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Analytic geometry/Differential topology

Scattering matrix and analytic torsion

Matrice de diffusion et torsion analytique

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ABSTRACT

We consider a compact manifold with a piece isometric to a (finite-length) cylinder. By making the length of the cylinder tend to infinity, we obtain an asymptotic gluing formula for the zeta determinant of the Hodge Laplacian and an asymptotic expansion of the L^2 torsion of the corresponding Mayer–Vietoris exact sequence. As an application, we give a purely analytic proof of the gluing formula for analytic torsion.

RÉSUMÉ

On considère une variété compacte ayant une partie isométrique à un cylindre fini. En faisant tendre la longueur du cylindre vers l'infini, on obtient une formule asymptotique pour le déterminant du laplacien de Hodge et un développement asymptotique de la torsion L^2 associée à la suite exacte de Mayer–Vietoris. On obtient une preuve analytique de la formule de recollement pour la torsion analytique.

Version française abrégée

Soit (Z, g^{TZ}) une variété riemannienne compacte et soit $Y \subset Z$ une hypersurface telle que $Z\backslash Y$ se compose de deux composantes connexes, notées Z_1 et Z_2. Soit (F, ∇^F) un fibré vectoriel complexe plat sur Z et soit h^F une métrique hermitienne sur F. On suppose que g^{TZ} et h^F sont produit près de Y.

À partir de Z_j $(j = 1, 2)$, on construit $Z_{j,R}$ en lui attachant un cylindre fini $Y \times [0, R]$. Alors $Z_{j,R}$ est encore une variété riemannienne compacte de bord Y. Soit $Z_R = Z_{1,R} \cup_Y Z_{2,R}$, qui est une variété riemannienne compacte sans bord. Les données (F, ∇^F, h^F) s'étendent alors à Z_R.

Soit $D^{F,2}_{Z_R}$ le laplacien de Hodge agissant sur $\Omega^\bullet(Z_R, F)$, l'espace vectoriel des formes différentielles sur Z_R à valeurs dans F, et soit $D^{F,2}_{Z_{j,R}}$ $(j = 1, 2)$ le laplacien de Hodge sur $Z_{j,R}$ muni de conditions aux limites relatives/absolues. Soient

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$\zeta_R(s)$, $\zeta_{1,R}(s)$ et $\zeta_{2,R}(s)$ les fonctions zêta associées à $D^{F,2}_{Z_R}$, $D^{F,2}_{Z_{1,R}}$ et $D^{F,2}_{Z_{2,R}}$. Dans cette note, on donne un développement asymptotique de $\zeta_R'(0) - \zeta_{1,R}'(0) - \zeta_{2,R}'(0)$ lorsque $R \to \infty$.

On considère ensuite la suite exacte de Mayer–Vietoris relative aux cohomologies de Z_R, $Z_{1,R}$ et $Z_{2,R}$ à valeurs dans F, cohomologies que l'on munit des métriques L^2 induites par la théorie de Hodge. Soit $\lambda_R(F)$ le produit alterné des déterminants des cohomologies en question, et soit $\|\cdot\|^{L^2}_{\lambda_R(F)}$ la métrique sur $\lambda_R(F)$ induite par les métriques L^2. Soit $\varrho_R \in \lambda_R(F)$ la section canonique induite par la suite exacte de Mayer–Vietoris. Dans cette note, on donne aussi un développement asymptotique de $\left\|\varrho_R\right\|^{L^2}_{\lambda_R(F)}$ lorsque $R \to \infty$.

Par application de ces résultats, on donne une preuve analytique de la formule de recollement pour la torsion analytique [11]. La preuve initiale [5] utilisait une version équivariante du théorème de Cheeger–Müller [6,8] pour se ramener à une formule de recollement pour la torsion de Reidemeister [12]. Parallèlement à la preuve initiale, la preuve donnée dans cette note a l'avantage d'être éventuellement généralisable aux formes de torsion analytique de Bismut–Lott [2].

1. Introduction

Given a manifold with cylindrical ends, that is, a non-compact Riemannian manifold whose non-compact part is isometric to a cylinder, the spectrum of its Laplacian has an absolutely continuous part, which is determined by a scattering matrix.

Here, we consider a compact Riemannian manifold which has a piece isometric to a (finite-length) cylinder. We stretch the cylinder so that its length tends to infinity. This procedure is known as taking the adiabatic limit. This adiabatic limit first appeared in the work of Douglas and Wojciechowski [7] for studying the η-invariant. Müller [9] studied the η-invariant of manifolds with cylindrical ends using scattering theory. Park and Wojciechowski [10] studied the adiabatic behavior of the spectrum of the Dirac operator using the scattering matrix.

