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Functional analysis

Positive block matrices and numerical ranges

Matrices par blocs positives et images numériques

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ABSTRACT

Any positive matrix M partitioned in four n-by-n blocks satisfies the unitarily invariant norm inequality $\|M\| \le \|M_{1,1} + M_{2,2} + \omega I\|$, where ω is the width of the numerical range of $M_{1,2}$. Some related inequalities and a reverse Lidskii majorization are given.

RÉSUMÉ

Toute matrice positive partitionnée en quatre blocs de même taille satisfait l'inégalité en norme unitairement invariante $\|M\| \le \|M_{1,1} + M_{2,2} + \omega I\|$, où ω est la largeur de l'image numérique de $M_{1,2}$.

1. Introduction

We shall prove an original inequality between positive semidefinite matrices partitioned into four blocks and the sums of their diagonal blocks. The numerical range of the off-diagonal block will play a central role. Our result leads to some unexpected norm inequalities. We will also discuss a reverse estimate to the famous Lidskii majorization.

Given a positive (semidefinite) matrix partitioned into four blocks of same size, it is well known that

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \le \|A\| + \|B\| \tag{1.1}$$

for all symmetric (i.e. unitarily invariant) norms, where $\|\cdot\|$ denotes a symmetric norm defined on $\mathbb{M}_{2n}$ and the induced norm on $\mathbb{M}_n$, the space of n-by-n matrices. In the special case of the Schatten p-norms, this easily yields

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_p \le 2^{1-1/p} \|A+B\|_p$$

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where the constant $2^{1-1/p}$ is the best possible one by considering the simple example of the block-matrix,

$$E = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{bmatrix}. \tag{1.2}$$

However, if the block-matrix is in the PPT class, that is if both

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$$

are positive, then Hiroshima [5] proved a stronger inequality than (1.1),

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \le \|A+B\|. \tag{1.3}$$

This happens in particular when the off-diagonal block X is Hermitian.

Our theorem states a companion inequality to (1.3) involving the numerical range $W(X)$ of X. Our main tool is a unitary orbit technique based on a useful decomposition [2, Lemma 3.4], which considerably strengthens (1.1): *for every positive matrix in $\mathbb{M}_{2n}$ partitioned into four blocks, we have*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^* \tag{1.4}$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

2. The width of the numerical range

By a strip $\mathcal{S}$ we mean the closed region between two parallel lines of the complex plane. The width of $\mathcal{S}$ is the distance between the two lines of its boundary. The identity matrix, of any size, is denoted by I.

Theorem 2.1. *Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive matrix partitioned into four blocks in $\mathbb{M}_n$. Suppose that $W(X)$ lies in a strip $\mathcal{S}$ of width ω. Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \le \|A+B+\omega I\|.$$

We can take ω as the width of $W(X)$, i.e. the smallest possible ω such that $W(X)$ is contained in a strip of width ω. The theorem is sharp with the block-matrix (1.2) and the operator norm $\|\cdot\|_\infty$, since the numerical range of the left lower block of (1.2) is a disc of diameter 1.

Proof. By using the unitary congruence implemented by

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & I \end{bmatrix}$$

we see that our block matrix is unitarily equivalent to

$$\begin{bmatrix} A & e^{i\theta}X \\ e^{-i\theta}X^* & B \end{bmatrix}.$$

As $W(e^{i\theta}X) = e^{i\theta}W(X)$, by choosing the adequate θ and replacing X by $e^{i\theta}X$, we may and do assume that $W(X)$ lies in a strip $\mathcal{S}$ of width ω and parallel to the imaginary axis,

$$\mathcal{S} = \{x + iy : \quad y \in \mathbb{R}, \quad r \le x \le r+\omega\}.$$

