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Partial differential equations

Well-posedness of the scalar and the vector advection–reaction problems in Banach graph spaces

Analyse des problèmes d'advection–réaction scalaire et vectoriel dans les espaces de Banach du graphe

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A B S T R A C T

An extension of the well-posedness analysis of the scalar and the vector advection–reaction problem in Banach graph spaces of power $p \in (1, \infty)$ is proposed. This analysis is based on the sign of the associated Friedrichs tensor, taking positive, null or reasonably negative values.

R É S U M É

Cette Note propose une extension de l'analyse de la bonne position des problèmes d'advection–réaction scalaire et vectorielle dans les espaces du graphe de Banach de puissance $p \in (1, \infty)$. Cette analyse étend l'hypothèse sur le signe du tenseur de Friedrichs associé à ces problèmes, permettant ainsi de considérer le cas où ce tenseur prend des valeurs positives, nulles ou raisonnablement négatives.

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1. Introduction

Let Ω be a domain of $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$ and consider $u : \Omega \to \mathbb{R}$ and $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ solving the following first-order homogeneous boundary valued problems

$$\boldsymbol{\beta}\cdot\nabla u + \mu u = s \ \text{ a.e. in } \ \Omega, \tag{1a}$$

$$u = 0 \ \text{ a.e. on } \partial\Omega^-, \tag{1b}$$

and

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$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{u}) + (\nabla\times\boldsymbol{u})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{u} = \boldsymbol{s} \text{ a.e. in } \Omega, \tag{2a}$$

$$\boldsymbol{u} = 0 \text{ a.e. on } \partial\Omega^-. \tag{2b}$$

In this paper, $\boldsymbol{\beta}$ denotes a Lipschitz-continuous $\mathbb{R}^3$-valued vector field on Ω, and μ and $\boldsymbol{\mu}$ denote two bounded reaction coefficients taking $\mathbb{R}$ and $\mathbb{R}^{3\times 3}$ values, respectively. The inflow boundary $\partial\Omega^-$ is defined as $\partial\Omega^- = \{\boldsymbol{x} \in \partial\Omega \,|\, \boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$, with $\boldsymbol{n}$ the exterior unit normal of $\partial\Omega$.

The first problem (1) has been studied several times in the literature. We mention in particular the pioneering work of Bardos [1] and Beirão da Veiga [2] for the well-posedness analysis in smooth domains with regular model parameters and also the work of DiPerna & Lions [6] when the problem is expressed in unbounded domains with irregular model parameters. More recently, Girault & Tartar [10] proved (using a viscous and a Yosida regularization) the well-posedness of these problem in $L^p(\Omega)$ for all $p > 2$ under the assumption $\boldsymbol{\beta} \in \boldsymbol{W}^{1,2}(\Omega)$, and also the $W^{1,p}(\Omega)$-regularity of solution to (1) if $s \in W^{1,p}(\Omega)$ and if $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ is sufficiently small. Regarding the problem (2), there is very little work in the literature. However, this problem models physical situations, such as the static advection of a magnetic field in a conductor of conductivity $\boldsymbol{\mu}$ and of velocity $\boldsymbol{\beta}$. In the context of the differential geometry, this problem is also very important, since it represents the Lie derivative of a so-called 1-form in a three-dimensional domain; see Bossavit [3].

In this paper, we analyze the well-posedness of problems (1) and (2) in Banach graph spaces of power $p \in (1, \infty)$. Observing that these problems define two Friedrichs systems (see Ern & al. [8,9]), the well-posedness is a consequence of the positivity of the $\mathbb{R}$-valued Friedrichs tensor

$$\sigma_{\boldsymbol{\beta},\mu;p} := \mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}, \tag{3}$$

for the first problem (1) and of the positivity of the lowest eigenvalue of the $\mathbb{R}^{3\times 3}$-valued Friedrichs tensor

$$\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p} := \frac{\boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}}}{2} + \frac{\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}}}{2} - \frac{1}{p}(\nabla\cdot\boldsymbol{\beta})\,\mathbf{Id}, \tag{4}$$

for the second problem (2). The first contribution of this work concerns the well-posedness in Banach graph spaces. Following the analysis of Friedrichs system in Hilbert space proposed in the aforementioned works, we establish the well-posedness of these two problems for positive (in the sense above) Friedrichs tensors (3) and (4). The second part of this work is devoted to the analysis when these assumptions are not satisfied. Introducing a so-called potential (whose existence follows from the regularity and the trajectory of the vector field in Ω, see Devinatz & al. [5]), we prove that one may extend the Friedrichs positivity assumptions so as to consider null or reasonably negative tensors.

This paper is organized as follows. First, some notations are introduced and we recall the classical statement of the Banach–Nečas–Babuška (BNB) theorem. Section 2 is concerned with the scalar problem (1); we prove that this problem is well posed in the Banach graph space of power $p \in (1, \infty)$ if the infimum of the Friedrichs tensor (3) takes positive values, and we extend this result to consider null or reasonably negative values. In Section 3, we extend these results to prove the well-posedness of the vector problem (2) under similar assumptions on the Friedrichs tensor (4).

1.1. Notations

In this paper, p denotes any real number in $(1, \infty)$ with p' its conjuguate number such that $\frac{1}{p} + \frac{1}{p'} = 1$. The inner, the cross and the tensor products in $\mathbb{R}^3$ are denoted by $\cdot$, $\times$ and $\otimes$ respectively. To alleviate the notation, $|\cdot|$ denotes either the Lebesgue measure of a set, the absolute value of a real number, the Euclidean norm of a vector or the Frobenius norm of a tensor. As usual, the Banach space $L^p(\Omega)$ collects all measurable functions $v : \Omega \to \mathbb{R}$, whose absolute value raised to the power p is Lebesgue integrable, i.e. $\|v\|_{L^p(\Omega)} = (\int_\Omega |v|^p)^{\frac{1}{p}} < \infty$. Similarly, the Banach space $\boldsymbol{L}^p(\Omega)$ collects all measurable functions $\boldsymbol{v} : \Omega \to \mathbb{R}^3$ whose Euclidean norm raised to the power p is Lebesgue integrable, i.e. $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} = (\int_\Omega |\boldsymbol{v}|^p)^{\frac{1}{p}} < \infty$. We denote by $\mathcal{C}^\infty(\Omega)$ (resp. $\boldsymbol{\mathcal{C}}^\infty(\Omega)$) the space of infinitely differentiable $\mathbb{R}$-valued functions (resp. $\mathbb{R}^3$-valued) on Ω and $\mathcal{C}_c^\infty(\Omega)$ (resp. $\boldsymbol{\mathcal{C}}_c^\infty(\Omega)$) the subspace of those that are compactly supported in Ω.

