrm{rank}(\alpha_i \times \alpha_{i+1}) = 2$ is simply $\det(\alpha_i \times \alpha_{i+1}) = 0$, and if we define

$$\widetilde{p}_{\mathbb{T}}(a_{ij}) = [\det(\alpha_i \times \alpha_{i+1})]^2 = \{(a_{12}a_{23}-a_{13}a_{22})\Delta_{11} + (a_{11}a_{23}-a_{13}a_{21})\Delta_{12} + (a_{11}a_{22}-a_{12}a_{2})\Delta_{13}\}^2, \tag{9}$$

Δ_{1l} cofactors of $(\alpha_i \times \alpha_{i+1})$, then $p_{\mathbb{T}}(a_{ij}) = 0$ if and only if $\widetilde{p}_{\mathbb{T}}(a_{ij}) = 0$. That is polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ is a polynomial of, in general, lower degree than $p_{\mathbb{T}}(a_{ij})$ with the same set of zeros.

4. Polynomial $\widetilde{p}_{\mathbb{T}}(a_{ij})$ in $\mathcal{B}(n,k,\mathcal{A}_\infty)$ in a real case

4.1. Case $\mathcal{B}(n, 3, \mathcal{A}_\infty)$

It is straightforward to generalize the example in section 3 to the case of n hyperplanes in $\mathbb{R}^3$. Denote by $(\alpha_{i_j} \times \alpha_{i_{j+1}})$ the matrix $\begin{pmatrix} \alpha_{i_1}\times\alpha_{i_2}\\ \alpha_{i_3}\times\alpha_{i_4}\\ \alpha_{i_5}\times\alpha_{i_6}\end{pmatrix}$, the following Theorem holds.

Theorem 4.1. *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{R}^3$ with normal vectors $\alpha_i = (a_{i1}, a_{i2}, a_{i3})$. Let $\mathbb{T} = \{L_1, L_2, L_3\}$ be a good 6-partition with a choice $L_1 = \{i_1, i_2, i_3, i_4\}$, $L_2 = \{i_3, i_4, i_5, i_6\}$ and $L_3 = \{i_1, i_2, i_5, i_6\}$ and $A_{\mathbb{T}}(\mathcal{A}_\infty)$ be the matrix with rows $\alpha_{L_1}, \alpha_{L_2}, \alpha_{L_3}$. Then the following statements are equivalent:*

(1) $\operatorname{rank} A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$;
(2) $p_{\mathbb{T}}(a_{ij}) = 0$;
(3) $\operatorname{rank}(\alpha_{i_j} \times \alpha_{i_{j+1}}) = 2$;
(4) $\widetilde{p}_{\mathbb{T}}(a_{ij}) = [\det(\alpha_{i_j} \times \alpha_{i_{j+1}})]^2 = 0$.

Proof. The equivalences (1) ⇔ (2) and (3) ⇔ (4) are obvious from the definitions of $p_{\mathbb{T}}(a_{ij})$ and $\widetilde{p}_{\mathbb{T}}(a_{ij})$. The proof that (1) ⇔ (3) can be obtained from the remarks in Section 3, relabeling indices $1, \ldots, 6$ with $i_1, \ldots, i_6$. □

Remark 4.2. Notice that, since $\widetilde{p}_{\mathbb{T}}(a_{ij}) = [\det(\alpha_{i_j} \times \alpha_{i_{j+1}})]^2$, then $\widetilde{p}_{\mathbb{T}}(a_{ij}) = 0$ if and only if $\det(\alpha_{i_j} \times \alpha_{i_{j+1}}) = 0$, the equivalence of conditions (1), (3) and (4) in Theorem 4.1 holds also for generic arrangements in $\mathbb{C}^3$.

