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Number theory

A remark on Liao and Rams' result on the distribution of the leading partial quotient with growing speed $e^{n^{1/2}}$ in continued fractions

Une remarque sur le résultat de Liao et Rams concernant la distribution des fractions continues dont le plus grand quotient partiel croît en $e^{n^{1/2}}$

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A R T I C L E   I N F O

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A B S T R A C T

For a real $x \in (0,1)\setminus\mathbb{Q}$, let $x = [a_1(x), a_2(x), \cdots]$ be its continued fraction expansion. Denote by

$$T_n(x) := \max\{a_k(x) : 1 \le k \le n\}$$

the maximum partial quotient up to n. For any real $\alpha \in (0,\infty)$, $\gamma \in (0,\infty)$, let

$$F(\gamma,\alpha) := \{x \in (0,1)\setminus\mathbb{Q} : \lim_{n\to\infty} \frac{T_n(x)}{e^{n^\gamma}} = \alpha\}.$$

For a set $E \subset (0,1)\setminus\mathbb{Q}$, let $\dim_H E$ be its Hausdorff dimension. Recently, Lingmin Liao and Michal Rams showed that

$$\dim_H F(\gamma,\alpha) = \begin{cases} 1 & \text{if } \gamma \in (0,1/2) \\ 1/2 & \text{if } \gamma \in (1/2,\infty) \end{cases}$$

for any $\alpha \in (0,\infty)$. In this paper, we show that $\dim_H F(1/2,\alpha) = 1/2$ for any $\alpha \in (0,\infty)$ following Liao and Rams' method, which supplements their result.

R É S U M É

Étant donné un réel $x \in (0,1)\setminus\mathbb{Q}$, soit $x = [a_1(x), a_2(x), \cdots]$ son développement en fraction continue. Soit

$$T_n(x) := \max\{a_k(x) : 1 \le k \le n\}$$

le plus grand quotient partiel jusqu'à n. Pour tout $\alpha \in (0,\infty)$, $\gamma \in (0,\infty)$, soit

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$$F(\gamma,\alpha) := \{x \in (0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} \frac{T_n(x)}{e^{n^\gamma}} = \alpha\}.$$

Pour un ensemble $E \subset (0,1) \setminus \mathbb{Q}$, soit $\dim_H E$ sa dimension de Hausdorff. Récemment, Lingmin Liao et Michal Rams ont montré que

$$\dim_H F(\gamma,\alpha) = \begin{cases} 1 & si\ \gamma \in (0,1/2) \\ 1/2 & si\ \gamma \in (1/2,\infty) \end{cases}$$

pour tout $\alpha \in (0,\infty)$. Dans cet article, nous montrons que $\dim_H F(1/2,\alpha) = 1/2$ pour tout $\alpha \in (0,\infty)$ en suivant la méthode de Liao et Rams, ce qui complète leur résultat.

1. Introduction

For a real $x \in (0,1) \setminus \mathbb{Q}$, let $x = [a_1(x), a_2(x), \cdots]$ be its regular continued fraction expansion. Denote by

$$T_n(x) := \max\{a_k(x) : 1 \le k \le n\}$$

the maximum partial quotient up to n. Let $S_n(x) = \sum_{k=1}^n a_k(x)$. We mainly focus on the limit behaviour of $T_n(x)$ in this paper, more precise, on the Hausdorff dimensions of sets under some limit behaviour. Erdös had ever conjectured that $\liminf_{n\to\infty} \frac{T_n(x)}{(n/\log\log n)} = 1$ almost everywhere, but later W. Philipp [7] showed that

$$\liminf_{n\to\infty} \frac{T_n(x)}{(n/\log\log n)} = \frac{1}{\log 2} \tag{1.1}$$

almost everywhere. Some explicit examples of reals x satisfying (1.1) are given in [6] by T. Okano. For a set $E \subset (0,1) \setminus \mathbb{Q}$, let $\dim_H E$ be its Hausdorff dimension. In 2008, Wu and Xu [8] first considered Hausdorff dimensions of some sets determined by some limit behaviour of $T_n(x)$. They showed that

$$\dim_H\{x \in (0,1) : \lim_{n\to\infty} \frac{T_n(x)}{\phi(n)} = \alpha\} = 1$$

for any $\alpha \ge 0$ and any monotone increasing sequence $\{\phi(n)\}_{n=1}^\infty$ with $\lim_{n\to\infty} \phi(n) = \infty$ and $\lim_{n\to\infty} \frac{\log\phi(n)}{\log n} < \infty$. In the case of some faster growing speed $\{\phi(n)\}_{n=1}^\infty$, let

$$F(\gamma,\alpha) := \{x \in (0,1) \setminus \mathbb{Q} : \lim_{n\to\infty} \frac{T_n(x)}{e^{n^\gamma}} = \alpha\}, \quad 0 < \alpha, \gamma < \infty.$$