1.2. Banach–Nečas–Babuška (BNB) theorem

Consider the following abstract variational problem

$$\text{Find } u \in U \quad \text{s.t.} \quad a(u, v) = \langle f, v\rangle_{V',V}, \quad \forall v \in V, \tag{5}$$

where U and V are two Banach spaces equipped with $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively, V is reflexive, $a \in \mathcal{L}(U\times V; \mathbb{R})$, $f \in V'$ and $\langle\cdot,\cdot\rangle_{V',V}$ is the duality pairing between $V' \equiv \mathcal{L}(V; \mathbb{R})$ and V. A necessary and sufficient condition for (5) to be well posed is given by the (BNB) theorem, see, e.g., Ern & Guermond [7].

Theorem 1.1 *(Banach–Nečas–Babuška). The abstract problem (5) is well posed if and only if:*
(BNB1) there exists $C_{\mathrm{BNB}} > 0$ such that

$$C_{\mathrm{BNB}}\|v\|_U \le \sup_{w\in V\setminus\{0\}} \frac{a(v,w)}{\|w\|_V}, \quad \forall v\in U;$$

(BNB2) for all $w\in V$, $(\forall v\in U,\ a(v,w)=0) \Longrightarrow (w=0)$.

2. Scalar advection–reaction problem

This section analyzes the well-posedness of the continuous problem (1) in Banach graph spaces and generalizes the sign condition on the Friedrichs tensor $\sigma_{\boldsymbol{\beta},\mu;p}$ defined by (3).

2.1. The graph space

The Banach graph space of power p associated with (1) is defined by

$$V_{\boldsymbol{\beta};p}(\Omega) := \left\{ v\in L^p(\Omega) \mid \boldsymbol{\beta}\cdot\nabla v \in L^p(\Omega)\right\}, \tag{6}$$

and is equipped with the norm $\|v\|_{V_{\boldsymbol{\beta};p}(\Omega)} := (\|v\|^p_{L^p(\Omega)} + \|\boldsymbol{\beta}\cdot\nabla v\|^p_{L^p(\Omega)})^{\frac{1}{p}}$ for all $v\in V_{\boldsymbol{\beta};p}(\Omega)$. This space defines a reflexive Banach space owing to the first and the second Clarkson inequalities (see Brezis [4]) where for all $v\in V_{\boldsymbol{\beta};p}(\Omega)$, $\boldsymbol{\beta}\cdot\nabla v\in L^p(\Omega)$ means that the linear form

$$\mathcal{C}_c^\infty(\Omega) \ni \varphi \mapsto -\int_\Omega v\,\nabla\cdot(\boldsymbol{\beta}\varphi), \tag{7}$$

is bounded in $L^{p'}(\Omega)$, so that $\boldsymbol{\beta}\cdot\nabla v$ is the Riesz representative of (7) in $L^p(\Omega)$. To specify the meaning of the trace of a function in $V_{\boldsymbol{\beta};p}(\Omega)$, we introduce the space $L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$ given by

$$L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega) := \left\{ v:\partial\Omega\to\mathbb{R} \mid v \text{ is Lebesgue measurable on } \partial\Omega \text{ and } \int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|v|^p < \infty\right\}, \tag{8}$$

which is a Banach space when equipped with the norm $\|v\|_{L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} := (\int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|v|^p)^{\frac{1}{p}}$ for all $v\in L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$. As observed by Ern & Guermond [8], the existence of traces in $L^2(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$ for a function in $V_{\boldsymbol{\beta};2}(\Omega)$ is not always guaranteed. A necessary and sufficient condition is the well-separation of the boundary $\partial\Omega$ with respect to the vector field $\boldsymbol{\beta}$, i.e.

$$\mathrm{dist}(\partial\Omega^-,\partial\Omega^+) > 0 \text{ with } \partial\Omega^\pm = \{\boldsymbol{x}\in\partial\Omega \mid \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}. \tag{9}$$

In this following, we always assume that this condition is satisfied. Let us adapt the proof of [8, Lemma 3.1] to the general case $p\in(1,\infty)$ to prove the existence of such traces.

Lemma 2.1 *(Trace in $L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$). The map $\gamma:\mathcal{C}^\infty(\overline{\Omega})\to L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$ with $\gamma(\varphi)=\varphi_{|\partial\Omega}$ for all $\varphi\in\mathcal{C}^\infty(\overline{\Omega})$, extends continuously to $V_{\boldsymbol{\beta};p}(\Omega)$, i.e. there exists $C_\gamma>0$ such that*

$$\|\gamma(v)\|_{L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} \le C_\gamma\|v\|_{V_{\boldsymbol{\beta};p}(\Omega)}, \quad \forall v\in V_{\boldsymbol{\beta};p}(\Omega).$$

Proof. Owing to the separation of the boundary from assumption (9), there exist $\psi^+,\psi^-\in\mathcal{C}^\infty(\overline{\Omega})$ such that $\psi^+ + \psi^- \equiv 1$ on $\partial\Omega$, $\psi^\pm\ge 0$, $\psi^+_{|\partial\Omega^-}\equiv 0$ and $\psi^-_{|\partial\Omega^+}\equiv 0$. Proceeding as in [8], we infer that

$$\int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|\varphi|^p = \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})|\varphi|^p(\psi^+ - \psi^-) = \int_\Omega \nabla\cdot(\boldsymbol{\beta}|\varphi|^p(\psi^+ - \psi^-)),\ \forall\varphi\in\mathcal{C}^\infty(\overline{\Omega}),$$

where we have used the partition of the unity on the boundary and the Stokes formula. Applying now the Leibniz product rule and recalling that $\nabla|\varphi|^p = p\varphi|\varphi|^{p-2}\nabla\varphi$, we obtain

$$\int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|\varphi|^p = p\int_\Omega (\psi^+ - \psi^-)(\boldsymbol{\beta}\cdot\nabla\varphi)\varphi|\varphi|^{p-2} + \int_\Omega |\varphi|^p\nabla\cdot(\boldsymbol{\beta}(\psi^+ - \psi^-)).$$

Next, Hölder's and Young's inequalities along with the identity $\|\varphi|\varphi|^{p-2}\|^{p'}_{L^{p'}(\Omega)} = \|\varphi\|^p_{L^p(\Omega)}$ yield

$$\int_\Omega \left|(\boldsymbol{\beta}\cdot\nabla\varphi)\varphi|\varphi|^{p-2}\right| \le \|\boldsymbol{\beta}\cdot\nabla\varphi\|_{L^p(\Omega)}\|\varphi\|^{p/p'}_{L^p(\Omega)} \le \frac{1}{p}\|\boldsymbol{\beta}\cdot\nabla\varphi\|^p_{L^p(\Omega)} + \frac{1}{p'}\|\varphi\|^p_{L^p(\Omega)}.$$