4.2. Generalization to $\mathcal{B}(n, k, \mathcal{A}_\infty)$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a generic arrangement of hyperplanes in $\mathbb{R}^k$ and $\mathbb{T} = \{L_1, L_2, L_3\}$ be a good $3s$-partition of indices in $[n]$. If α_τ are normal vectors to $H_\tau \in \mathcal{A}$, $\tau = 1, \ldots, n$, $T = \{j_1, \cdots, j_t\}$ a subset of $[n]$ that has empty intersection with $L_1 \cup L_2 \cup L_3$, define vector spaces

$$U^{\perp}_{i,j} = \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in L_i \cap L_j\},$$

where $v \cdot \alpha_\tau$ is the scalar product of v and α_τ, and

$$W_T = \begin{cases} \mathbb{R}^k & (T = \emptyset) \\ \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in T\} & (T \neq \emptyset) \end{cases} \quad . \tag{10}$$

Then W_T is the vector space associated with $\bigcap_{\tau \in T} H_\tau$ and $U^{\perp}_{i,j} \cap W_T = \{v \in \mathbb{R}^k \mid v \cdot \alpha_\tau = 0, \tau \in (L_i \cap L_j) \cup T\}$ is a vector space of dimension $k - (s + t)$, where s and t are, respectively, cardinalities of $L_i \cap L_j$ and T. With the above notations, define the polynomial

$$\widetilde{p}_{\mathbb{T},T}(a_{ij}) = \sum_{U \in \mathbb{U}_{\mathbb{T},T}} [\det U]^2,$$

where $\mathbb{U}_{\mathbb{T},T}$ is the set of all $k \times k$ submatrices of the $3(k - s - t) \times k$ matrix having as rows the vector spanning $U^{\perp}_{i,j} \cap W_T$.

If $k = 2s - 1$ and $n = 3s$, $s \geq 2$, we have $T = \emptyset$, and hence $U^{\perp}_{i,j} \cap W_T = U^{\perp}_{i,j}$ is a space of dimension $\dim U^{\perp}_{i,j} = s - 1$. $\mathbb{U}_{\mathbb{T},\emptyset}$ is the set of all $(2s - 1) \times (2s - 1)$ submatrices of the $3(s - 1) \times (2s - 1)$ matrix having as rows the vectors spanning $U^{\perp}_{i,j}$ and the following lemma, equivalent to Lemma 2.1 holds.

Lemma 4.3. *Let $s \geq 2$, $n = 3s$, $k = 2s - 1$, i.e. $T = \emptyset$, and $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{R}^k$. Given a good $3s$-partition $\mathbb{T} = \{L_1, L_2, L_3\}$ of $[3s] = [n]$, $U^{\perp}_{i,j}$ span a proper subspace of $\mathbb{R}^k$ if and only if the rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is 2, that is, $\widetilde{p}_{\mathbb{T},\emptyset}(a_{ij}) = 0$ if and only if $p_{\mathbb{T}}(a_{ij}) = 0$.*

Proof. Since $\mathbb{T}$ is a good $3s$-partition and $A_{\mathbb{T}}(\mathcal{A}_\infty) = (\alpha_L)_{L \in \mathbb{T}}$ is a $3 \times n$ matrix, the rank of the matrix $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if α_L, $L \in \mathbb{T}$, are linearly dependent, that is, the intersection $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ of hyperplanes in $\mathcal{B}(n, k, \mathcal{A}_\infty)$ is a space of codimension 2. Then, by Lemma 2.1, this corresponds to $H_{\infty,i,j} = \bigcap_{\tau \in L_i \cap L_j} \bar{H}_\tau \cap H_\infty \subset H_\infty$ span a proper subspace in H_∞. Let V_τ be the vector spaces associated with the hyperplanes H_τ, hence $V_{i,j} = \bigcap_{\tau \in L_i \cap L_j} V_\tau$ are the vector spaces associated with $H_{i,j} = \bigcap_{\tau \in L_i \cap L_j} H_\tau$ and $V_{i,j} = U^{\perp}_{i,j}$ since $v \in V_{i,j}$ if and only if $v \cdot \alpha_\tau = 0$ for any $\tau \in L_i \cap L_j$. It follows that $H_{\infty,i,j}$ span a proper subspace of H_∞ if and only if $U^{\perp}_{i,j}$ span a proper subspace of $\mathbb{R}^k$. That is, $\det U = 0$ for any $U \in \mathbb{U}_{\mathbb{T},\emptyset}$ or, equivalently, $\widetilde{p}_{\mathbb{T},\emptyset}(a_{ij}) = 0$. □