Recently Liao and Rams [5] showed that

$$\dim_H F(\gamma,\alpha) = \begin{cases} 1 & \text{if } \gamma \in (0,1/2) \\ 1/2 & \text{if } \gamma \in (1/2,\infty) \end{cases} \tag{1.2}$$

for any $\alpha \in (0,\infty)$. In their proof, the case of $\gamma \in (0,1/2)$ follows from [9, Section 4]. In the case of $\gamma \in (1/2,\infty)$, the lower bound is obtained by constructing a large subset of $F(\gamma,\alpha)$ with H-dimension 1/2 (see [5, Lemma 2.3] and [2, Lemma 3.2]). The argument applies to all $\gamma \in (0,1)$ in fact. The upper bound 1/2 is obtained by transferring the situation to the distribution of $S_n(x)$, as

$$F(\gamma,\alpha) \subset \{x : \alpha(1-\epsilon)e^{n^\gamma} \le S_n(x) \le \alpha(1+\epsilon)e^{n^\gamma}\} \tag{1.3}$$

for any $\epsilon > 0$. This relates closely the distribution of the two terms $T_n(x)$ and $S_n(x)$ in continued fractions. We will not discuss the dimensions of level sets determined by $S_n(x)$ here, but we will recommend [1,4,5,9,10] to interested readers. The jump of dimensions in (1.2) is interesting, we will deal with the case $\gamma = 1/2$ in this paper. We follow Liao and Rams' method to show the following theorem.

1.1. Theorem. *For any real* $\alpha > 0$, *we have* $\dim_H F(1/2,\alpha) \le 1/2$.

Considering the results mentioned before, this will force the following theorem.

1.2. Theorem. *For any real* $\alpha > 0$, *we have* $\dim_H F(1/2,\alpha) = 1/2$.

For dimensions of the set $\{x \in (0,1) : \lim_{n\to\infty} \frac{T_n(x)}{\phi(n)} = \alpha\}$ with doubly exponential increasing rate $\{\phi(n)\}_{n=1}^\infty$, see [3]. There are more introductions of metric results on the sets related with T_n in [3].

2. The proof of Theorem 1.1

We follow Liao and Rams' notations [5] throughout the proof. We only prove

$$\dim_H F(1/2, 1) \le 1/2,$$

as one can show the theorem for any $\alpha \in \mathbb{R}^+ := (0, \infty)$ by the same process. In order to do this, we first show that

2.1. Lemma. *Let $L \in \mathbb{R}^+$ be a constant. Let $n_k := [(\frac{k}{L})^2]$ (the integer part of $(\frac{k}{L})^2$), $k \in \mathbb{N}$. Then for any $x \in F(1/2, 1)$ and k large enough, there exists an integer j_k, $n_{k-1} < j_k \le n_k$, such that*

$$T_{n_k}(x) = a_{j_k}(x).$$

Proof. We prove this by reduction to absurdity. Suppose that there exist infinitely many integers k_i, j_{k_i}, $i \in \mathbb{N}$, $k_i > k_{i-1}$, $j_{k_i} \le n_{k_i - 1}$, such that

$$T_{n_{k_i}}(x) = a_{j_{k_i}}(x)$$

for some $x \in F(1/2, 1)$. Note that, in this case, we have

$$T_{n_{k_i - 1}}(x) = a_{j_{k_i}}(x).$$

Then, for the sequence $\{n_{k_1 - 1}, n_{k_2 - 1}, \cdots\}$, we have

$$\lim_{i\to\infty} \frac{T_{n_{k_i-1}}(x)}{e^{n_{k_i-1}^{1/2}}} = \lim_{i\to\infty} \frac{T_{n_{k_i}}(x)}{e^{[(k_i-1)^2/L^2]^{1/2}}} = \lim_{i\to\infty} \frac{T_{n_{k_i}}(x)}{e^{n_{k_i}^{1/2}}} \frac{e^{[k_i^2/L^2]^{1/2}}}{e^{[(k_i-1)^2/L^2]^{1/2}}} = 1 \cdot e^{1/L} \ne 1,$$

which contradicts the fact that

$$\lim_{k\to\infty} \frac{T_k(x)}{e^{k^{1/2}}} = 1$$

as $x \in F(1/2, 1)$. So our conclusion holds for any sufficiently large k. □

In the following, we will omit the integer notation [ ] for simplicity, as the results will not be affected. By Lemma 2.1,

2.2. Corollary. *For $x \in F(1/2, 1)$ and $n_k := (\frac{k}{L})^2$, we have*

$$(1 - \epsilon)e^{k/L} \le S_{n_k}(x) - S_{n_{k-1}}(x) \le (1 + \epsilon)(\tfrac{k}{L})^2 e^{k/L}$$

for a small $\epsilon \in \mathbb{R}^+$ and any k large enough.