In the present note, we study the adiabatic properties of the Hodge–de Rham operator. The scattering matrix plays a central role in our work.

One of the main ingredients in this note is an asymptotic estimate of the spectrum of the Hodge–de Rham operator in the adiabatic limit. As a consequence, we obtain an asymptotic gluing formula for the ζ-determinant of the Hodge Laplacian.

Another main ingredient in this note is an asymptotic estimate of the L^2-metric on de Rham cohomology in the adiabatic limit. Rather than considering a single manifold, we consider two manifolds glued together. Then the cohomologies in question fit into a Mayer–Vietoris exact sequence. As a consequence, we obtain the adiabatic limit of the L^2 torsion associated with this exact sequence. As an application of our results, we give a purely analytic proof of the gluing formula for analytic torsion [5,11]. This strategy has been used by Zhu [13] to prove the gluing formula for the Bismut–Lott torsion form [2] under the assumption that the cohomology of the boundary vanishes. Our strategy could also be used to prove the gluing formula for the Bismut–Lott torsion form [2] in the general case, which remains an open problem.

1.1. Asymptotics of the determinant of the Hodge Laplacian

Let (Z, g^{TZ}) be a closed Riemannian manifold. Let $Y \subseteq Z$ be a hypersurface cutting Z into two pieces, say Z_1 and Z_2. Then $\partial Z_1 = \partial Z_2 = Y$ and $Z = Z_1 \cup_Y Z_2$. Let (F, ∇^F) be a flat complex vector bundle over Z. Let h^F be a Hermitian metric on F. The restriction of F to Z_1 or Z_2 is still denoted by F.

Let $U =]-1,1[\times Y$ be a collar neighborhood of ∂X. Let $\pi_Y :]-1,1[\times Y \to Y$ be the natural projection. Using parallel transport along $u \in]-1,1[$, $(F|_U, \nabla^F|_U)$ is identified with $\pi_Y^*(F|_Y, \nabla^F|_Y)$. We suppose that each of g^{TZ} and h^F is product near Y, i.e.

$$g^{TX}\big|_U = du^2 + g^{TY} , \qquad h^F\big|_U = \pi_Y^* \left(h^F\big|_Y\right) . \tag{1}$$

Set

$$\begin{aligned} &Z_{1,R} = Z_1 \cup_Y [0,R] \times Y , \qquad Z_{2,R} = Z_2 \cup_Y [-R,0] \times Y , \qquad \text{for } R \in [0,\infty[, \\ &Z_{1,\infty} = Z_1 \cup_Y [0,\infty[\times Y , \qquad Z_{2,\infty} = Z_2 \cup_Y]-\infty,0] \times Y , \end{aligned} \tag{2}$$

where the gluings identify $\{0\} \times Y$ with ∂Z_1 and ∂Z_2. For $R \in [0,\infty[$, set $Z_R = Z_{1,R} \cup_Y Z_{2,R}$ (see Fig. 1). Then (F, ∇^F, h^F) extends to Z_R in the sense of (1).

Let $d^F : \Omega^\bullet(Z_R, F) \to \Omega^{\bullet+1}(Z_R, F)$ be the de Rham operator induced by ∇^F. Let $d^{F,*}$ be the formal adjoint of d^F. Set

$$D^F_{Z_R} = d^F + d^{F,*} , \tag{3}$$

which is the Hodge–de Rham operator. Its square $D^{F,2}_{X_R}$ is the Hodge Laplacian. We consider equally the Hodge Laplacians $D^{F,2}_{Z_{1,R}}$ and $D^{F,2}_{Z_{2,R}}$ with

$$\begin{aligned} &\mathrm{Dom}\left(D^{F,2}_{Z_{1,R}}\right) = \left\{\omega \in \Omega^\bullet(Z_{1,R}, F) \ : \ du \wedge \omega = 0 \, , \ du \wedge d^{F,*}\omega = 0 \text{ on } Y\right\} , \\ &\mathrm{Dom}\left(D^{F,2}_{Z_{2,R}}\right) = \left\{\omega \in \Omega^\bullet(Z_{2,R}, F) \ : \ i_{\frac{\partial}{\partial u}}\omega = 0 \, , \ i_{\frac{\partial}{\partial u}} d^F\omega = 0 \text{ on } Y\right\} , \end{aligned} \tag{4}$$

where $\frac{\partial}{\partial u} \in TZ \cap TY^\perp|_Y$ is the unit normal vector, du is its dual, and $i_{\frac{\partial}{\partial u}}$ is the contraction by $\frac{\partial}{\partial u}$.