Notice that if $s = 2$, i.e. the case $\mathcal{B}(6, 3, \mathcal{A}_\infty)$, $\widetilde{p}_{\mathbb{T},\emptyset}(a_{ij})$ coincides with $\widetilde{p}_{\mathbb{T}}(a_{ij})$, defined in Section 3. In this case, 1-dimensional subspaces $U^{\perp}_{1,2}$, $U^{\perp}_{1,3}$ and $U^{\perp}_{2,3}$ are spanned, respectively, by $\alpha_1 \times \alpha_2, \alpha_3 \times \alpha_4$ and $\alpha_5 \times \alpha_6$, that is, they are the lines drawn in Fig. 1.

Analogously to [9], we call a generic arrangement $\mathcal{A} = \{W_1, \cdots, W_{3s}\}$ in $\mathbb{R}^{2s-1}$, $s \geq 2$, dependent if there exists a good $3s$-partition such that $U^{\perp}_{i,j}$ span a proper subspace of $\mathbb{R}^{2s-1}$. With this notation, by Lemma 2.1 and Theorem 2.2, the following theorem holds.

Theorem 4.4. *Let $\mathcal{A}$ be a generic arrangement of n hyperplanes in $\mathbb{R}^k$, $\mathbb{T}$ a good 3s-partition, $3s \le n$, and $T = [n] \setminus \cup_{L\in\mathbb{T}} L$. If W_T is the vector space defined in equation (10), then the rank of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ is equal to 2 if and only if the restriction arrangement*

$$\mathcal{A}_{W_T} = \{H \cap \bigcap_{\tau\in T} H_\tau \mid H \in \mathcal{A}\backslash\{H_i\}_{i\in T}\}$$

is dependent. With this choice of T and $\mathbb{T}$, we get that $p_{\mathbb{T}}(a_{ij}) = 0$ if and only if $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$.

Remark 4.5. For a fixed good 3s-partition $\mathbb{T}$, equation $p_{\mathbb{T}}(a_{ij}) = 0$ corresponds to $\binom{n}{3s}\binom{3s}{s}$ nonlinear relations on Plücker coordinates β_I, $(2s-1)\times(2s-1)$ minors of the matrix $A = (a_{ij})$.

On the other hand, $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$ is equivalent to vanishing of $(2s-1)\times(2s-1)$ minors of the matrix with rows given by solutions to the system $A_I \cdot x = 0$, $A_I = (a_{ij})_{i\in I}$, i.e. the $\binom{n}{3s}\binom{3s-3}{2s-1}$ equations on a_{ij}. That is $\widetilde{p}_{\mathbb{T},T}(a_{ij}) = 0$ is a reduced form of $p_{\mathbb{T}}(a_{ij}) = 0$.

5. Hypersurfaces in complex Grassmannian $Gr(3, n)$

Let now $\mathcal{A}$ be a generic arrangement of six hyperplanes in $\mathbb{C}^3$ (i.e. the example in Section 3 in $\mathbb{C}^3$ instead of $\mathbb{R}^3$) and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & \vdots & \vdots \\ a_{61} & a_{62} & a_{63} \end{pmatrix} \tag{11}$$

be the matrix having in each row normal vectors α_i to hyperplanes $H_i^0 \in \mathcal{A}$. Since $\mathcal{A}$ is generic, the columns of A are independent vectors in $\mathbb{C}^6$ and they span a subspace of dimension 3 in $\mathbb{C}^6$, i.e. an element in the Grassmannian $Gr(3,6)$. The non-null 3×3 minors of A are Plücker coordinates β_{ijk}, and the matrix $A(\mathcal{A}_\infty)$ is the matrix of the map

$$\begin{aligned} \varphi_x : \mathbb{C}^6 &\to \bigwedge^4 \mathbb{C}^6 \\ v &\mapsto v \wedge x, \end{aligned}$$