The rest of the work goes similarly as the estimation of the upper bound for E_φ when $\gamma > 1/2$ in [5, Proof of Theorem 1.1]. For the length of the rank-n fundamental interval

$$I_n(a_1, \cdots, a_n) := \{x \in (0, 1) \setminus \mathbb{Q} : a_1(x) = a_1, \cdots, a_n(x) = a_n\},$$

we have

$$\prod_{i=1}^n \frac{1}{(a_i+1)^2} \le |I_n(a_1, \cdots, a_n)| \le \prod_{i=1}^n \frac{1}{a_i^2}.$$

Let

$$A(m, n) := \{(i_1, \cdots, i_n) \in \{1, \cdots, m\}^n : \textstyle\sum_{j=1}^n i_j = m\}.$$

Let $\zeta(\cdot)$ be the Riemann zeta function. We quote [5, Lemma 2.1] as following.

2.3. Lemma *(Liao and Rams). For $s \in (1/2, 1)$ and $m \ge n$, we have*

$$\sum_{(i_1,\cdots,i_n)\in A(m,n)} \prod_{j=1}^n \frac{1}{i_j^{2s}} \le (\frac{9}{2}(2 + \zeta(2s)))^n \frac{1}{m^{2s}}.$$

Now we are in a position to bound the Hausdorff dimension of $F(1/2,1)$ above.

Proof of Theorem 1.1. Let D_l be the integers in the interval $[(1-\epsilon)e^{l/L}, (1+\epsilon)(\frac{l}{L})^2 e^{l/L}], l > L$. Let $B(1/2, N)$ be the union of the intervals $\{I_{n_k}(a_1, a_2, \cdots, a_{n_k})\}_{k \geq N}$ such that the quotients satisfy

$$\sum_{j=n_{l-1}+1}^{n_l} a_j = m \text{ with } m \in D_l$$

for any $N \leq l \leq k$. By Corollary 2.2, one can see that

$$F(1/2,1) \subset \cup_{N=1}^{\infty} B(1/2, N).$$

Now we show that $\dim_H B(1/2,1) \leq 1/2$. A similar method implies $\dim_H B(1/2, N) \leq 1/2$ for any $N \in \mathbb{N}$, which is enough to prove our Theorem 1.1. By Lemma 2.3,

$$\sum_{I_{n_k} \subset B(1/2,1)} |I_{n_k}|^s \leq \prod_{l=1}^{k} \sum_{m \in D_l} (\frac{9}{2}(2+\zeta(2s)))^{n_l - n_{l-1}} \frac{1}{m^{2s}}.$$

Note that $|D_l| \leq (1+\epsilon)(\frac{k}{L})^2 e^{k/L}, m > (1-\epsilon)e^{k/L}$, so

$$\begin{aligned} &\sum_{I_{n_k} \subset B(1/2,1)} |I_{n_k}|^s \\ \leq\ & \prod_{l=1}^{k} (1+\epsilon)(1-\epsilon)^{2s} (l/L)^2 e^{(1-2s)l/L} (\frac{9}{2}(2+\zeta(2s)))^{\frac{2l-1}{L^2}} \\ \leq\ & \prod_{l=1}^{k} \left(\left((1+\epsilon)(1-\epsilon)^{2s} (l/L)^2 \right)^{1/l} e^{(1-2s)/L} (\frac{9}{2}(2+\zeta(2s)))^{3/L^2} \right)^l . \end{aligned}$$

Solve the equation

$$\frac{9}{2}(2+\zeta(2s)) = \frac{1}{2} e^{\frac{2s-1}{3}L}$$

regarding the main terms, we get a unique solution $s_L \in (1/2, 1)$ when L is large enough. $s_L \to 1/2$ as $L \to \infty$ since $\zeta(2 \cdot \frac{1}{2}) = \zeta(1) = \infty$. Then $\sum_{I_{n_k} \subset B(1/2,1)} |I_{n_k}|^s < \infty$, which forces $\dim_H B(1/2,1) \leq 1/2$. □

Remark. Our Corollary 2.2 sharpens the estimation on $S_{n_k}(x) - S_{n_{k-1}}(x)$ in [5, Proof of Theorem 1.3] for $x \in F(1/2,1)$. In fact, we can do similar estimations for any $x \in F(\gamma, \alpha), \gamma \in \mathbb{R}^+, \alpha \in \mathbb{R}^+, n_k = k^{1/\gamma}$. This enables us to give better estimation on $\sum_{I_{n_k} \subset B(\gamma, N)} |I_{n_k}|^s$, $\gamma \in [1/2, 1)$. By virtue of it, when estimating the upper bound in [5, Proof of Theorem 1.3] for the H-dimension of $F(\gamma, \alpha), \gamma \in (1/2, 1)$, we can simply take $n_k = k^{1/\gamma}$ instead of $k^{1/\gamma}(\log k)^{1/\gamma^2}$.

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