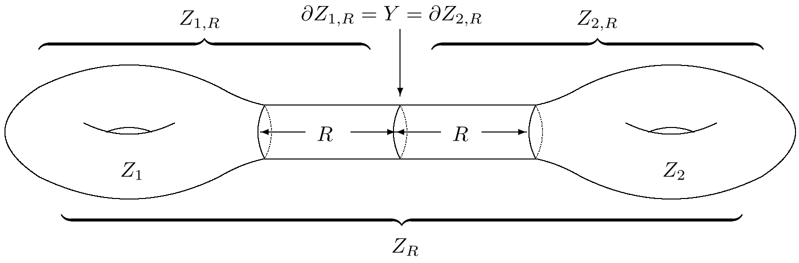

Fig. 1. An illustration of Z_R.

Let N be the number operator on $\Omega^\bullet(Z_R, F)$, i.e. for $\omega \in \Omega^p(Z_R, F)$, $N\omega = p\omega$. Let $P : \Omega^\bullet(Z_R, F) \to \ker\left(D^{F,2}_{Z_R}\right)$ be the orthogonal projection with respect to the L^2-metric. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{1}{2}\dim Z$, the ζ function associated with $D^{F,2}_{Z_R}$ is defined by

$$\zeta_R(s) = -\mathrm{Tr}\left[(-1)^N N\left(D^{F,2}_{Z_R}\right)^{-s}(1-P)\right]. \tag{5}$$

Then ζ_R admits a meromorphic continuation to the whole complex plane $\mathbb{C}$, which is regular at $0 \in \mathbb{C}$. In the same way, we define $\zeta_{j,R}(s)$ $(j = 1, 2)$, the ζ function associated with $D^{F,2}_{Z_{j,R}}$ (cf. (4)).

Let $\mathscr{H}^\bullet(Y, F) \subseteq \Omega^\bullet(Y, F)$ be the kernel of D^F_Y, the Hodge–de Rham operator on Y. Set $\mathscr{H}^\bullet(Y, F[du]) = \mathscr{H}^\bullet(Y, F) \oplus \mathscr{H}^\bullet(Y, F)du$. We fix $\delta_Y > 0$ such that $]-\delta_Y, \delta_Y[\cap \mathrm{Sp}(D^F_Y) \subseteq \{0\}$. For $j = 1, 2$, let

$$C_j(\lambda) \in \mathrm{End}\big(\mathscr{H}^\bullet(Y, F[du])\big), \qquad \lambda \in]-\delta_Y, \delta_Y[\,, \tag{6}$$

be the scattering matrix (cf. [9, §4]) associated with the action of $D^F_{j,\infty}$ on $\Omega^\bullet(Z_{j,\infty}, F)$. Set

$$C^p_{12} = C_2^{-1}(0)C_1(0)\big|_{\mathscr{H}^p(Y,F)\oplus\mathscr{H}^{p-1}(Y,F)du}\,. \tag{7}$$

Set

$$\begin{aligned}
\chi'(C_{12}) &= \sum_{p=0}^{\dim Z} p(-1)^p \dim\ker\left(C^p_{12} - 1\right), \\
\chi' &= \sum_{p=0}^{\dim Z} p(-1)^p \Big\{\dim H^p(Z, F) - \dim H^p(Z_1, \partial Z_1, F) - \dim H^p(Z_2, F)\Big\}, \\
\chi(Y) &= \sum_{p=0}^{\dim Y} (-1)^p \dim H^p(Y),
\end{aligned} \tag{8}$$

where $H^\bullet(\cdot, F)$ (resp. $H^\bullet(\cdot, \partial\cdot, F)$) is the cohomology (resp. relative cohomology) with values in F.

For a Hermitian matrix A, we denote by $\det{}^*(A)$ the product of its nonzero eigenvalues.

Theorem 1.1. *For any $\varepsilon > 0$, we have, as $R \to +\infty$,*

$$\begin{aligned}
\zeta_R{}'(0) - \zeta_{1,R}{}'(0) - \zeta_{2,R}{}'(0) = 2\chi' \log R + \big(\chi(Y)\mathrm{rk}(F) + \chi'(C_{12})\big)\log 2 \\
+ \sum_{p=0}^{\dim Z} \frac{p}{2}(-1)^p \log\det{}^*\Big(\frac{2 - C^p_{12} - (C^p_{12})^{-1}}{4}\Big) + \mathscr{O}(R^{-1+\varepsilon}).
\end{aligned} \tag{9}$$