It follows that $\|\varphi\|_{L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} \leq C'(\|\boldsymbol{\beta}\cdot\nabla\varphi\|^p_{L^p(\Omega)} + p\|\varphi\|^p_{L^p(\Omega)})^{\frac{1}{p}}$ with the constant $C' = 2^{\frac{1}{p}}(\|\psi^+ - \psi^-\|_{L^\infty(\Omega)} + \|\nabla\cdot(\boldsymbol{\beta}(\psi^+ - \psi^-))\|_{L^\infty(\Omega)})^{\frac{1}{p}}$. Then, observing that $p^{\frac{1}{p}} \leq \mathrm{e}^{\frac{p-1}{p}} \leq \mathrm{e}$, we obtain

$$\|\varphi\|_{L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} \leq C_\gamma \|\varphi\|_{V_{\boldsymbol{\beta};p}(\Omega)}, \quad \forall \varphi \in \mathcal{C}^\infty(\overline{\Omega}),$$

with $C_\gamma = \mathrm{e}C'$. Finally, recalling that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $V_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,\infty)$ (see Jensen [12, Theorem 2]), this inequality holds as well for any function in $V_{\boldsymbol{\beta};p}(\Omega)$.

Owing to the existence of traces in $L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$, the following integration by parts formulae hold.

Lemma 2.2 *(Integration by parts). For all $v \in V_{\boldsymbol{\beta};p}(\Omega)$ and for all $w \in V_{\boldsymbol{\beta};p'}(\Omega)$,*

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla v)\, w + \int_\Omega (\boldsymbol{\beta}\cdot\nabla w)\, v + \int_\Omega (\nabla\cdot\boldsymbol{\beta})\, v w = \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, v w. \tag{10a}$$

In addition, for all $v \in V_{\boldsymbol{\beta};p}(\Omega)$ and for all $z \in W^{1,\infty}(\Omega)$,

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla v)\, v|v|^{p-2}z + \frac{1}{p}\int_\Omega (\nabla\cdot\boldsymbol{\beta})\,|v|^p z + \frac{1}{p}\int_\Omega (\boldsymbol{\beta}\cdot\nabla z)\,|v|^p = \frac{1}{p}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})|v|^p z. \tag{10b}$$

Proof. These formulae follow from the density of $\mathcal{C}^\infty(\overline{\Omega})$ in $V_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,\infty)$. The first one results from the Leibniz product rule, while the second one is a consequence of the identity

$$\boldsymbol{\beta}\cdot\nabla(\varphi|\varphi|^{p-2}z) = \varphi|\varphi|^{p-2}\boldsymbol{\beta}\cdot\nabla z + (p-1)|\varphi|^{p-2}z\,\boldsymbol{\beta}\cdot\nabla\varphi, \tag{11}$$

for all $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ and for all $z \in W^{1,\infty}(\Omega)$.

2.2. Weak formulation

To examine the well-posedness of (1), we introduce the bilinear form $a_{\boldsymbol{\beta},\mu;p} \in \mathcal{L}(V^0_{\boldsymbol{\beta};p}(\Omega)\times L^{p'}(\Omega);\mathbb{R})$, where $V^0_{\boldsymbol{\beta};p}(\Omega) := \{w \in V_{\boldsymbol{\beta};p}(\Omega) \mid w_{|\partial\Omega^-} = 0\}$, and such that for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$ and all $w \in L^{p'}(\Omega)$,

$$a_{\boldsymbol{\beta},\mu;p}(v,w) := \int_\Omega (\boldsymbol{\beta}\cdot\nabla v)\, w + \int_\Omega \mu\, v\, w. \tag{12}$$

Observe that, for all $p \in (1,\infty)$, $V^0_{\boldsymbol{\beta};p}(\Omega)$ is a closed subspace of $V_{\boldsymbol{\beta};p}(\Omega)$ owing to Lemma 2.1. Assuming that $s \in L^p(\Omega)$, the weak formulation of (1) in the graph space $V^0_{\boldsymbol{\beta};p}(\Omega)$ is:

$$\text{Find } u \in V^0_{\boldsymbol{\beta};p}(\Omega) \ \text{ s.t. } \ a_{\boldsymbol{\beta},\mu;p}(u,v) = \int_\Omega s\, v, \quad \forall v \in L^{p'}(\Omega). \tag{13}$$

It is readily seen that if $u \in V^0_{\boldsymbol{\beta};p}(\Omega)$ solves (13), the equation (1a) holds in $L^p(\Omega)$ and the boundary condition (1b) holds in $L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)$. Note that the boundary conditions are *strongly* enforced in (13).

2.3. Well-posedness for positive Friedrichs tensor

To examine the uniqueness of the weak solution u to (13) in the graph space $V_{\boldsymbol{\beta};p}(\Omega)$, we recall the $\mathbb{R}$-valued Friedrichs tensor

$$\sigma_{\boldsymbol{\beta},\mu;p} := \mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}. \tag{14}$$

Hereafter, we assume that this tensor satisfies the so-called Friedrichs positivity assumption $(\mathcal{H}_p)$:

$(\mathcal{H}_p)$ ess $\inf_\Omega \sigma_{\boldsymbol{\beta},\mu;p} > 0$. We define the reference time $\tau = \left(\text{ess}\inf_\Omega \sigma_{\boldsymbol{\beta},\mu;p}\right)^{-1}$.

Proposition 2.3 *(Uniqueness under $(\mathcal{H}_p)$). Assume that $(\mathcal{H}_p)$ holds. Then*

$$a_{\boldsymbol{\beta},\mu;p}(v, v|v|^{p-2}) \geq \tau^{-1}\|v\|^p_{L^p(\Omega)}, \quad \forall v \in V^0_{\boldsymbol{\beta};p}(\Omega). \tag{15}$$

Proof. Let $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$. Observing that $v|v|^{p-2} \in L^{p'}(\Omega)$, the quantity $a_{\boldsymbol{\beta},\mu;p}(v, v|v|^{p-2})$ is well defined. Owing to the integration by parts formula (10b) with $z \equiv 1$ on Ω (so that $\boldsymbol{\beta}\cdot\nabla z \equiv 0$), we infer that

$$a_{\boldsymbol{\beta},\mu;p}(v, v|v|^{p-2}) = \int_\Omega \left(\mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}\right)|v|^p + \frac{1}{p}\int_{\partial\Omega}(\boldsymbol{\beta}\cdot\boldsymbol{n})|v|^p,$$

whence, using the definition (14) of the Friedrichs tensor $\sigma_{\boldsymbol{\beta},\mu;p}$ and the fact that $v_{|\partial\Omega^-} = 0$, we obtain

$$a_{\boldsymbol{\beta},\mu;p}(v, v|v|^{p-2}) = \int_\Omega \sigma_{\boldsymbol{\beta},\mu;p}|v|^p + \frac{1}{p}\int_{\partial\Omega^+}(\boldsymbol{\beta}\cdot\boldsymbol{n})|v|^p.$$

The desired bound then follows from $(\mathcal{H}_p)$ and the definition of $\partial\Omega^+$.