where $x = \sum_{1\le i<j<k\le n} \beta_{ijk}(e_i \wedge e_j \wedge e_k)$. If $\mathcal{A}_\infty$ is dependent, then β_{ijk} have to satisfy both, classical Plücker relations and relations in equation (8). Notice that since the relations in equation (8) come directly from condition rank $A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$, we get exactly same relations in the real and complex cases. The latter can be simplified as:

$$(I): \begin{cases} (a): \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0 \\ (b): \beta_{124}\beta_{356} - \beta_{123}\beta_{456} = 0 \\ (c): \beta_{125}\beta_{346} - \beta_{126}\beta_{345} = 0 \end{cases} \quad \text{and} \quad (II): \begin{cases} (d): \beta_{234}\beta_{126}\beta_{456} + \beta_{124}\beta_{256}\beta_{346} = 0 \\ (e): \beta_{234}\beta_{125}\beta_{456} + \beta_{124}\beta_{256}\beta_{345} = 0 \\ (f): \beta_{234}\beta_{126}\beta_{356} + \beta_{123}\beta_{256}\beta_{346} = 0 \\ (g): \beta_{234}\beta_{125}\beta_{356} + \beta_{123}\beta_{256}\beta_{345} = 0 \\ (h): \beta_{134}\beta_{126}\beta_{456} + \beta_{124}\beta_{156}\beta_{346} = 0 \\ (i): \beta_{134}\beta_{125}\beta_{456} + \beta_{124}\beta_{156}\beta_{345} = 0 \\ (j): \beta_{134}\beta_{126}\beta_{356} + \beta_{123}\beta_{156}\beta_{346} = 0 \\ (k): \beta_{134}\beta_{125}\beta_{356} + \beta_{123}\beta_{156}\beta_{345} = 0 \end{cases}.$$

Where equation $(I)(a)$ is obtained dividing the first four equations in system (I) in (8) respectively by $-\beta_{456}, \beta_{356}, -\beta_{346}, \beta_{345} \neq 0$ and, similarly, equations $(I)(b)$ and (c) are obtained dividing, respectively, equations (5) to (8) and equations (9) to (12) in system (I) in (8) by, respectively, $-\beta_{256}, \beta_{156}, -\beta_{126}, \beta_{125} \neq 0$ and $-\beta_{234}, \beta_{134}, -\beta_{124}, \beta_{123} \neq 0$, while equations in (II) (8) are left unchanged, except for a change of sign. Remark that this is only possible since $\mathcal{A}$ is a generic arrangement, which implies that all $\beta_{ijk} \neq 0$ and hence we can divide equations in (8) (I) opportunely by them. In the following, we refer to the equations in (I) and (II) by using the corresponding letters, for example (a) will refer to equation $\beta_{134}\beta_{256} - \beta_{234}\beta_{156}$. The Plücker relations in equation (7) for $k = 3$ become:

$$\beta_{i_1 i_2 k_0}\beta_{k_1 k_2 k_3} - \beta_{i_1 i_2 k_1}\beta_{k_0 k_2 k_3} + \beta_{i_1 i_2 k_2}\beta_{k_0 k_1 k_3} - \beta_{i_1 i_2 k_3}\beta_{k_0 k_1 k_2} = 0.$$

Fixing $i_1 = 1, i_2 = 2, k_0 = 4, k_1 = 3, k_2 = 5, k_3 = 6$, we obtain

$$\beta_{124}\beta_{356} - \beta_{123}\beta_{456} + \beta_{125}\beta_{436} - \beta_{126}\beta_{435} = 0,$$

that is, $(b) = (c)$, and fixing $i_1 = 5, i_2 = 6, k_0 = 2, k_1 = 1, k_2 = 3, k_3 = 4$, we get $(a) = (b)$. This means that the relations in (I) are equivalent.

