



Functional analysis

Extended Caffarelli–Kohn–Nirenberg inequalities and superweights for L^p -weighted Hardy inequalities



Inégalités de Caffarelli–Kohn–Nirenberg étendues et super-poids des inégalités de Hardy L^p -pondérées

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ABSTRACT

In this paper, we give an extension of the classical Caffarelli–Kohn–Nirenberg inequalities with respect to the range of parameters. We also establish best constants for large families of parameters. Moreover, we also obtain anisotropic versions of these inequalities which can be conveniently formulated in the language of Folland and Stein's homogeneous groups. We also establish sharp Hardy type inequalities in L^p , $1 < p < \infty$, with superweights.

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R É S U M É

Dans cet article, nous donnons une extension des inégalités classiques de Caffarelli–Kohn–Nirenberg relativement à l'étendue du domaine des paramètres. Nous établissons également les meilleures constantes pour les grandes familles de paramètres. De plus, nous obtenons des versions anisotropes de ces inégalités qui peuvent être commodément formulées dans le langage des groupes homogènes de Folland et Stein. Nous établissons aussi des inégalités de type Hardy dans L^p , $1 < p < \infty$, avec des super-poids.

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Le premier objectif de cet article est d'améliorer les inégalités de Caffarelli–Kohn–Nirenberg (CKN) pour ce qui est de l'étendue du domaine des indices dans (1). Une autre avancée vient du remplacement du gradient plein ∇f en (2) par la dérivée radiale $\mathcal{R}f = \frac{\partial f}{\partial r}$. Ces améliorations sont obtenues avec des constantes précises, aussi bien dans le cas isotrope que

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dans celui anisotrope. Nous obtenons également le cas critique de l'inégalité de Caffarelli–Kohn–Nirenberg avec des termes logarithmiques.

Comme les inégalités de Caffarelli–Kohn–Nirenberg, les inégalités classiques de Hardy et leurs extensions font généralement apparaître un poids de la forme $\frac{1}{|x|^m}$. Dans cet article, nous considérons aussi le poids de la forme $\frac{(a+b|x|^\alpha)^\beta}{|x|^m}$, qui permettent des choix différents de α et de β . Si $\alpha = 0$ ou $\beta = 0$, ces poids se réduisent aux poids traditionnels.

1. Introduction

Let us restate the classical Caffarelli–Kohn–Nirenberg (CKN) inequality from the celebrated paper [1].

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $p, q, r, a, b, d, \delta \in \mathbb{R}$ be such that $p, q \geq 1, r > 0, 0 \leq \delta \leq 1$, and*

$$\frac{1}{p} + \frac{a}{n}, \frac{1}{q} + \frac{b}{n}, \frac{1}{r} + \frac{c}{n} > 0 \tag{1}$$

where $c = \delta d + (1 - \delta)b$. Then there exists a positive constant C such that

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq C \| |x|^a |\nabla f| \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta} \tag{2}$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

$$\frac{1}{r} + \frac{c}{n} = \delta \left(\frac{1}{p} + \frac{a-1}{n} \right) + (1-\delta) \left(\frac{1}{q} + \frac{b}{n} \right), \tag{3}$$

$$a-d \geq 0 \quad \text{if } \delta > 0, \tag{4}$$

$$a-d \leq 1 \quad \text{if } \delta > 0 \quad \text{and} \quad \frac{1}{r} + \frac{c}{n} = \frac{1}{p} + \frac{a-1}{n}. \tag{5}$$

The first goal of the present paper is to extend the CKN inequalities with respect to widening the range of indices in (1). Moreover, another improvement will be achieved by replacing the full gradient ∇f in (2) by the radial derivative $\mathcal{R}f = \frac{\partial f}{\partial r}$. It turns out that such generalised versions hold both in the isotropic and anisotropic settings with sharp constants. We also establish the critical case of the CKN inequality with logarithmic terms. These results are presented in Section 2.

Although the obtained results in this paper are new already in the usual setting of \mathbb{R}^n , our techniques apply well also for anisotropic structures. Consequently, it is convenient to work in the setting of homogeneous groups developed by Folland and Stein [3] with an idea of emphasising general results of harmonic analysis depending only of the group and dilation structures. In particular, in this way we obtain results on the anisotropic \mathbb{R}^n , on the Heisenberg group, general stratified groups, graded groups, etc. In the special case of stratified groups (or homogeneous Carnot groups), other formulations using horizontal gradient are possible, and we refer to [7] (see also [5]) and especially to [6] for versions of such results and the discussion of the corresponding literature. Let us very briefly discuss some key notions of homogeneous groups following Folland and Stein [3] as well as a recent treatise [2]. A connected simply connected Lie group \mathbb{G} is called a *homogeneous group* if its Lie algebra \mathfrak{g} is equipped with a family of the following dilations:

$$D_\lambda = \text{Exp}(A \ln \lambda) = \sum_{k=0}^\infty \frac{1}{k!} (\ln(\lambda)A)^k,$$

where A is a diagonalisable positive linear operator on \mathfrak{g} , and every D_λ is a morphism of \mathfrak{g} , that is, $\forall X, Y \in \mathfrak{g}, \lambda > 0, [D_\lambda X, D_\lambda Y] = D_\lambda [X, Y]$, holds. We recall that $Q := \text{Tr } A$ is called the homogeneous dimension of \mathbb{G} .