To prove the well-posedness of (13) under the assumption $(\mathcal{H}_p)$, we need to introduce the two Lipschitz spaces

$$\mathrm{Lip}_0(\Omega) := \left\{v : \Omega \to \mathbb{R} \mid v \in \mathrm{Lip}(\Omega) \text{ and } v_{|\partial\Omega^-} \equiv 0\right\}, \tag{16}$$

and

$$\mathrm{Lip}_c(\partial\Omega^+) := \left\{v : \partial\Omega \to \mathbb{R} \mid v \in \mathrm{Lip}(\partial\Omega) \text{ and } v \text{ is compactly supported on } \partial\Omega^+\right\}, \tag{17}$$

which satisfy the following Proposition.

Proposition 2.4 *(Surjectivity of traces). For all $v \in Lip_c(\partial\Omega^+)$, there is $w \in Lip_0(\Omega)$ such that $w_{|\partial\Omega} = v$.*

Proof. In order to stay general, we denote $d = 3$ the dimension of Ω. Let $v \in \mathrm{Lip}_c(\partial\Omega^+)$ and denote K its compact support on $\partial\Omega^+$. Owing to the Borel–Lebesgue property, we define $\{B_i\}_{1\le i\le N}$ the finite family of open sets in $\mathbb{R}^d$ covering K, i.e.

$$K \subset \bigcup_{1\le i\le N}(B_i \cap \partial\Omega^+) \subsetneq \partial\Omega^+, \tag{18}$$

and we denote $\{\theta_i\}_{1\le i\le N}$ the partition of the unity subordinate to this covering, i.e. for all $1 \le i \le N$, $0 \le \theta_i \le 1$, $\theta_i \in \mathcal{C}^\infty_c(B_i)$ and $\sum_{1\le i\le N}\theta_{i|K} \equiv 1$.

Recalling that the boundary $\partial\Omega$ is assumed to be Lipschitz-continuous, we introduce, for all $1 \le i \le N$, the local bi-Lipschitz charts $\psi_i : Q \to B_i$ where $Q := \{(\boldsymbol{x}', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \mid \left|\boldsymbol{x}'\right| < 1 \text{ and } |x_d| < 1\}$, such that $\psi_i(Q_+) = B_i \cap \Omega$ with $Q_+ = \{(\boldsymbol{x}', x_d) \in \mathbb{R}^{d-1}\times\mathbb{R} \mid \left|\boldsymbol{x}'\right| < 1 \text{ and } 0 < x_d < 1\}$ and $\psi_i(Q_0) = B_i \cap \partial\Omega^+$ with $Q_0 = \{(\boldsymbol{x}', 0) \in \mathbb{R}^{d-1}\times\mathbb{R} \mid \left|\boldsymbol{x}'\right| < 1\}$. Denoting $v_i = v_{|B_i\cap\partial\Omega^+}$, we introduce the function $\tilde{v}_i : Q_+ \to \mathbb{R}$ defined as the extrusion of $v_i \circ \psi_i$ in Q_+, i.e. for all $(\boldsymbol{x}', x_d) \in Q_+$, $\tilde{v}_i(\boldsymbol{x}', x_d) = v_i \circ \psi_i(\boldsymbol{x}', 0)$. Next, mapping back to Ω, we consider $w_i : \overline{\Omega} \to \mathbb{R}$ such that $w_i(\boldsymbol{x}) = \tilde{v}_i \circ \psi_i^{-1}(\boldsymbol{x})$ for all $\boldsymbol{x} \in B_i \cap \overline{\Omega}$ and $w_i(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \overline{\Omega}\backslash B_i \cap \overline{\Omega}$. Finally, collecting these functions $\{w_i\}_{1\le i\le N}$, we observe that the function $w = \sum_{1\le i\le N}\theta_i w_i$ satisfies the desired conditions.

Theorem 2.5 *(Well-posedness under $(\mathcal{H}_p)$). Assume that $(\mathcal{H}_p)$ holds. Then the problem (13) is well-posed.*

Proof. We apply Theorem 1.1. Adapting the proof of Ern & Guermond [7, Theorem 5.7], we first consider $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$ and we denote

$$\mathbb{S}_p := \sup_{w\in L^{p'}(\Omega)\backslash\{0\}} \frac{a_{\boldsymbol{\beta},\mu;p}(v, w)}{\|w\|_{L^{p'}(\Omega)}}.$$

Owing to Proposition 2.3, we infer that $\|v\|^p_{L^p(\Omega)} \le \tau\,\mathbb{S}_p\|v\|^{\frac{p}{p'}}_{L^p(\Omega)}$, recalling that $\|v\,|v|^{p-2}\|^{p'}_{L^{p'}(\Omega)} = \|v\|^p_{L^p(\Omega)}$. Hence, we obtain $\|v\|_{L^p(\Omega)} \le \tau\,\mathbb{S}_p$. To bound the second part of the graph norm, i.e. the advective derivative, we observe that

$$\|\boldsymbol{\beta}\cdot\nabla v\|_{L^p(\Omega)} := \sup_{w\in L^{p'}(\Omega)\backslash\{0\}} \frac{a_{\boldsymbol{\beta},0;p}(v, w)}{\|w\|_{L^{p'}(\Omega)}} = \sup_{w\in L^{p'}(\Omega)\backslash\{0\}} \frac{a_{\boldsymbol{\beta},\mu;p}(v, w) - \int_\Omega \mu v w}{\|w\|_{L^{p'}(\Omega)}}.$$

Then, we obtain

$$\|\boldsymbol{\beta}\cdot\nabla v\|_{L^p(\Omega)} \le \mathbb{S}_p + \|\mu\|_{L^\infty(\Omega)}\|v\|_{L^p(\Omega)} \le \mathbb{S}_p(1 + \|\mu\|_{L^\infty(\Omega)}\tau),$$

yielding the first condition (BNB1) with the constant $C_{\text{BNB}} = (\tau^p + (1 + \|\mu\|_{L^\infty(\Omega)}\tau)^p)^{\frac{1}{p}}$.