Next, we focus on type-(II) relations and on the vanishing of all 4×4 minors of the Plücker matrix. We fix a good 6-partition $\mathbb{T} = \{L_1, L_2, L_3\}$, for any subset $L_4 \subset [6]$ of cardinality 4 such that $L_4 \notin \mathbb{T}$, and we define the submatrix

$$Pl_{\mathbb{T}}(D_{L_4}) = (\alpha_{L_i})_{1\le i\le 4} \tag{12}$$

of $A(\mathcal{A}_\infty)$. The matrix $Pl_{\mathbb{T}}(D_{L_4})$ is obtained by adding one row to the matrix $A_{\mathbb{T}}(\mathcal{A}_\infty)$. Hence, since the relations in equation (8) correspond to the vanishing of 3×3 minors of $A_{\mathbb{T}}(\mathcal{A}_\infty)$, $\mathbb{T} = \{\{1,2,3,4\},\{1,2,5,6\},\{3,4,5,6\}\}$, then zero of 4×4 minors of matrix $Pl_{\mathbb{T}}(D_{L_4})$ for same fixed $\mathbb{T}$ naturally give rise to relations among the relations in (8). For example $(d) = 0$ and $(e) = 0$ correspond to vanishing of minors obtained considering, respectively, 1st, 3rd and 5th columns and 1st, 3rd and 6th columns of $A_{\mathbb{T}}(\mathcal{A}_\infty)$. Adding to $A_{\mathbb{T}}(\mathcal{A}_\infty)$ the normal vector to the hyperplane $D_{\{2,4,5,6\}}$ as 4th row, we get

$$Pl_{\mathbb{T}}(D_{\{2,4,5,6\}}) = \begin{pmatrix} -\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0 \\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125} \\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345} \\ 0 & -\beta_{456} & 0 & \beta_{256} & -\beta_{246} & \beta_{245} \end{pmatrix}$$

and calculating the determinant of the submatrix obtained by the 1st, 3rd, 5th, and 6th columns, we get the relation among (e) and (d):

$$\beta_{246}\cdot(e) - \beta_{245}\cdot(d) = 0 \quad . \tag{13}$$

Analogously, the vanishing of minor obtained by 1st, 4th, 5th and 6th columns gives:

$$\beta_{256}\beta_{234}\cdot(c) - \beta_{246}\cdot(g) + \beta_{245}\cdot(f) = 0 \quad . \tag{14}$$

Applying similar considerations to opportunely chosen $L_4 \notin \mathbb{T}$ we get the following additional syzygies.

The vanishing of minors obtained considering 1st, 4th, 5th and 6th columns and 1st, 3rd, 5th and 6th columns of

$$Pl_{\mathbb{T}}(D_{\{2,3,5,6\}}) = \begin{pmatrix} -\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0 \\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125} \\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345} \\ 0 & -\beta_{356} & \beta_{256} & 0 & -\beta_{236} & \beta_{235} \end{pmatrix}$$

leads, respectively, to relations $\beta_{236}\cdot(g) - \beta_{235}\cdot(f) = 0$ and $\beta_{256}\beta_{234}\cdot(c) + \beta_{236}\cdot(e) - \beta_{235}\cdot(d) = 0$. Those relations, jointly with the one in equations (13) and (14), state dependency of $(d), (e), (f)$ and (g) from (c) which, in turn, is equivalent to (a), i.e. they are all zero if and only if (a) is zero.

By vanishing of the minors given by the 2nd, 3rd, 5th and 6th columns and 2nd, 4th, 5th and 6th columns of the submatrix

$$Pl_{\mathbb{T}}(D_{\{1,4,5,6\}}) = \begin{pmatrix} -\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0 \\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125} \\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345} \\ -\beta_{456} & 0 & 0 & \beta_{156} & -\beta_{146} & \beta_{145} \end{pmatrix}$$

we get, respectively, $\beta_{146}\cdot(i) - \beta_{145}\cdot(h) = 0$ and $\beta_{156}\beta_{134}\cdot(c) - \beta_{146}\cdot(k) + \beta_{145}\cdot(j) = 0$.