The projection property for the real part $\operatorname{Re} W(X) = W(\operatorname{Re} X)$ then ensures that

$$rI \le \operatorname{Re} X \le (r+\omega)I. \tag{2.1}$$

Now we use the decomposition [3, Corollary 2.1] derived from (1.4),

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + \operatorname{Re} X & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - \operatorname{Re} X \end{bmatrix} V^* \tag{2.2}$$

for some unitaries $U, V \in \mathbb{M}_{2n}$. Note that the two matrices in the right-hand side of (2.2) are positive since so are

$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & -I \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} I \\ -I \end{bmatrix}.$$

Combining (2.1) and (2.2) yields

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \le U \begin{bmatrix} \frac{A+B}{2} + (r+\omega)I & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - rI \end{bmatrix} V^* \tag{2.3}$$

where the two matrices of the right-hand side are positive. From each Ky Fan k-norm, $k = 1, 2, \dots, 2n$, we then have

$$\begin{aligned} \left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\|_{(k)} &\le \left\| \frac{A+B}{2} + (r+\omega)I \right\|_{(k)} + \left\| \frac{A+B}{2} - rI \right\|_{(k)} \\ &= \| A + B + \omega I \|_{(k)} . \end{aligned}$$

The Ky Fan principle [1, p. 93] then ensures that this inequality holds for all symmetric norms. □

The important special case of Theorem 2.1 with $\omega = 0$, first proved in [10], reads as: *let* $\begin{bmatrix} A & L \\ L^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$*. Suppose that* $W(L)$ *is a line segment. Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & L \\ L^* & B \end{bmatrix} \right\| \le \|A + B\|.$$

This contains (1.3) when $X = X^*$. A simple example of such a block matrix is

$$\begin{bmatrix} 2.6 & 2 & 1-\mathrm{i} & -\mathrm{i} \\ 2 & 3.6 & \mathrm{i} & 1-\mathrm{i} \\ 1+\mathrm{i} & -\mathrm{i} & 3.6 & 0 \\ \mathrm{i} & 1+\mathrm{i} & 0 & 2.6 \end{bmatrix}.$$

This matrix is not PPT, hence Theorem 2.1 provides more block-matrices for which the Hiroshima majorization (1.3) holds.

Theorem 2.1 also contains some interesting norm inequalities. For any symmetric norm and $A \in \mathbb{M}_n$, we have $\|A^*A\| = \|AA^*\|$ as A^*A and AA^* are unitarily invariant. Our first corollary yields an interesting comparison between the similar expressions for two operators, $A^*A + B^*B$ and $AA^* + BB^*$.

Corollary 2.2. *Let $A, B \in \mathbb{M}_n$ and let ω be the width of $W(AB^*)$. Then, for all symmetric norms,*

$$\left\| A^*A + B^*B \right\| \le \left\| AA^* + BB^* + \omega I \right\| .$$

Corollary 2.2 is sharp for the operator norm with A and B in place of the upper left corner and lower left corner of (1.2), respectively. It seems difficult to find an analogous statement involving three, or more, matrices. It would be interesting to have a detailed description of $W(AB^*)$ when both A and B are partial isometries; the special case of two projections is nicely studied in [8].

Proof. Note that

$$\left\| A^*A + B^*B \right\| = \|T^*T\| = \|TT^*\|$$

with $T = \begin{bmatrix} A \\ B \end{bmatrix}$ and

$$TT^* = \begin{bmatrix} AA^* & AB^* \\ BA^* & BB^* \end{bmatrix}$$

so that Theorem 2.1 yields the result. □

Corollary 2.3. *Let $H, K, X \in \mathbb{M}_n$ be Hermitian. If X is invertible, then, for all symmetric norms,*

$$\left\| XH^2X + X^{-1}K^2X^{-1} \right\| \le \left\| HX^2H + KX^{-2}K + \gamma I \right\| ,$$

where the constant γ can be chosen as

(*a*) *$\gamma = 2$, if H and K are two contractions;*

(*b*) $\gamma = 1$, *if* H *is positive and* HK *is a contraction;*
(*c*) $\gamma = 1/2$, *if* H *and* K *are two positive contractions;*
(*d*) $\gamma = 0$, *if* $HK = KH$.

