C. R. Acad. Sci. Paris, Ser. I 355 (2017) 310–317

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations

Decay of solutions to a new Hall–MHD system in $\mathbb{R}^3$

Décroissance des solutions d'un nouveau système d'équations magnétohydrodynamiques de Hall dans $\mathbb{R}^3$

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A R T I C L E   I N F O

Article history:
Received 30 August 2016
Accepted after revision 26 January 2017
Available online 11 February 2017

Presented by the Editorial Board

A B S T R A C T

This paper discusses the large-time behavior of solutions for a new Hall–MHD system in $\mathbb{R}^3$. Using the Fourier splitting method, we establish the upper bound of the time-decay rate in $L^2(\mathbb{R}^3)$ for weak solutions.

R É S U M É

Cette Note traite du comportement à long terme des solutions d'un nouveau système d'équations magnétohydrodynamiques de Hall dans $\mathbb{R}^3$. Utilisant la méthode de décomposition de Fourier, nous donnons une borne supérieure du taux de décroissance en temps dans $L^2(\mathbb{R}^3)$ pour les solutions faibles.

1. Introduction

In this paper, we study the following new Hall–MHD system [5,6,11]:

$$\mathrm{div}u = \mathrm{div}b = 0, \tag{1}$$

$$\partial_t u + u\cdot\nabla u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b\cdot\nabla b, \tag{2}$$

$$\partial_t b - \left(\frac{\delta_e}{L_0}\right)^2 \partial_t \Delta b + u\cdot\nabla b - b\cdot\nabla u - \Delta b = \frac{\delta_i}{L_0}\mathrm{rot}(b\times \mathrm{rot}b) - \left(\frac{\delta_e}{L_0}\right)^2 \mathrm{rot}((u\cdot\nabla)\mathrm{rot}b), \tag{3}$$

$$(u,b)(\cdot,0) = (u_0,b_0)(\cdot) \quad \text{in } \mathbb{R}^3. \tag{4}$$

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http://dx.doi.org/10.1016/j.crma.2017.01.019

Here $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, π is the pressure and b is the magnetic field. In addition, L_0, δ_e, δ_i and ρ denote the normalizing length limit, the electron inertia, the ion inertia and the fluid density, respectively. For simplicity, we set $L_0 = \delta_e = \delta_i = \rho = 1$.

When $\delta_e = 0$, system (1)–(4) reduces to the standard Hall–MHD system. There is much literature concerned with this system; for more recent results, we refer the reader to [3,4,8,17–19] and the references therein.

In [7], Fan, Ahmad, Hayat and Zhou studied the global existence of weak solutions for the new Hall–MHD system in $\mathbb{R}^3$. They point out that if $u_0 \in L^2$, $b_0 \in H^1$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, then there exists a weak solution (u, b) for system (1)–(4), which satisfies the energy inequality

$$\int_{\mathbb{R}^3} (|u|^2 + |b|^2 + |\nabla b|^2)\mathrm{d}x + 2\int_0^T \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla b|^2)\mathrm{d}x\mathrm{d}t \le \int_{\mathbb{R}^3} (|u_0|^2 + |b_0|^2 + |\nabla b_0|^2)\mathrm{d}x.$$

In addition, the authors also established some blow-up criteria.

The goal of this paper is to investigate the time-decay rate of solutions for system (1)–(4). By using the Fourier splitting method and the properties of decay character r^*, we prove the upper bound of the decay rate in $L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ for solutions to system (1)–(4). The first definition of the decay character r^* can be traced back to Bjorland and Schonbek [1]. The authors introduced the idea of the decay indicator $P_r^s(u_0)$ and decay character $r^* = r^*(u_0)$ of a function $u_0 \in L^2(\mathbb{R}^3)$ to study the decay rates of the heat equation. In [2,12], the authors considered the sharp decay estimates for solutions to the heat equation

$$\frac{\partial w}{\partial t} + \Delta w = 0, \quad w(\cdot, 0) = u_0, \tag{5}$$

in terms of $r^* = r^*(u_0)$. Later, Niche [13] characterized the decay of

$$\partial_t (v - \Delta v) - \Delta v = 0, \quad v(\cdot, 0) = b_0, \tag{6}$$

and studied the upper bound of decay rate for Navier–Stokes–Voigt equations.

Now, we give the definitions of the decay indicator $P_r^s(u_0)$ and of the decay character r^*.