Finally, by vanishing of minors given by 2nd, 4th, 5th and 6th columns and 2nd, 3rd, 5th and 6th columns of

$$Pl_{\mathbb{T}}(D_{\{1,3,5,6\}}) = \begin{pmatrix} -\beta_{234} & \beta_{134} & -\beta_{124} & \beta_{123} & 0 & 0 \\ -\beta_{256} & \beta_{156} & 0 & 0 & -\beta_{126} & \beta_{125} \\ 0 & 0 & -\beta_{456} & \beta_{356} & -\beta_{346} & \beta_{345} \\ -\beta_{356} & 0 & \beta_{156} & 0 & -\beta_{136} & \beta_{135} \end{pmatrix}$$

give relations $\beta_{136}\cdot(k) - \beta_{135}\cdot(j) = 0$ and $-\beta_{156}\beta_{134}\cdot(c) - \beta_{136}\cdot(i) + \beta_{135}\cdot(h) = 0$.

That is, the relations in equation (8) are all equivalent, and we are left with only one independent relation

$$(a) = 0: \quad \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0. \tag{15}$$

This degree-2 homogeneous polynomial defines a degree-2 hypersurface on the projective variety $Gr(3,6)$.

The above computations are a direct consequence of the following more general Lemma.

Lemma 5.1. *Let $A(\mathcal{A}_\infty)$ be the Plücker matrix associated with a generic arrangement $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ and $\mathbb{T}$ a good 6-partition of indices $i_1,\dots,i_6 \in [n]$. If the entries β_I of the matrix $A(\mathcal{A}_\infty)$ satisfy the Plücker relations, then* rank $A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$ *if and only if one of its 3×3 minors vanishes.*

Proof. $\Rightarrow$) Since rank $A_{\mathbb{T}}(\mathcal{A}_\infty) = 2$ if and only if all 3×3 minors of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ vanish, it is obvious.

$\Leftarrow$) Entries β_I of $A(\mathcal{A}_\infty)$ satisfy the Plücker relations if and only if any 4×4 minor in $A(\mathcal{A}_\infty)$ vanishes. For any 4 columns $s_1 < s_2 < s_3 < s_4 \in \{i_1,\dots,i_6\}$ of matrix $A(\mathcal{A}_\infty)$ let M_i and M_j be the two 3×3 minors in $A_{\mathbb{T}}(\mathcal{A}_\infty)$ obtained considering, respectively, the columns $\{s_1,s_2,s_3,s_4\}\backslash\{s_i\}$ and $\{s_1,s_2,s_3,s_4\}\backslash\{s_j\}$. If we add to the submatrix $A_{\mathbb{T}}(\mathcal{A}_\infty)$ the row of $A(\mathcal{A}_\infty)$

corresponding to the vector α_L, $L = \{s_i, s_j, s_5, s_6\}$, with $\{s_5, s_6\} = \{i_1, \dots, i_6\} \setminus \{s_1, s_2, s_3, s_4\}$, then the 4×4 minor of the matrix $\begin{pmatrix} A_{\mathbb{T}}(\mathcal{A}_\infty) \\ \alpha_L \end{pmatrix}$ obtained considering the columns $\{s_1, s_2, s_3, s_4\}$ vanishes, that is,

$$\beta_{L\setminus\{s_i\}} M_i \pm \beta_{L\setminus\{s_j\}} M_j = 0 \tag{16}$$

where $\beta_{L\setminus\{s_t\}}$ is the entry of the row α_L in the column $s_t, t = i, j$. Dividing by $\beta_{L\setminus\{s_i\}} \neq 0$ (entries of $A(\mathcal{A}_\infty)$ are all not zero by $\mathcal{A}$ generic), we get

$$M_i = \pm M_j \cdot \frac{\beta_{L\setminus\{s_j\}}}{\beta_{L\setminus\{s_i\}}} \tag{17}$$

that is, $M_i = 0$ if and only if $M_j = 0$. Applying the above considerations to any subset $\{s_1 < s_2 < s_3 < s_4\} \subset \{i_1, \dots, i_6\}$ and transitivity of equality, we get that if a 3×3 minor of $A_{\mathbb{T}}(\mathcal{A}_\infty)$ vanishes, then all minors vanish. $\square$