Since both $\|X\|_\infty$ and $\|X^{-1}\|_\infty$ may be arbitrarily large, it is rather surprising that we have bounded constants γ in Corollary 2.3. However, we note that (*d*) is a byproduct of Hiroshima's theorem as a special case of [9].

Proof. Setting $A = HX$ and $B = KX^{-1}$, Corollary 2.2 shows that it suffices to check that $W(HK)$ has width less or equal than γ. This follows from the estimate for the imaginary part of HK,

$$-\frac{\gamma}{2}I \le \operatorname{Im} HK \le \frac{\gamma}{2}I, \tag{2.4}$$

which does hold. Indeed, in case of assumption (*a*) or (*d*), this is obvious. If (*b*) holds, this follows from a beautiful result of Kittaneh for commutators involving a positive matrix [7]. If (*c*) holds, this is also well known, see for instance [4] where a proof of Fuzhen Zhang is proposed. □

For any pair $A, B \in \mathbb{M}_n$ of positive matrices, Lidskii's majorization holds: *for all symmetric norms*,

$$\|A + B\| \ge \left\| A^{\uparrow} + B^{\downarrow} \right\|.$$

Here, as usual, $A^{\uparrow}$ is the diagonal matrix whose diagonal entries are the eigenvalues of A arranged in increasing order, and similarly $B^{\downarrow}$ is the diagonal matrix whose diagonal entries are the eigenvalue of B arranged in decreasing order. A proof is in [1, p 98].

Proposition 2.4. *Let* $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive matrix partitioned into four blocks in* $\mathbb{M}_n$. *Let* $\nu = \|X\|_\infty$. *Then, for all symmetric norms,*

$$\left\| \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \right\| \le \|A^{\uparrow} + B^{\downarrow} + \nu I\|.$$

Note that by Lidskii's majorization, we have

$$\|A^{\uparrow} + B^{\downarrow} + \nu I\| \le \|A + B + \nu I\|.$$

Hence the proposition says that Theorem 2.1 is significant when ω is not too large, $\omega \le \nu$. ($\omega \le 2\nu$ always holds.) If $0 \notin W(X)$, then necessarily $\omega < \nu$. However, Theorem 2.1 is more informative than Proposition 2.4; the case $\omega = 0$ is already interesting, while the case $\nu = 0$ is trivial.

Proof. Note that, with adequate unitary matrices $U, V \in \mathbb{M}_n$ and the unitary congruence implemented by

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

our block matrix is unitarily equivalent to

$$\begin{bmatrix} A^{\uparrow} & UXV^* \\ VX^*U^* & B^{\downarrow} \end{bmatrix}.$$

Since $Z \le |Z|$ for any Hermitian matrix Z, we have

$$\begin{aligned} \begin{bmatrix} A^{\uparrow} & UXV^* \\ VX^*U^* & B^{\downarrow} \end{bmatrix} &\le \begin{bmatrix} A^{\uparrow} & 0 \\ 0 & B^{\downarrow} \end{bmatrix} + \begin{bmatrix} |VX^*U^*| & 0 \\ 0 & |UXV^*| \end{bmatrix} \\ &\le \begin{bmatrix} A^{\uparrow} & 0 \\ 0 & B^{\downarrow} \end{bmatrix} + \begin{bmatrix} \nu I & 0 \\ 0 & \nu I \end{bmatrix}. \end{aligned}$$

Therefore, for each Ky Fan k-norm, $k = 1, 2, \dots, n$, we have

$$\begin{aligned} \left\| \begin{bmatrix} A^{\uparrow} & UXV^* \\ VX^*U^* & B^{\downarrow} \end{bmatrix} \right\|_{(k)} &\le \left\| \begin{bmatrix} A^{\uparrow} & 0 \\ 0 & B^{\downarrow} \end{bmatrix} \right\|_{(k)} + \left\| \begin{bmatrix} \nu I & 0 \\ 0 & \nu I \end{bmatrix} \right\|_{(k)} \\ &\le \|A^{\uparrow} + B^{\downarrow}\|_{(k)} + \|\nu I\|_{(k)} \\ &= \|A^{\uparrow} + B^{\downarrow} + \nu I\|_{(k)} \end{aligned}$$

thanks to the basic majorization $\|A^{\uparrow} \oplus B^{\downarrow}\| \le \|A^{\uparrow} + B^{\downarrow}\|$. This also obviously holds for $n \le k \le 2n$ because the block matrix has same trace than $A^{\uparrow} + B^{\downarrow}$. Applying the Ky Fan principle then completes the proof. □

