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Harmonic analysis

On a discrete bilinear singular operator

Sur un opérateur bilinéaire discret singulier

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ABSTRACT

We prove that for a large class of functions P and Q, the discrete bilinear operator $T_{P,Q}(f,g)(n) = \sum_{m\in\mathbb{Z}\setminus\{0\}} f(n-P(m))g(n-Q(m))\frac{1}{m}$ is bounded from $l^2 \times l^2$ into $l^{1+\epsilon,\infty}$ for any $\epsilon \in (0,1]$.

RÉSUMÉ

Nous montrons, que pour une grande classe de fonctions P et Q, l'opérateur bilinéaire discret $T_{P,Q}(f,g)(n) = \sum_{m\in\mathbb{Z}\setminus\{0\}} f(n-P(m))g(n-Q(m))\frac{1}{m}$ est borné de $l^2 \times l^2$ dans $l^{1+\epsilon,\infty}$, pour tout $\epsilon \in (0,1]$.

1. Introduction

The Hilbert transform (HT for short) is defined by

$$H(f)(x) = \int f(x-t)\frac{\mathrm{d}t}{t}, \quad f \in \mathcal{S}(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R}^n)$, $n \in \mathbb{N}$, is the Schwartz space on $\mathbb{R}^n$. It was proved in 1928 ([21]) that HT is bounded on L^p for $p \in (1,\infty)$. An interesting generalization of HT is the so-called HT along curves:

$$H_C(f)(x) = \int f(x-\gamma(t))\frac{\mathrm{d}t}{t}, f \in \mathcal{S}(\mathbb{R}^n).$$

Here $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a well-behaved curve. The L^p boundedness of H_C has been obtained for various curves. See [22] for a comprehensive survey and [2] for a generalization of H_C to the non-translation-invariant setting. When γ is a polynomial with integer coefficients, there is a discrete version of H_C defined by

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$$H_C^{\text{dis}}(f)(n) = \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n - \gamma(m)) \frac{1}{m}, \ f \in D(\mathbb{Z}^n),$$

where $D(\mathbb{Z}^n)$ is the space of compactly supported complex-valued functions defined on $\mathbb{Z}^n$. On the one hand, H_C^{dis} has many applications in ergodic theory ([6,16–18,20]), but on the other hand this discrete operator is more subtle to handle than its continuous counterpart H_C, as many number theoretical tools are involved. H_C^{dis} was at first proved to be bounded on l^p for $p \in (\frac{3}{2}, 3)$ ([23]). This restricted range was extended to the full range $(1, \infty)$ a long time later ([7,15]).

Another direction of generalizing HT is to consider its bilinear analogue, which is significantly more difficult to analyze since Plancherel Theorem is unavailable in the bilinear setting. The bilinear Hilbert transform (BHT for short) can be defined as

$$B(f, g)(x) = \int f(x-t) g(x+t) \frac{\mathrm{d}t}{t}, \ \ f, g \in \mathcal{S}(\mathbb{R}).$$

It was about 70 years after the first proof of the boundedness of HT that Lacey and Thiele ([9,10]) obtained the L^p estimates for BHT. Very recently, L^p estimates for BHT along curves

$$B_C(f, g)(x) = \int f(x-t) g(x - \gamma(t)) \frac{\mathrm{d}t}{t}, \ \ f, g \in \mathcal{S}(\mathbb{R}),$$

were also established when γ is a polynomial ([14]). Note that B_C is a natural bilinear version of H_C.

Following the development of the linear case, in this paper we consider the discrete version of B_C, that is,

$$B_C^{\text{dis}}(f, g)(n) = \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n-m) g(n - P(m)) \frac{1}{m}, \ \ f, g \in D(\mathbb{Z}),$$

where P is a polynomial with integer coefficients. This operator can also be viewed as a bilinear analogue of H_C^{dis}. As H_C^{dis} is harder to handle than H_C, it is reasonable to expect that proving the boundedness of B_C^{dis} should be more difficult than that of B_C. As a starting point of the long journey of investigation on B_C^{dis}, in this article we show the $l^2 \times l^2 \to l^{1+\epsilon, \infty}$ boundedness of B_C^{dis} (Theorem 1.1).