Let us prove now the second condition (BNB2). Consider $w \in L^{p'}(\Omega)$ such that $a_{\boldsymbol{\beta},\mu;p}(v,w) = 0$ for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$. Owing to the inclusion $\mathcal{C}^\infty_c(\Omega) \subset V^0_{\boldsymbol{\beta};p}(\Omega)$, it follows that $\mu w - \nabla\cdot(\boldsymbol{\beta} w) = 0$ a.e. in Ω, so that the dense inclusion $\mathcal{C}^\infty_c(\Omega) \subset L^{p'}(\Omega)$ implies that $\boldsymbol{\beta}\cdot\nabla w = (\mu - \nabla\cdot\boldsymbol{\beta}) w \in L^{p'}(\Omega)$, whence $w \in V_{\boldsymbol{\beta};p'}(\Omega)$. Applying now the integration by parts formula (10a), we observe that

$$\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, v w = a_{\boldsymbol{\beta},\mu;p}(v,w) - \int_{\Omega} (\mu - \nabla\cdot\boldsymbol{\beta})\, v w + \int_{\Omega} (\boldsymbol{\beta}\cdot\nabla w)\, v = 0,$$

for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$. In particular, observing that $\mathrm{Lip}_0(\Omega) \subset V^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,\infty)$, and owing to Proposition 2.4, we have

$$\int_{\partial\Omega^+} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, v w = 0, \quad \forall v \in \mathrm{Lip}_c(\partial\Omega^+).$$

Owing to the vanishing integral property, this identity yields $w_{|\partial\Omega^+} = 0$. Now, testing the identity $\mu w - \nabla\cdot(\boldsymbol{\beta} w) = 0$ by an arbitrary $y \in L^p(\Omega)$ and using the chain rule $\nabla\cdot(\boldsymbol{\beta} w) = \boldsymbol{\beta}\cdot\nabla w + (\nabla\cdot\boldsymbol{\beta}) w$, we infer that

$$0 = \int_{\Omega} (\mu w - \nabla\cdot(\boldsymbol{\beta} w))\, y = \int_{\Omega} (\mu - \nabla\cdot\boldsymbol{\beta}) w y - \int_{\Omega} \boldsymbol{\beta}\cdot\nabla w\, y.$$

Hence, the particular choice $y = w\,|w|^{p'-2}$ along with the identity (10b) with p replaced by p' and with $z \equiv 1$ yields

$$\begin{aligned} 0 &= \int_{\Omega} (\mu - \nabla\cdot\boldsymbol{\beta})\, |w|^{p'} + \frac{1}{p'} \int_{\Omega} (\nabla\cdot\boldsymbol{\beta})\, |w|^{p'} - \frac{1}{p'} \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, |w|^{p'} \\ &= \int_{\Omega} \sigma_{\boldsymbol{\beta},\mu;p}\, |w|^{p'} - \frac{1}{p'} \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, |w|^{p'} \geq \tau^{-1} \|w\|^{p'}_{L^{p'}(\Omega)}, \end{aligned}$$

where we have used that $w_{|\partial\Omega^+} = 0$ and the assumption $(\mathcal{H}_p)$. As a result, $w = 0$ a.e. in Ω, so that the condition (BNB2) is satisfied. Owing to Theorem 1.1, there exists a unique solution solving the problem (13).

2.4. Well-posedness for non-positive Friedrichs tensor

Summarizing the results obtained so far, we have proved under assumption $(\mathcal{H}_p)$ the well-posed of (1) in the graph space $V^0_{\boldsymbol{\beta};p}(\Omega)$. This section aims to extend this result under the new assumption $(\mathcal{H}'_p)$ so as to include the situation where the infimum of the Friedrichs tensor $\sigma_{\boldsymbol{\beta},\mu;p}$ takes null or slightly negative values.

$(\mathcal{H}'_p)$ ess $\inf_\Omega \sigma_{\boldsymbol{\beta},\mu;p} \leq 0$ and there exists a non-dimensional function $\zeta \in \mathrm{Lip}(\Omega)$ such that

$$\operatorname{ess\,inf}_\Omega e^\zeta \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p} \boldsymbol{\beta}\cdot\nabla\zeta \right) > 0. \tag{19}$$

We define the reference time $\tau = \left(\operatorname{ess\,inf}_\Omega e^\zeta \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta \right) \right)^{-1}$.

Proposition 2.6 *(Uniqueness under $(\mathcal{H}'_p)$). Assume that $(\mathcal{H}'_p)$ holds. Then*

$$a_{\boldsymbol{\beta},\mu;p}(v, e^\zeta v|v|^{p-2}) \geq \tau^{-1} \|v\|^p_{L^p(\Omega)}, \quad \forall v \in V^0_{\boldsymbol{\beta};p}(\Omega). \tag{20}$$

Proof. Let $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$. Observing that

$$a_{\boldsymbol{\beta},\mu;p}(v, e^\zeta v|v|^{p-2}) = a_{\widetilde{\boldsymbol{\beta}},\widetilde{\mu};p}(v, v|v|^{p-2}),$$

where we have denoted $\widetilde{\boldsymbol{\beta}} = e^\zeta \boldsymbol{\beta}$ and $\widetilde{\mu} = e^\zeta \mu$, the inequality (20) follows from Proposition 2.3 if ess $\inf_\Omega \sigma_{\widetilde{\boldsymbol{\beta}},\widetilde{\mu};p} > 0$, i.e. if $(\mathcal{H}'_p)$ holds, since we have

$$\sigma_{\widetilde{\boldsymbol{\beta}},\widetilde{\mu};p} = e^\zeta \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta \right).$$

Remark 1 *(Using integration by parts).* Instead of using Proposition 2.3 in the proof of Proposition 2.6, it is also possible to obtain this result by applying the general integration by parts formula (10b) with $z = e^\zeta v|v|^{p-2}$.

Example 1. Assumption $(\mathcal{H}'_p)$ indeed generalizes the assumption $(\mathcal{H}_p)$ since it is now possible to consider situations that cannot be handled under $(\mathcal{H}_p)$. For example, considering the rotating field $\boldsymbol{\beta} = (y, -x, z+1)$ expressed in the Cartesian coordinates of $\mathbb{R}^3$ and a reaction coefficient $\mu \in \mathbb{R}$, we have $\sigma_{\boldsymbol{\beta},\mu;p} = \mu - \frac{1}{p}$. If $\mu \leq \frac{1}{p}$, this Friedrichs tensor does not satisfy $(\mathcal{H}_p)$, whereas $(\mathcal{H}'_p)$ does, for example with the potential $\zeta(\boldsymbol{x}) = \alpha(1+z)^2$ for all $\alpha \in \mathbb{R}$ such that $8\alpha < \mu p - 1$.

Remark 2 *(Existence of ζ).* Following Devinatz & al. [5] and considering a continuously differentiable field $\boldsymbol{\beta} \in \mathcal{C}^1(\mathbb{R}^d)$, the existence of the potential ζ relies on the assumption that every solution to the Cauchy problem $d_t\boldsymbol{x}(t) = \boldsymbol{\beta}(\boldsymbol{x}(t))$, $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \Omega$ remains in the domain Ω for a finite time only. Observing that the proof in this reference is based on the flow box theorem, the extension to a less regular field (e.g. $\boldsymbol{\beta} \in \mathbf{Lip}(\Omega)$) is a priori not obvious.