If $C \in \mathbb{M}_n$ is positive, we define $C' := C - \lambda_n(C)I$ where $\lambda_n(\cdot)$ is the smallest eigenvalue. Hence, if C is singular, $C' = C$.

Corollary 2.5. *Let $A, B \in \mathbb{M}_n$ be positive and let ρ' be the spectral radius of $A'B'$. Then, for all symmetric norms,*

$$\|A+B\| \le \left\| A^{\uparrow} + B^{\downarrow} + \sqrt{\rho'}I \right\|.$$

Proof. First, observe that for each Ky Fan k-norms, $k = 1, 2, \dots, n$

$$\|A+B\|_{(k)} = \|A'+B'\|_{(k)} + k(\lambda_n(A) + \lambda_n(B))$$

and

$$\left\| A^{\uparrow} + B^{\downarrow} + \sqrt{\rho'}I \right\|_{(k)} = \left\| A'^{\uparrow} + B'^{\downarrow} + \sqrt{\rho'}I \right\|_{(k)} + k(\lambda_n(A) + \lambda_n(B)).$$

Hence we may assume that $A = A'$ and $B = B'$. Since

$$\left\| A' + B' \right\| = \|T^*T\| = \|TT^*\|$$

with $T = \begin{bmatrix} A'^{1/2} \\ B'^{1/2} \end{bmatrix}$ and $TT^* = \begin{bmatrix} A' & A'^{1/2}B'^{1/2} \\ B'^{1/2}A'^{1/2} & B' \end{bmatrix}$, we may apply Proposition 2.4 with $\nu = \|A'^{1/2}B'^{1/2}\|_\infty = \sqrt{\rho'}$. □

The following consequence of Corollary 2.5 was first obtained by Kittaneh in [6, Corollary 2].

Corollary 2.6. *Let $A, B \in \mathbb{M}_n$ be two positive matrices. Then, the spectral radius of AB satisfies*

$$\rho(AB) \ge \{\|A+B\|_\infty - \|A \oplus B\|_\infty\}^2.$$

Proof. By adding some zero columns and rows we may assume that $A, B \in \mathbb{M}_m$ with $m = 2n$, hence $A = A'$, $B = B'$, and $\operatorname{rank} A + \operatorname{rank} B \le m$ so that $\|A^{\uparrow} + B^{\downarrow}\|_\infty = \|A \oplus B\|_\infty$. Corollary 2.5 with $\rho' = \rho(AB)$ then yields

$$\|A+B\|_\infty \le \|A \oplus B\|_\infty + \sqrt{\rho(AB)}$$

which proves our last corollary. □

Denote by $\lambda_1(M) \ge \lambda_2(M) \ge \dots \ge \lambda_n(M)$ the eigenvalues in non-increasing order of a positive matrix $M \in \mathbb{M}_n$. From (2.3) and the classical Weyl inequalities [1, p. 62], we infer several estimates which complete the majorization stated in Theorem 2.1.

Proposition 2.7. *Let $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive matrix partitioned into four blocks in $\mathbb{M}_n$. Let ω be the width of $W(X)$. Then, for all $j, k \in \{0, 1, \dots, n-1\}$,*

$$\lambda_{1+j+k}\left(\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\right) \le \frac{\lambda_{1+j}(A+B) + \lambda_{1+k}(A+B)}{2} + \omega.$$

It follows that Corollaries 2.2–2.3 have similar eigenvalue versions.

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