We will study an operator that is more general than B_C^{dis} (see (1.1)). Given two functions P and Q that map $\mathbb{Z}$ into $\mathbb{Z}$, define

$$A^{P,Q} := \left\{ (m_1, m_2) \in (\mathbb{Z} \setminus \{0\})^2 : P(m_1) - Q(m_1) = P(m_2) - Q(m_2) \right\}.$$

We say that the pair of functions (P, Q) satisfies **condition** $(\star)$ if there are constants D_1 and D_2 such that $\frac{|m_1|}{|m_2|} \le D_1$ for all $(m_1, m_2) \in A^{P,Q}$ and for each $m_1 \in \mathbb{Z}$, there are at most D_2 pairs (m_1, m_2) in the set $A^{P,Q}$.

Theorem 1.1. *Given two functions P and Q that map $\mathbb{Z}$ into $\mathbb{Z}$, let*

$$T_{P,Q}(f, g)(n) := \sum_{m \in \mathbb{Z} \setminus \{0\}} f(n - P(m)) g(n - Q(m)) \frac{1}{m}, \ \ f, g \in D(Z). \tag{1.1}$$

Assume that (P, Q) satisfies condition $(\star)$. Then for any $\epsilon \in (0, 1]$, there is a constant C_ϵ depending only on ϵ, D_1 and D_2 such that

$$\|T_{P,Q}(f, g)\|_{l^{1+\epsilon,\infty}} \le C_\epsilon \|f\|_{l^2} \|g\|_{l^2}. \tag{1.2}$$

Remarks. (1). Condition $(\star)$ is mild. A pair of polynomials with integer coefficients (P, Q) satisfies condition $(\star)$ as long as $P - Q$ is not constant. Note that D_1 depends on the coefficients of P and Q, so does C_ϵ in the theorem. It is natural to expect that this dependence can be removed, as uniform estimates exist for related operators ([3,4,11,12,14,23,24]). We shall not pursuit this here.

(2). We conjecture that at least for some special pairs of P and Q (for example, $P(t) = t$ and $Q(t) = t^2$), $T_{P,Q}$ is bounded from $l^p \times l^q$ into l^r, where $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This problem is very difficult and currently out of reach.

(3). A useful operator related with $T_{P,Q}$ is the corresponding maximal operator $T^*_{P,Q}(f, g)(n) = \sup_{M \in [1,\infty)} |\frac{1}{M} \times \sum_{m=1}^{M} f(n - P(m)) g(n - Q(m))|$, which is at first proved to be bounded from $l^2 \times l^2$ to l^r when $r > 1$ ([5]). By using Hölder inequality and boundedness of the corresponding discrete linear maximal function $f \to \sup_{M \in [1,\infty)]} \frac{1}{M} \sum_{m=1}^{M} |f(n - P(m))|$ (see, for example, [1,8,19,26]), we can prove that $T^*_{P,Q}$ is bounded from $l^p \times l^q$ into l^r, whenever $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $r > 1$ (see p. 75 in [25] for a similar trick). Whether the restriction $r > 1$ can be dropped is still unknown.

(4). See [13] for a discussion about an ergodic analogue of $T_{P,Q}$.

The rest of the paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

We will use $A \lesssim B$ to denote the statement that $A \le CB$ for some positive constant C. When the implied constant C depends on r, we write $A \lesssim_r B$. All the constants may depend on D_1 and D_2 (appeared in the definition of condition ($\star$)), but this dependence will be suppressed since D_1 and D_2 are often fixed in applications. $A \simeq B$ is short for $A \lesssim B$ and $B \lesssim A$. For any set of integers E, $|E|$ and χ_E will be used to denote the counting measure and the indicator function of E, receptively.