Definition 1 *([2,12,13]).* Suppose that $v_0 \in L^2(\mathbb{R}^n)$, $\Lambda = (-\Delta)^{\frac{1}{2}}$ and that

$$P_r^s(v_0) = \lim_{\rho \to 0} \rho^{-2r-n} \int_{B(\rho)} |\xi|^{2s} |\widehat{v_0}(\xi)|^2 \mathrm{d}\xi, \quad s \ge 0,$$

exists, for $r \in (-\frac{n}{2} + s, \infty)$, and denote by $B(\rho)$ the ball at the origin with radius ρ. Then, $P_r^s(v_0)$ is the s-decay indicator corresponding to $\Lambda^s v_0$.

Definition 2 *([12,13]).* The decay character of $\Lambda^s v_0$, denoted by $r_s^* = r_s^*(v_0)$ is the unique $r \in (-\frac{n}{2} + s, \infty)$ such that $0 < P_r^s(v_0) < \infty$, provided that this number exists. If such $P_r^s(v_0)$ does not exist, set $r_s^* = -\frac{n}{2} + s$, when $P_r^s(v_0) = \infty$ for all $r \in (-\frac{n}{2} + s, \infty)$ or $r_s^* = \infty$, if $P_r^s(v_0) = 0$ for all $r \in (-\frac{n}{2} + s, \infty)$.

The following lemma describes the L^2 decay characterization of solutions to (5) and (6) in terms of the decay character r^*.

Lemma 3 ([2,13]). *Assume that $u_0 \in L^2(\mathbb{R}^3)$, $b_0 \in H^1(\mathbb{R}^3)$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ is the decay character. Suppose that w is a solution to (5) and v is a solution to (6). Then,*

$$C_1(1+t)^{-(r^*+\frac{3}{2})} \le \|w(t)\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \le C_2(1+t)^{-(r^*+\frac{3}{2})}, \quad \forall t > 0, \tag{7}$$

where C_1 and C_2 are two positive constants.

The following theorem is the main result and will be proved in Section 2.

Theorem 4. *Assume that $(u_0, b_0) \in L^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, $\mathrm{div} u_0 = \mathrm{div} b_0 = 0$ and $r^* = r^*(u_0) = r^*(b_0) \in (-\frac{3}{2}, +\infty)$ is the decay character. Let (u, b) be the solution to system (1)–(4) with initial value (u_0, b_0). Then*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \le C(1+t)^{-\min\{\frac{3}{2}+r^*, \frac{5}{2}\}}, \quad \forall t > 0,$$

where the constant C depends essentially on $\|u_0\|_{L^2}$, $\|b_0\|_{L^2}$ and $\|\nabla b_0\|_{L^2}$.

Remark 5. The Fourier splitting method was introduced by Schonbek in the 1980s (see [14,15]), then it becomes a standard way (also a powerful tool) to establish the decay rate of solutions. In 2007, Zhou [20] introduced a new method (some people called Zhou's method) to handle decay rate problems. One can refer to [9,10,21] for details and developments.

Remark 6. In [4], Chae and Schonbek established the temporal decay estimates for weak solutions to the classical Hall–MHD system with the initial data in $L^1(\mathbb{R}^3)\bigcap L^2(\mathbb{R}^3)$. Only recently, Weng [18] generalized Chae and Schonbek's results to cover more classes of initial data. Comparing with [4,18], we find out that the last term of (3) does not affect the time asymptotic behavior, and the $L^2\times H^1$-decay rate behave like it of the Hall–MHD system.

Throughout this paper, we use C to denote the generic constant that can take different values in different places. In addition, $L^p(\mathbb{R}^3)$ $(1\le p\le\infty)$ represents the 3D vector Lebesgue space with norm

$$\|u\|_{L^p}=\left(\int_{\mathbb{R}^3}|u(x,t)|^p\mathrm{d}x\right)^{\frac{1}{p}},\quad \|u\|_{L^\infty}=\operatorname*{ess\,sup}_{x\in\mathbb{R}^3}|u(x,t)|.$$

2. Proof of Theorem 4

In this section, we consider the upper bound of the time-decay rate for the solutions to system (1)–(4).

Lemma 7. *Let $(u_0,b_0)\in L^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ with $\mathrm{div}u_0=\mathrm{div}b_0=0$. Suppose that (u,b) is the solution to system (1)–(4) with initial values (u_0,b_0). Then*

$$|\widehat{u}(\xi,t)|^2\le C\left[\mathrm{e}^{-2|\xi|^2t}|\widehat{u_0}(\xi)|+|\xi|^2\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\right],\tag{8}$$

and

$$|\widehat{b}(\xi,t)|^2\le C\left[\mathrm{e}^{-\frac{2|\xi|^2t}{1+|\xi|^2}}|\widehat{b_0}(\xi)|^2+(|\xi|^2+|\xi|^4+|\xi|^6)\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\right].\tag{9}$$