We are now in a position to state the well-posedness of (13) under assumption $(\mathcal{H}'_p)$.

Theorem 2.7 *(Well-posedness under $(\mathcal{H}'_p)$). Assume that $(\mathcal{H}'_p)$ holds. Then the problem (13) is well posed.*

Proof. We follow the same ideas as in the proof of Theorem 2.5. The condition (BNB1) is inferred from Proposition 2.6 with $C_{\text{BNB}} = (\|e^\zeta\|^p_{L^\infty(\Omega)}\tau^p + (1+\|\mu\|_{L^\infty(\Omega)}\tau\|e^\zeta\|_{L^\infty(\Omega)})^p)^{\frac{1}{p}}$. Turning to the second condition (BNB2), we consider $w \in L^{p'}(\Omega)$ such that $a_{\boldsymbol{\beta},\mu;p}(v, w) = 0$ for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$. Proceeding as in the proof of Theorem 2.5, this implies that w belongs to the graph space $V_{\boldsymbol{\beta};p'}(\Omega)$ and that it satisfies $\mu w - \nabla\cdot(\boldsymbol{\beta} w) = 0$ a.e. in Ω and $w_{|\partial\Omega^+} \equiv 0$. Let us prove that $w \equiv 0$ a.e. in Ω. First, we observe that replacing p by p' and choosing $z = e^{\zeta(1-p')}$ in (10b) yields

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla w)\, w\, |w|^{p'-2}\, e^{\zeta(1-p')} = \frac{1}{p'}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, |w|^{p'}\, e^{\zeta(1-p')} - \frac{1}{p'}\int_\Omega (\nabla\cdot\boldsymbol{\beta})\, |w|^{p'}\, e^{\zeta(1-p')} - \frac{1-p'}{p'}\int_\Omega (\boldsymbol{\beta}\cdot\nabla\zeta)\, |w|^{p'}\, e^{\zeta(1-p')}.$$

Hence, testing the identity $\mu w - \nabla\cdot(\boldsymbol{\beta} w) = 0$ with the function $w\,|w|^{p'-2}\,e^{\zeta(1-p')}$, we infer that

$$\begin{aligned} 0 &= \int_\Omega (\mu - \nabla\cdot\boldsymbol{\beta})\, |w|^{p'}\, e^{\zeta(1-p')} - \int_\Omega (\boldsymbol{\beta}\cdot\nabla w)\, w\, |w|^{p'-2}\, e^{\zeta(1-p')} \\ &= \int_\Omega (\mu - \nabla\cdot\boldsymbol{\beta})\, |w|^{p'}\, e^{\zeta(1-p')} + \frac{1}{p'}\int_\Omega (\nabla\cdot\boldsymbol{\beta})\, |w|^{p'}\, e^{\zeta(1-p')} - \frac{1}{p}\int_\Omega (\boldsymbol{\beta}\cdot\nabla\zeta)\, |w|^{p'}\, e^{\zeta(1-p')} - \frac{1}{p'}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, |w|^{p'}\, e^{\zeta(1-p')}. \end{aligned}$$

Then, collecting these terms and using the fact that $w_{|\partial\Omega^+} = 0$, we obtain

$$\begin{aligned} 0 &= \int_\Omega e^{\zeta(1-p')} \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) |w|^{p'} - \frac{1}{p'}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, |w|^{p'}\, e^{\zeta(1-p')} \\ &\geq \int_\Omega e^{\zeta(1-p')} \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) |w|^{p'}. \end{aligned}$$

As a result, owing to $(\mathcal{H}_{p'})$ and the fact that $w_{|\partial\Omega^+} \equiv 0$, it follows that $w \equiv 0$ a.e. in Ω. Owing to Theorem 1.1, we conclude that (13) is well posed under assumption $(\mathcal{H}_{p'})$.

3. Vector advection–reaction problem

In this section, we apply similar ideas to analyze the well-posedness of the vector-valued problem (2) in Banach graph spaces where we generalize the assumption of the sign of the Friedrichs tensor $\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p}$ defined by (4). For the sake of brevity, the proofs are omitted if they are straightforwardly adapted from those of the scalar case in Section 2.

3.1. The graph space

Let us introduce the graph space

$$\boldsymbol{V}_{\boldsymbol{\beta};p}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \mid (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\right\}, \tag{21}$$

where the i-th component in the Cartesian basis of $(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}$ is given by $\beta_j \partial_j \boldsymbol{v}_i$ (where repeated indices are summed) and where $(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ means that the linear form

$$\boldsymbol{\mathcal{C}}_c^\infty(\Omega) \ni \boldsymbol{\varphi} \mapsto -\int_\Omega \nabla\cdot(\boldsymbol{\beta}\otimes\boldsymbol{\varphi})\cdot\boldsymbol{v}, \tag{22}$$

is bounded in $\boldsymbol{L}^{p'}(\Omega)$, so that $(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}$ is the Riesz representative of (22) in $\boldsymbol{L}^p(\Omega)$. Equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}_{\beta;p}(\Omega)} := (\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)} + \|(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)})^{\frac{1}{p}}$ for all $\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega)$, this space defines a reflexive Banach space. The following proposition states that the problem (2) is well defined in the graph space $\boldsymbol{V}_{\beta;p}(\Omega)$.

Proposition 3.1 *(Equivalent definition of $\boldsymbol{V}_{\beta;p}(\Omega)$). The following holds:*

$$\boldsymbol{V}_{\beta;p}(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \mid \nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} \in \boldsymbol{L}^p(\Omega)\right\},$$

where $\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} \in \boldsymbol{L}^p(\Omega)$ means that the linear form

$$\boldsymbol{\mathcal{C}}_c^\infty(\Omega) \ni \boldsymbol{\varphi} \mapsto -\int_\Omega (\boldsymbol{\beta}\,\nabla\cdot\boldsymbol{\varphi} + \nabla\times(\boldsymbol{\varphi}\times\boldsymbol{\beta}))\cdot\boldsymbol{v}, \tag{23}$$

is bounded in $\boldsymbol{L}^{p'}(\Omega)$.