Let P and Q be a pair of functions satisfying condition ($\star$). For notational convenience, we will simply write T for $T_{P,Q}$ and $r := 1+\epsilon$. For any $\lambda > 0$ and $f, g \in D(\mathbb{Z})$, define the level set $E_\lambda = \{n \in \mathbb{Z} : |T(f,g)(n)| > \lambda\}$. Our goal is to prove the following level set estimate

$$|E_\lambda| \lesssim_r \frac{1}{\lambda^r}, \text{ whenever } \|f\|_{l^2} = \|g\|_{l^2} = 1. \tag{2.3}$$

We first write $T = \sum_{m\in\mathbb{Z}\setminus\{0\}} f(n-P(m))g(n-Q(m))\frac{1}{m}$ as a bilinear multiplier operator. Recall the Fourier transform for any $f \in D(\mathbb{Z})$ is defined by $\hat{f}(\xi) := \sum_{m\in\mathbb{Z}} f(m)\mathrm{e}^{-2\pi\mathrm{i}\xi m}$. Hence

$$T(f,g)(n) = \int_{\mathbf{T}}\int_{\mathbf{T}} \hat{f}(\xi)\hat{g}(\eta)\mathrm{e}^{2\pi\mathrm{i}(\xi+\eta)n}\sigma(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

where $\mathbf{T}$ is the unit circle and σ is the periodic multiplier (a.k.a. symbol) given by

$$\sigma(\xi,\eta) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{m}\mathrm{e}^{-2\pi\mathrm{i}(P(m)\xi+Q(m)\eta)}.$$

Then we decompose dyadically the symbol σ as follows. Pick an odd function $\rho \in \mathcal{S}(\mathbb{R})$ supported in the set $\{x : |x| \in (\frac{1}{2}, 2)\}$ with the property that

$$\frac{1}{x} = \sum_{j=0}^{\infty} \frac{1}{2^j}\rho\left(\frac{x}{2^j}\right) \text{ for any } x \in \mathbb{R} \text{ with } |x| \ge 1.$$

So the symbol σ can be written as $\sigma(\xi,\eta) = \sum_{j=0}^{\infty}\sigma_j(\xi,\eta)$, where

$$\sigma_j(\xi,\eta) := \frac{1}{2^j}\sum_{m\in\mathbb{Z}} \rho\left(\frac{m}{2^j}\right)\mathrm{e}^{-2\pi\mathrm{i}(P(m)\xi+Q(m)\eta)}.$$

Correspondingly $T = \sum_{j=0}^{\infty} T_j$, where

$$\begin{aligned} T_j(f,g)(n) &= \int_{\mathbf{T}}\int_{\mathbf{T}} \hat{f}(\xi)\hat{g}(\eta)\mathrm{e}^{2\pi\mathrm{i}(\xi+\eta)n}\sigma_j(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{1}{2^j}\sum_{m\in\mathbb{Z}} \rho\left(\frac{m}{2^j}\right) f(n-P(m))g(n-Q(m)). \end{aligned}$$

By the support of ρ and Hölder inequality, it is easy to see $\|T_j(f,g)\|_{l^1} \lesssim \|f\|_{l^2}\|g\|_{l^2}$. So we have the following level set estimate for each T_j.

Lemma 2.1. *For any $f, g \in D(\mathbb{Z})$ with l^2-norm 1, $j \in \mathbb{N}$, and $\lambda > 0$, we have*

$$|\{n \in \mathbb{Z} : |T_j(f,g)(n)| > \lambda\}| \lesssim \frac{1}{\lambda}.$$

This lemma says that each single T_j is under good (and uniform) control. The difficulty is how to get the desired estimates for the sum of T_j's. In the following, we will apply the idea of the TT^* method.