1.2. Analytic torsion and the Mayer–Vietoris exact sequence

For a complex line λ, let $\lambda^{-1} = \lambda^*$ be its dual. For a finite dimensional complex vector space E, its determinant line is defined as $\det E = \Lambda^{\max} E$. More generally, for a $\mathbb{Z}$-graded finite dimensional complex vector space $E^\bullet = \bigoplus_{k=0}^n E^k$, we define

$$\det E^\bullet = \bigotimes_{k=0}^{n} \left(\det E^k\right)^{(-1)^k}. \tag{10}$$

We consider the Mayer–Vietoris exact sequence

$$\cdots \to H^p(Z_{1,R}, \partial Z_{1,R}, F) \to H^p(Z_R, F) \to H^p(Z_{2,R}, F) \to \cdots , \tag{11}$$

where each of the cohomology groups is equipped with the L^2-metric induced by Hodge Theory (cf. [5, Theorem 1.1]). Let $\|\cdot\|^{L^2}_{\lambda_R(F)}$ be the induced metric on

$$\lambda_R(F) := \Big(\det H^\bullet(Z_R, F)\Big)^{-1} \otimes \det H^\bullet(Z_{1,R}, \partial Z_{1,R}, F) \otimes \det H^\bullet(Z_{2,R}, F) . \tag{12}$$

By [1, Definition 1.1], $\lambda_R(F)$ is canonically identified with $\mathbb{C}$, i.e. there is a canonical section $\varrho_R \in \lambda_R(F)$.

Theorem 1.2. *As $R \to \infty$, we have*

$$\big\|\varrho_R\big\|^{L^2}_{\lambda_R(F)} = 2^{\chi'(C_{12})/2} R^{\chi'} \prod_{p=0}^{\dim Z} \det{}^*\Big(\frac{2 - C^p_{12} - (C^p_{12})^{-1}}{4}\Big)^{\frac{p}{4}(-1)^p} + \mathscr{O}(R^{\chi'-1}) . \tag{13}$$

We use the conventions that $Z_0 = Z$ and

$$H^\bullet_{\mathbf{bd}}(Z_0, F) = H^\bullet(Z, F) , \qquad H^\bullet_{\mathbf{bd}}(Z_1, F) = H^\bullet(Z_1, \partial Z_1, F) , \qquad H^\bullet_{\mathbf{bd}}(Z_2, F) = H^\bullet(Z_2, F) . \tag{14}$$

Let $\zeta_j(s)$ $(j = 0, 1, 2)$ be the ζ-functions (cf. (5)) associated with the Hodge Laplacian $D^{F,2}_{Z_j}$ (cf. (4)). Let $\|\cdot\|^{L^2}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)}$ be the L^2-metric on $\det H^\bullet_{\mathbf{bd}}(Z_j, F)$.

The Ray–Singer metric on $\det H^\bullet_{\mathbf{bd}}(Z_j, F)$ $(j = 0, 1, 2)$ is defined by

$$\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)} = \|\cdot\|^{L^2}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)} \exp\Big(\frac{1}{2}\zeta_j{}'(0)\Big) . \tag{15}$$

Let $\|\cdot\|^{\mathrm{RS}}_{\lambda(F)}$ be the product metric on $\lambda(F) := \lambda_0(F)$ induced by $\|\cdot\|^{\mathrm{RS}}_{\det H^\bullet_{\mathbf{bd}}(Z_j,F)}$. The following theorem was first proved by Brüning and Ma [5, Theorem 0.3].

Theorem 1.3. *We have*

$$\big\|\varrho_0\big\|^{\mathrm{RS}}_{\lambda(F)} = 2^{-\frac{1}{2}\chi(Y)\mathrm{rk}(F)} . \tag{16}$$

We reformulate (16) as follows,

$$\frac{1}{2}\zeta'(0) - \frac{1}{2}\zeta_1{}'(0) - \frac{1}{2}\zeta_2{}'(0) - \log\big\|\varrho_0\big\|^{L^2}_{\lambda(F)} = \frac{1}{2}\chi(Y)\mathrm{rk}(F)\log 2 . \tag{17}$$

Here, we give a direct proof of (17): by Theorems 1.1 and 1.2, we know that

$$\frac{1}{2}\zeta'_R(0) - \frac{1}{2}\zeta'_{1,R}(0) - \frac{1}{2}\zeta'_{2,R}(0) - \log\big\|\varrho_R\big\|^{L^2}_{\lambda_R(F)} \to \frac{1}{2}\chi(Y)\mathrm{rk}(F)\log 2 \tag{18}$$

as $R \to \infty$. Meanwhile, using the anomaly formula for analytic torsion [3, Theorem 0.1] [4, Theorem 0.1], we know that the left hand side of (18) is independent of R. This completes the proof of (17).

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