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega)$. By definition, we have

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{\varphi} = -\int_\Omega \nabla\cdot(\boldsymbol{\beta}\otimes\boldsymbol{\varphi})\cdot\boldsymbol{v}, \quad \forall\boldsymbol{\varphi} \in \boldsymbol{\mathcal{C}}_c^\infty(\Omega).$$

Now, recalling the identity $\nabla\cdot(\boldsymbol{\beta}\otimes\boldsymbol{\varphi}) = \boldsymbol{\beta}\,\nabla\cdot\boldsymbol{\varphi} + \nabla\times(\boldsymbol{\varphi}\times\boldsymbol{\beta}) + (\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\beta}$ for all $\boldsymbol{\beta} \in \mathbf{Lip}(\Omega)$ and for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{C}}_c^\infty(\Omega)$, it follows that

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{\varphi} = -\int_\Omega (\boldsymbol{\beta}\,\nabla\cdot\boldsymbol{\varphi} + \nabla\times(\boldsymbol{\varphi}\times\boldsymbol{\beta}))\cdot\boldsymbol{v} - \int_\Omega ((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\beta})\cdot\boldsymbol{v}.$$

Hence, observing that $((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\beta})\cdot\boldsymbol{v} = ((\nabla\boldsymbol{\beta})\boldsymbol{\varphi})\cdot\boldsymbol{v}$, we obtain

$$\int_\Omega \left((\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + (\nabla\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{v}\right)\cdot\boldsymbol{\varphi} = -\int_\Omega (\boldsymbol{\beta}\,\nabla\cdot\boldsymbol{\varphi} + \nabla\times(\boldsymbol{\varphi}\times\boldsymbol{\beta}))\cdot\boldsymbol{v}, \quad \forall\boldsymbol{\varphi} \in \boldsymbol{\mathcal{C}}_c^\infty(\Omega).$$

Hence, the linear form (23) is bounded, yielding $\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} \in \boldsymbol{L}^p(\Omega)$, so that the inclusion holds. Note that we have the identity $(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + (\nabla\boldsymbol{\beta})^{\mathrm{T}}\boldsymbol{v} = \nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta}$ a.e. in Ω. Since the proof of the converse inclusion is similar, the proof is completed.

Recalling now that the boundary $\partial\Omega$ is well separated in the sense of (9), functions in the graph space $\boldsymbol{V}_{\beta;p}(\Omega)$ have a trace in the space

$$\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega) := \left\{\boldsymbol{v} : \partial\Omega \to \mathbb{R}^3 \mid v \text{ is Lebesgue measurable on } \partial\Omega \text{ and } \int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|\boldsymbol{v}|^p < \infty\right\}. \tag{24}$$

Equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} := (\int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|\boldsymbol{v}|^p)^{\frac{1}{p}}$ for all $\boldsymbol{v} \in \boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$, this space defines a Banach space.

Lemma 3.2 *(Trace in $\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$). The map $\boldsymbol{\gamma} : \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega}) \to \boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$ with $\boldsymbol{\gamma}(\boldsymbol{\varphi}) = \boldsymbol{\varphi}_{|\partial\Omega}$ for all $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{C}}^\infty(\overline{\Omega})$ extends continuously to $\boldsymbol{V}_{\beta;p}(\Omega)$, i.e. there exists $C_{\boldsymbol{\gamma}} > 0$ such that*

$$\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} \le C_{\boldsymbol{\gamma}}\|\boldsymbol{v}\|_{\boldsymbol{V}_{\beta;p}(\Omega)}, \quad \forall\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega).$$

Owing to the existence of a trace in $\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$, we now extend Lemma 2.2 to a vector-valued function.

Lemma 3.3 *(Integration by parts). For all $\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega)$ and for all $\boldsymbol{w} \in \boldsymbol{V}_{\beta;p'}(\Omega)$,*

$$\int_\Omega \boldsymbol{w}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + \int_\Omega \boldsymbol{v}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{w} + \int_\Omega (\nabla\cdot\boldsymbol{\beta})\,\boldsymbol{v}\cdot\boldsymbol{w} = \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\;\boldsymbol{v}\cdot\boldsymbol{w}. \tag{25a}$$

In addition, for all $\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega)$ and for all $z \in W^{1,\infty}(\Omega)$,

$$\int_\Omega |\boldsymbol{v}|^{p-2} z\,\boldsymbol{v}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + \frac{1}{p}\int_\Omega (\nabla\cdot\boldsymbol{\beta})|\boldsymbol{v}|^p z + \frac{1}{p}\int_\Omega \boldsymbol{\beta}\cdot\nabla z\,|\boldsymbol{v}|^p = \frac{1}{p}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})|\boldsymbol{v}|^p. \tag{25b}$$

3.2. Weak formulation

Following the route of Section 2, we introduce the bilinear form $\boldsymbol{a}_{\beta,\mu;p} \in \mathcal{L}(\boldsymbol{V}^0_{\beta;p}(\Omega)\times\boldsymbol{L}^{p'}(\Omega);\mathbb{R})$ with $\boldsymbol{V}^0_{\beta;p}(\Omega) = \{\boldsymbol{w} \in \boldsymbol{V}_{\beta;p}(\Omega) \mid \boldsymbol{w}_{|\partial\Omega^-} = \boldsymbol{0}\}$ such that for all $\boldsymbol{v} \in \boldsymbol{V}_{\beta;p}(\Omega)$ and for all $\boldsymbol{w} \in \boldsymbol{L}^{p'}(\Omega)$,

$$\boldsymbol{a}_{\beta,\mu;p}(\boldsymbol{v},\boldsymbol{w}) := \int_\Omega (\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta})\cdot\boldsymbol{w} + \int_\Omega \boldsymbol{\mu}\boldsymbol{v}\cdot\boldsymbol{w}, \tag{26}$$

with $\boldsymbol{\mu} : \Omega \to \mathbb{R}^{3\times3}$ the reaction tensor. Following the proof of Proposition 3.1, we observe that the bilinear form (26) can be reformulated as

$$\boldsymbol{a}_{\beta,\mu;p}(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \boldsymbol{w}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + \int_\Omega \boldsymbol{w}\cdot(\nabla\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\mu})\boldsymbol{v}, \tag{27}$$

and the writing $\boldsymbol{\mu}' = \nabla\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\mu}$ yields $\boldsymbol{\mu}' \in \boldsymbol{L}^\infty(\Omega)$ and

$$\boldsymbol{a}_{\beta,\mu;p}(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{w} + \int_\Omega \boldsymbol{\mu}'\boldsymbol{v}\cdot\boldsymbol{w}. \tag{28}$$

Assuming now that $\boldsymbol{s} \in \boldsymbol{L}^p(\Omega)$, the weak formulation of (2) in the graph space $\boldsymbol{V}^0_{\beta;p}(\Omega)$ is:

$$\text{Find } \boldsymbol{u} \in \boldsymbol{V}^0_{\beta;p}(\Omega) \quad \text{s.t.} \quad \boldsymbol{a}_{\beta,\mu;p}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{s}\cdot\boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega). \tag{29}$$

We readily see that if $\boldsymbol{u} \in \boldsymbol{V}^0_{\beta;p}(\Omega)$ solves (29), the problem (2a) holds in $\boldsymbol{L}^p(\Omega)$, and the boundary condition (2b) holds in $\boldsymbol{L}^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$.