A homogeneous group is a nilpotent (Lie) group and exponential mapping $\exp_{\mathbb{G}} : \mathfrak{g} \rightarrow \mathbb{G}$ of this group is a global diffeomorphism. Thus, this implies the dilation structure, and this dilation is denoted by $D_\lambda x$ or just by λx , on homogeneous groups. Moreover, every homogeneous group \mathbb{G} has a homogeneous quasi-norm, which will be denoted by $|\cdot|$. Another key property of a homogeneous group is that it allows polarisation. We also use a derivative operator \mathcal{R} on \mathbb{G} with respect to the corresponding quasi-norm.

The classical Hardy inequalities and their sharp extensions, such as the sharp CKN inequalities, usually involve the weights of the form $\frac{1}{|x|^m}$. In this note, we also consider the weights of the form $\frac{(a+b|x|^\alpha)^\beta}{|x|^m}$ allowing for different choices of α and β , which give sharp constants. If $\alpha = 0$ or $\beta = 0$, this reduces to traditional weights. So, we are interested in Section 3 in the situation when $\alpha\beta \neq 0$ and, in fact, we obtain two families of sharp inequalities depending on whether $\alpha\beta > 0$ or $\alpha\beta < 0$. Moreover, $|\cdot|$ in these expressions can be an arbitrary homogeneous quasi-norm and the constants for the obtained inequalities are sharp. The freedom in choosing parameters α, β, a, b, m and a quasi-norm led us to calling these weights the ‘superweights’ in this context.

2. Extended Caffarelli–Kohn–Nirenberg inequalities

In this section, we introduce new KKN-type inequalities in the Euclidean setting of \mathbb{R}^n as well as on homogeneous groups.

Theorem 2.1. *Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a - 1) + b(1 - \delta)$. Then, for any homogeneous quasi-norm $|\cdot|$ on a homogeneous group \mathbb{G} of homogeneous dimension Q we have the following KKN-type inequalities for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$:*

If $Q \neq p(1 - a)$, then

$$\| |x|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{Q - p(1 - a)} \right|^\delta \| |x|^a \mathcal{R} f \|_{L^p(\mathbb{G})}^\delta \| |x|^b f \|_{L^q(\mathbb{G})}^{1-\delta}. \tag{6}$$

If $Q = p(1 - a)$, then

$$\| |x|^c f \|_{L^r(\mathbb{G})} \leq p^\delta \| |x|^a \log |x| \mathcal{R} f \|_{L^p(\mathbb{G})}^\delta \| |x|^b f \|_{L^q(\mathbb{G})}^{1-\delta}, \tag{7}$$

where $\mathcal{R} := \frac{d}{d|x|}$. The constant in the inequality (6) is sharp for $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1 - a) + bq \neq 0$. Moreover, the constants in (6) and (7) are sharp for $\delta = 0$ or $\delta = 1$.

Note that the conditions $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a - 1) + b(1 - \delta)$ imply the condition (3) of Theorem 1.1, and in our case $a - d = 1$. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (7) implies a new type of the KKN inequality for any quasi-norm on \mathbb{R}^n : Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, $n = p(1 - a)$ and $c = \delta(a - 1) + b(1 - \delta)$. Then we have the KKN-type inequality for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any homogeneous quasi-norm $|\cdot|$:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \| |x|^a \log |x| \left(\frac{x}{|x|} \cdot \nabla f \right) \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \tag{8}$$

By the Schwarz inequality with the standard Euclidean distance given by $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, we obtain the Euclidean form of the KKN-type inequality:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \| |x|^a \log |x| \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}, \tag{9}$$

where ∇ is the standard gradient in \mathbb{R}^n . Similarly, we write the inequality (6) in the Abelian case: Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, $n \neq p(1 - a)$ and $c = \delta(a - 1) + b(1 - \delta)$. Then we have a KKN-type inequality for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any homogeneous quasi-norm $|\cdot|$:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n - p(1 - a)} \right|^\delta \| |x|^a \left(\frac{x}{|x|} \cdot \nabla f \right) \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \tag{10}$$