Remark 5.2. Recall that if $\mathcal{A}$ is an arrangement of n hyperplanes in $\mathbb{C}^3$, then the matrix $A(\mathcal{A}_\infty)$ is an $\binom{n}{4} \times n$ matrix such that, for any $L = \{s_1 < s_2 < s_3 < s_4\}$, the entries $(x_1, \dots, x_n)$ of the row vector α_L are all zeros, except $x_{i_j} = (-1)^j \beta_{I_j}$, $I_j = L \setminus \{s_j\}$, $j = 1, \dots, 4$. Hence, for any fixed six indices $s_1 < \dots < s_6 \in [n]$, we get a $\binom{6}{4} \times 6$ submatrix of $A(\mathcal{A}_\infty)$ obtained considering all rows α_L, $L \subset \{s_1, \dots, s_6\}$, $|L| = 4$ and columns $\{s_1, \dots, s_6\}$ (all columns $j \notin \{s_1, \dots, s_6\}$ of the matrix $(\alpha_L)_{L\subset\{s_1,\dots,s_6\},|L|=4}$ are zero). It follows that the general case of n hyperplanes in $\mathbb{C}^3$ essentially reduce to the case $n = 6$.

On the other hand, one can easily remark that, if $s_1 < \dots < s_6 \in [n]$ are six fixed indices and if $\mathbb{T} = \{\{s_1, s_2, s_3, s_4\}, \{s_1, s_2, s_5, s_6\}, \{s_3, s_4, s_5, s_6\}\}$ (which is analogous to a good 6-partition $\{\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 5, 6\}\}$ of indices $\{1, \dots 6\}$), then any other good 6-partition on indices $\{s_1, \dots, s_6\}$ is of the form

$$\sigma.\mathbb{T} = \{\{i_1, i_2, i_3, i_4\}, \{i_1, i_2, i_5, i_6\}, \{i_3, i_4, i_5, i_6\}\} \tag{18}$$

where $i_j = \sigma(s_j)$, $\sigma \in \mathbf{S_6}$, $\mathbf{S_6}$ being the group of all permutations of indices $\{s_1, \dots, s_6\}$. Note that in general i_j are not ordered, and that we can have $i_j > i_{j+1}$.

The following Lemma holds.

Lemma 5.3. *Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^3$ and $\sigma.\mathbb{T} = \{\{i_1, i_2, i_3, i_4\}, \{i_1, i_2, i_5, i_6\}, \{i_3, i_4, i_5, i_6\}\}$, a good 6-partition of indices $s_1 < \dots < s_6 \in [n]$ such that* rank $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = 2$*, then $\mathcal{A}$ is a point in the hypersurface*

$$\beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0 \quad . \tag{19}$$

Proof. Let $\sigma.\mathbb{T} = \{L'_1 = \{i_1, i_2, i_3, i_4\}, L'_2 = \{i_1, i_2, i_5, i_6\}, L'_3 = \{i_3, i_4, i_5, i_6\}\}$ be a good 6-partition of indices $s_1 < \dots < s_6 \in [n]$ and denote by $(L'_1) = (i_1, i_2, i_3, i_4)$, $(L'_2) = (i_1, i_2, i_5, i_6)$ and $(L'_3) = (i_3, i_4, i_5, i_6)$ the ordered 4-tuples of indices. Then, there exist unique permutations τ_i, $i = 1, 2, 3$ of indices $s_1 < \dots < s_6$ such that τ_i fixes indices outside L'_i and, if $L'_i = \{s_{j_1} < s_{j_2} < s_{j_3} < s_{j_4}\}$, then $(L'_i) = \tau_i.L'_i = (\tau_i(s_{j_1}), \tau_i(s_{j_2}), \tau_i(s_{j_3}), \tau_i(s_{j_4}))$, $i = 1, 2, 3$. By the determinant rule on permutations of columns, we have that