Define an auxiliary function $h(n) = \frac{\overline{T(f,g)(n)}}{T(f,g)(n)}\chi_{E_\lambda}(n)$. It is easy to verify that

$$\lambda^2|E_\lambda|^2 \le \left(\sum_{n\in\mathbb{Z}} T(f,g)(n)h(n)\right)^2. \tag{2.4}$$

By Fubini theorem and the definition of Fourier transform,

$$\sum_{n\in\mathbb{Z}} T(f,g)(n)h(n) = \int_{\mathbf{T}}\int_{\mathbf{T}} \hat{f}(\xi)\hat{g}(\eta)\sigma(\xi,\eta)\hat{h}(-(\xi+\eta))\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$= \int_{\mathbf{T}}\int_{\mathbf{T}} \hat{f}(\xi-\eta)\hat{g}(\eta)\sigma(\xi-\eta,\eta)\hat{h}(-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Apply the Cauchy–Schwarz inequality and the Plancherel Theorem, and we get

$$\left(\sum_{n\in\mathbb{Z}} T(f,g)(n)h(n)\right)^2 \le B|E_\lambda|, \tag{2.5}$$

where

$$B := \sup_{\xi\in\mathbf{T}} \int_{\mathbf{T}} |\sigma(\xi-\eta,\eta)|^2 \mathrm{d}\eta.$$

Combining (2.4) and (2.5), we see that $|E_\lambda| \le \frac{B}{\lambda^2}$. Hence, to prove (2.3), it suffices to obtain the estimate

$$B \lesssim_r \lambda^{2-r}. \tag{2.6}$$

To control B, we make use of the dyadic decomposition of σ, aiming for some cancellations. For any $\xi\in\mathbf{T}$,

$$\begin{aligned}\int_{\mathbf{T}} |\sigma(\xi-\eta,\eta)|^2\,\mathrm{d}\eta &= \int_{\mathbf{T}} \left|\sum_{j=0}^{\infty}\sigma_j(\xi-\eta,\eta)\right|^2 \mathrm{d}\eta \\ &\le \sum_{j_1,j_2=0}^{\infty} \frac{1}{2^{j_1}}\frac{1}{2^{j_2}} \sum_{m_1,m_2\in\mathbb{Z}} \left|\rho\left(\frac{m_1}{2^{j_1}}\right)\rho\left(\frac{m_2}{2^{j_2}}\right)\right| \chi_{A^{P,Q}}(m_1,m_2).\end{aligned} \tag{2.7}$$

By condition ($\star$), $\frac{|m_1|}{|m_2|} \le D_1$ for all $(m_1,m_2)\in A^{P,Q}$. The support of ρ forces $|m_1|\simeq 2^{j_1}$ and $|m_2|\simeq 2^{j_2}$. These facts show that $|j_1-j_2|\lesssim 1$. Also note that for each m_1, there are only bounded number of m_2's such that $(m_1,m_2)\in A^{P,Q}$. Thus (2.7) implies

$$B = \sup_{\xi\in\mathbf{T}} \int_{\mathbf{T}} |\sigma(\xi-\eta,\eta)|^2\,\mathrm{d}\eta \lesssim \sum_{j=0}^{\infty}\frac{1}{2^j}.$$

When $\lambda\ge 1$, as $r\in(1,2]$, trivially $B\lesssim\lambda^{2-r}$ and we are done. Let $M=[(2-r)\log_2\frac{1}{\lambda}]+1$, where $[x]$ denotes the integer part of x. In the case $\lambda<1$, since $\sum_{j=M+1}^{\infty}\frac{1}{2^j}\lesssim_r\lambda^{2-r}$, the above method still gives the desired estimate for $\sum_{j=M+1}^{\infty}T_j$, the operator associated with the symbol $\sum_{j=M+1}^{\infty}\sigma_j$. It remains to control the level set of the operator $\sum_{j=0}^{M}T_j$ for $\lambda<1$. Applying Lemma 2.1, we have

$$\left|\left\{n\in\mathbb{Z} : \left|\sum_{j=0}^{M} T_j(f,g)(n)\right| > \lambda\right\}\right| \lesssim \frac{M^2}{\lambda} \lesssim_r \frac{1}{\lambda^r},$$

where we used the facts $r>1$ and $\lambda<1$ in the last inequality. This finishes the proof of Theorem 1.1.

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