3.3. Well-posedness for positive and non-positive Friedrichs tensors

The uniqueness of the solution to problem (29) relies on the sign of the lowest eigenvalue of the $\mathbb{R}^{3\times3}$-valued Friedrichs tensor

$$\boldsymbol{\sigma}_{\beta,\mu;p} := \frac{\boldsymbol{\mu}+\boldsymbol{\mu}^{\mathrm{T}}}{2} + \frac{\nabla\boldsymbol{\beta}+\nabla\boldsymbol{\beta}^{\mathrm{T}}}{2} - \frac{1}{p}(\nabla\cdot\boldsymbol{\beta})\,\mathbf{Id}. \tag{30}$$

For all $\boldsymbol{x} \in \Omega$, this lowest eigenvalue is denoted by $\aleph_p(\boldsymbol{x})$ and is defined as

$$\aleph_p(\boldsymbol{x}) = \min\left\{\left(\boldsymbol{\sigma}_{\beta,\mu;p}(\boldsymbol{x})\boldsymbol{y},\boldsymbol{y}\right) \mid \boldsymbol{y} \in \mathbb{R}^3 \text{ s.t. } |\boldsymbol{y}| = 1\right\},$$

where $(\cdot,\cdot)$ denotes the classical Euclidean inner product in $\mathbb{R}^3$. Hereafter, we assume that this eigenvalue satisfies the following assumption.

$(\mathcal{H}_p)$ ess $\inf_\Omega \aleph_p > 0$. We define $\tau = \left(\text{ess}\inf_\Omega \aleph_p\right)^{-1}$.

Proposition 3.4 *(Uniqueness under $(\mathcal{H}_p)$). Assume that $(\mathcal{H}_p)$ holds. Then,*

$$\boldsymbol{a}_{\beta,\mu;p}(\boldsymbol{v},\boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) \geq \tau^{-1}\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0_{\beta;p}(\Omega).$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ and consider $\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2})$ (since $\boldsymbol{v}\,|\boldsymbol{v}|^{p-2} \in \boldsymbol{L}^{p'}(\Omega)$). Owing to the identity (27), we infer that

$$\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) = \int_\Omega |\boldsymbol{v}|^{p-2}\, \boldsymbol{v}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} + \int_\Omega |\boldsymbol{v}|^{p-2}\, \boldsymbol{v}\cdot \left(\nabla\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\mu}\right)\boldsymbol{v}.$$

Using now the integration by parts formula (25b) with $z \equiv 1$, we obtain

$$\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) = \int_\Omega |\boldsymbol{v}|^{p-2}\, \boldsymbol{v}\cdot\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p}\cdot\boldsymbol{v} + \frac{1}{p}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\,|\boldsymbol{v}|^p\,, \tag{31}$$

whence the result follows using $(\boldsymbol{\mathcal{H}}_p)$ and recalling that $\boldsymbol{v}_{|\partial\Omega^-} = 0$.

To take into account the situation where the smallest eigenvalue $\aleph_p$ takes null or slightly negative values in Ω, we consider the new assumption $(\boldsymbol{\mathcal{H}}'_p)$.

$(\boldsymbol{\mathcal{H}}'_p)$ ess $\inf_\Omega \aleph_p \le 0$ and there exists a non-dimensional function $\zeta \in \mathrm{Lip}(\Omega)$ such that

$$\text{ess}\inf_\Omega e^\zeta \left(\aleph_p - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) > 0.$$

We define $\tau^{-1} = \text{ess}\inf_\Omega e^\zeta \left(\aleph_p - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right)$.

Proposition 3.5 *(Uniqueness under $(\boldsymbol{\mathcal{H}}'_p)$). Assume that $(\boldsymbol{\mathcal{H}}'_p)$ holds. Then,*

$$\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, e^\zeta\,\boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) \ge \tau^{-1}\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega).$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$. Denoting $\widetilde{\boldsymbol{\beta}} = e^\zeta\boldsymbol{\beta}$, $\widetilde{\boldsymbol{\mu}} = e^\zeta\boldsymbol{\mu}$ and observing that

$$\nabla(\widetilde{\boldsymbol{\beta}}\cdot\boldsymbol{v}) = e^\zeta\,(\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\zeta\otimes\boldsymbol{\beta})\boldsymbol{v}) \quad \text{a.e. in } \Omega,$$

we infer that $\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, e^\zeta\,\boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) = \boldsymbol{a}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) - \boldsymbol{a}_{\boldsymbol{0},\nabla\zeta\otimes\boldsymbol{\beta};p}(\boldsymbol{v}, e^\zeta\,\boldsymbol{v}\,|\boldsymbol{v}|^{p-2})$. Owing to (31), this identity yields

$$\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, e^\zeta\,\boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) \ge \int_\Omega |\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\boldsymbol{\sigma}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p}\cdot\boldsymbol{v} - \int_\Omega e^\zeta\,|\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\left(\frac{\nabla\zeta\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\nabla\zeta}{2}\right)\cdot\boldsymbol{v}.$$

In addition, observing that

$$\boldsymbol{\sigma}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p} = e^\zeta\left(\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p} - \frac{1}{p}(\boldsymbol{\beta}\cdot\nabla\zeta)\mathbf{Id}\right) + e^\zeta\left(\frac{\nabla\zeta\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\nabla\zeta}{2}\right),$$

the expected result follows from assumption $(\boldsymbol{\mathcal{H}}'_p)$.

Finally, the well-posedness of (2) holds under assumption $(\boldsymbol{\mathcal{H}}_p)$ or $(\boldsymbol{\mathcal{H}}'_p)$. The proof follows the same ideas used to prove Theorems 2.5 and 2.7, this time using Propositions 3.4 and 3.5, respectively.

Theorem 3.6 *(Well-posedness). Assume that $(\boldsymbol{\mathcal{H}}_p)$ or $(\boldsymbol{\mathcal{H}}'_p)$ holds. Then the problem (29) is well posed.*

4. Conclusion

In this paper, we have extended the well-posedness of problems (1) and (2), not only in the Banach graph space of exponent $p \in (1, \infty)$, but also under new assumptions regarding the classical Friedrichs tensor, so as to consider the situation when it takes positive, null, or slightly negative values. Observing that equations (1a) and (2a) are the proxy of the Lie derivative in $\mathbb{R}^3$ of a 0- and a 1-form respectively, the present analysis could be extended in a future work within the more general framework proposed by Heumann [11] to treat the advection of a differential k-form within a manifold of $\mathbb{R}^d$. However, the question of the existence of the potential ζ in such a more general context is still open. Another extension of this work concerns the case of the non-homogeneous Dirichlet boundary condition, requiring to establish the surjectivity of the trace maps in these Banach graph spaces.

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