$$\begin{aligned}
\sum_{j=1}^{4}(-1)^j \det(\alpha_{\tau(1)}, \dots, \hat{\alpha_{\tau(j)}}, \dots, \alpha_{\tau(4)}) e_{\tau(j)} &= \begin{vmatrix} a_{\tau(1)1} & a_{\tau(2)1} & a_{\tau(3)1} & a_{\tau(4)1} \\ a_{\tau(1)2} & a_{\tau(2)2} & a_{\tau(3)2} & a_{\tau(4)2} \\ a_{\tau(1)3} & a_{\tau(2)3} & a_{\tau(3)3} & a_{\tau(4)3} \\ e_{\tau(1)} & e_{\tau(2)} & e_{\tau(3)} & e_{\tau(4)} \end{vmatrix} \\
&= \operatorname{sign}(\tau) \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ e_1 & e_2 & e_3 & e_4 \end{vmatrix} \\
&= \operatorname{sign}(\tau) \sum_{j=1}^{4}(-1)^j \det(\alpha_1, \dots, \hat{\alpha_j}, \dots, \alpha_4) e_j \quad .
\end{aligned}$$

Hence, if we define the matrix $\sigma.A_{\mathbb{T}}$ as the matrix having in its rows respectively the coefficients of the three vectors

$$\begin{aligned}
\tau_1.\alpha_{L'_1} &= \sum_{j=1}^{4}(-1)^j \det(\alpha_{i_1}, \dots, \hat{\alpha_{i_j}}, \dots, \alpha_{i_4}) e_{i_j}, \\
\tau_2.\alpha_{L'_2} &= \sum_{j\in\{1,2,5,6\}}(-1)^j \det(\alpha_{i_1}, \dots, \hat{\alpha_{i_j}}, \dots, \alpha_{i_6}) e_{i_j},
\end{aligned}$$

$$\tau_3.\alpha_{L'_3} = \sum_{j=3}^{6}(-1)^j \det(\alpha_{i_3}, \ldots, \hat{\alpha_{i_j}}, \ldots, \alpha_{i_6})e_{i_j}$$

with respect to the ordered basis $\{e_{i_1}, \ldots, e_{i_6}\}$, then the i-th row of $\sigma.A_{\mathbb{T}}$ is obtained from the i-th row of $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ by a σ column permutation and multiplication by $\text{sign}(\tau_i)$ (notice that $\sigma_{|_{L'_i}} = \tau_i$). That is, $\text{rank}\sigma.A_{\mathbb{T}} = \text{rank}\, A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ and, more in details, the 3×3 minor given by columns $\{i, j, k\}$ in $A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty)$ vanishes if and only if the 3×3 minor of columns $\{\sigma(i), \sigma(j), \sigma(k)\}$ in $\sigma.A_{\mathbb{T}}$ vanishes. Hence, by Lemma 5.1 $\text{rank}\, A_{\sigma.\mathbb{T}}(\mathcal{A}_\infty) = \text{rank}\sigma.A_{\mathbb{T}} = 2$ if and only if one minor vanishes. In particular, the first three columns $\{i_1, i_2, i_3\}$ in $\sigma.A_{\mathbb{T}}$ are of the form

$$\begin{pmatrix} -\beta_{i_2 i_3 i_4} \\ -\beta_{i_2 i_5 i_6} \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \beta_{i_1 i_3 i_4} \\ \beta_{i_1 i_5 i_6} \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -\beta_{i_1 i_2 i_4} \\ 0 \\ -\beta_{i_4 i_5 i_6} \end{pmatrix}$$

from which we get that the 3×3 minor corresponding to them vanishes if and only if

$$\beta_{i_1 i_3 i_4}\beta_{i_2 i_5 i_6} - \beta_{i_2 i_3 i_4}\beta_{i_1 i_5 i_6} = 0$$

(recall that all entries β_I in the matrix $A(\mathcal{A}_\infty)$ verify $\beta_I \neq 0$). $\square$

By Remark 5.2 and Lemma 5.3, the following main Theorem follows.

Theorem 5.4. *The set of generic arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{C}^3$ that contains a dependent sub-arrangement is the set of points in an hypersurface in the Grassmannian $Gr(3, n)$ such that each component is the intersection of the Grassmannian with a quadric.*

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