Then, using the Schwarz inequality with the standard Euclidean distance, we obtain the Euclidean form of the KKN-type inequality:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n - p(1 - a)} \right|^\delta \| |x|^a \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \tag{11}$$

Note that if the conditions (1) hold, then the inequality (11) is contained in the family of Caffarelli–Kohn–Nirenberg inequalities in Theorem 1.1. However, already in this case, if we require $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1 - a) + bq \neq 0$, then (11) yields the inequality (2) with sharp constant. Moreover, the constants $\left| \frac{p}{n - p(1 - a)} \right|^\delta$ and p^δ are sharp for $\delta = 0$ or $\delta = 1$. Our conditions $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a - 1) + b(1 - \delta)$ imply the condition (3) of the Theorem 1.1, as well as (4)–(5) which are all necessary for having estimates of this type, at least under the conditions (1).

If the conditions (1) are not satisfied, then the inequality (11) is not covered by Theorem 1.1. So, this gives an extension of Theorem 1.1 with respect to the range of parameters. Let us give an example.

Example 1. Let us take $1 < p = q = r < \infty$, $a = -\frac{n-2p}{p}$, $b = -\frac{n}{p}$ and $c = -\frac{n-\delta p}{p}$. Then, by (11), for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have the inequality

$$\left\| \frac{f}{|x|^{\frac{n-\delta p}{p}}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x|^{\frac{n-2p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^\delta \left\| \frac{f}{|x|^{\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta}, \quad 1 < p < \infty, 0 \leq \delta \leq 1, \tag{12}$$

with sharp constant, where ∇ is the standard gradient in \mathbb{R}^n . Since we have $\frac{1}{q} + \frac{b}{n} = \frac{1}{p} + \frac{1}{n} \left(-\frac{n}{p}\right) = 0$, we see that (1) fails, so that the inequality (12) is not covered by Theorem 1.1.

3. L^p -Hardy inequalities with super weights

We now give versions of Hardy inequalities with more general weights, that we call superweights, due to their rather general form and the freedom in choosing the parameters.

Theorem 3.1. *Let $a, b > 0$ and $1 < p < \infty$. For any homogeneous quasi-norm $|\cdot|$ on a homogeneous group \mathbb{G} of homogeneous dimension Q .*

(i) *If $\alpha\beta > 0$ and $pm \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have*

$$\frac{Q - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}f \right\|_{L^p(\mathbb{G})}; \tag{13}$$

if $Q \neq pm + p$, then the constant $\frac{Q - pm - p}{p}$ is sharp.

(ii) *If $\alpha\beta < 0$ and $pm - \alpha\beta \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have*

$$\frac{Q - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}f \right\|_{L^p(\mathbb{G})}; \tag{14}$$

if $Q \neq pm + p - \alpha\beta$, then the constant $\frac{Q - pm + \alpha\beta - p}{p}$ is sharp.

As it is mentioned in the introduction, the weights in the inequalities (13) and (14) are called superweights, since the constants both in (13) and in (14) are sharp for an arbitrary homogeneous quasi-norm $|\cdot|$ of \mathbb{G} and a wide range of choices of the allowed parameters α, β, a, b and m . Directly from the inequalities (13) and (14), choosing different α, β, a, b, m and Q , one can obtain a number of Hardy-type inequalities that have various consequences and applications. For instance, in the Abelian (isotropic or anisotropic) case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, so for any quasi-norm $|\cdot|$ on \mathbb{R}^n , all $a, b > 0$ and $1 < p < \infty$ these imply new inequalities. Thus, if $\alpha\beta > 0$ and $pm \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \frac{df}{d|x|} \right\|_{L^p(\mathbb{R}^n)} \tag{15}$$

with the constant being sharp for $n \neq pm + p$.

If $\alpha\beta < 0$ and $pm - \alpha\beta \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \frac{df}{d|x|} \right\|_{L^p(\mathbb{R}^n)} \tag{16}$$

with the sharp constant for $n \neq pm + p - \alpha\beta$. In the case of the standard Euclidean distance $|x| = \sqrt{x_1^2 + \dots + x_n^2}$, by using the Schwartz inequality from the inequalities (15) and (16), we obtain that if $\alpha\beta > 0$ and $pm \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$\frac{n - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \nabla f \right\|_{L^p(\mathbb{R}^n)} \tag{17}$$

with the constant sharp for $n \neq pm + p$.

If $\alpha\beta < 0$ and $pm - \alpha\beta \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \nabla f \right\|_{L^p(\mathbb{R}^n)} \tag{18}$$

with the sharp constant for $n \neq pm + p - \alpha\beta$. The L^2 -version, that is, for $p = 2$ the inequalities (17) and (18) were obtained in [4]. We also shall note that these inequalities have interesting applications in the theory of ODEs (see [4, Theorem 2.1]).

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