Proof. Taking the Fourier transform for (2), we derive that

$$\partial_t\widehat{u}(\xi,t)+|\xi|^2\widehat{u}(\xi,t)=H(\xi,t),\tag{10}$$

where

$$H(\xi,t)=-\widehat{u\cdot\nabla u}(\xi,t)-\widehat{\nabla\pi}(\xi,t)-\widehat{(\mathrm{rot}b)\times b}(\xi,t).\tag{11}$$

Integrating in time from 0 to t, we get

$$\partial_t\widehat{u}(\xi,t)=\mathrm{e}^{-|\xi|^2t}\widehat{u_0}(\xi)+\int_0^t\mathrm{e}^{-|\xi|^2(t-s)}H(\xi,s)\mathrm{d}s.\tag{12}$$

Therefore,

$$|\widehat{u}(\xi,t)|\le|\mathrm{e}^{-|\xi|^2t}\widehat{u_0}(\xi)|+\int_0^t\mathrm{e}^{-|\xi|^2(t-s)}|H(\xi,s)|\mathrm{d}s.\tag{13}$$

We next estimate the terms in $H(\xi,s)$. Since $\mathrm{div}u=\mathrm{div}b=0$, applying the divergence operator to the first set of the system gives

$$-\Delta\pi=\sum_{k,j=1}^{3}\frac{\partial^2}{\partial x_k\partial x_j}(u_ku_j-b_kb_j),$$

which means

$$\widehat{\pi}(\xi,t) = -\frac{1}{|\xi|^2}\sum_{k,j=1}^{3}\xi_k\xi_j(\widehat{u_k u_j} - \widehat{b_k b_j}).$$

Note that the Fourier transform is a bounded map from L^1 into L^∞. It follows that

$$\begin{aligned}|\nabla\widehat{\pi}(\xi,t)| &\le \sum_{i,j=1}^{3}\frac{|\xi_i\xi_j|}{|\xi|}(|\widehat{u_i u_j}(\xi,t)| + |\widehat{b_i b_j}(\xi,t)|)\\ &\le C|\xi|(\|u(t)u(t)\|_{L^1} + \|b(t)b(t)\|_{L^1})\\ &\le C|\xi|(\|u\|_{L^2}^2 + \|b\|_{L^2}^2).\end{aligned}$$

Similarly, for the other terms, we have by using the divergence free condition

$$|\widehat{u\cdot\nabla u}(\xi,t)| \le \sum_{i=1}^{3}|\xi||\widehat{u_i u}(\xi,t)| \le C|\xi|\|u\|_{L^2}^2,$$

and

$$|(\widehat{\mathrm{rot} b)\times b}(\xi,t)| \le \sum_{i=1}^{3}|\xi||\widehat{b_i b}(\xi,t)| \le C|\xi|\|b\|_{L^2}^2.$$

Summing up, we have

$$|H(\xi,t)| \le C|\xi|(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2). \tag{14}$$

By (13) and (14), we obtain (8). Taking the Fourier transform for (3), we derive that

$$\partial_t[(1+|\xi|^2)\widehat{b}(\xi,t)] + |\xi|^2\widehat{b}(\xi,t) = G(\xi,t), \tag{15}$$

where

$$G(\xi,t) = \widehat{b\cdot\nabla u}(\xi,t) - \widehat{u\cdot\nabla b}(\xi,t) + \mathrm{rot}(\widehat{b\times\mathrm{rot} b})(\xi,t) - \mathrm{rot}(\widehat{(u\cdot\nabla)\mathrm{rot} b})(\xi,t). \tag{16}$$

Integrating in time from 0 to t, we deduce that

$$|\widehat{b}(\xi,t)| \le \mathrm{e}^{-\frac{|\xi|^2 t}{1+|\xi|^2}}|\widehat{b_0}(\xi)| + \int_0^t \mathrm{e}^{-\frac{|\xi|^2}{1+|\xi|^2}(t-s)}|G(\xi,s)|\mathrm{d}s. \tag{17}$$

Note that

$$\begin{aligned}|\widehat{b\cdot\nabla u}(\xi,t)| + |\widehat{u\cdot\nabla b}(\xi,t)| &\le \sum_{i=1}^{3}|\xi|(|\widehat{u_i b}(\xi,t)| + |\widehat{b_i u}(\xi,t)|)\\ &\le C|\xi|(\|u(t)b(t)\|_{L^1} + \|b(t)u(t)\|_{L^1})\\ &\le C|\xi|\|u\|_{L^2}\|b\|_{L^2}.\end{aligned}$$

We also have

$$|\mathrm{rot}(\widehat{b\times\mathrm{rot} b})(\xi,t)| \le |\xi\times\sum_{i=1}^{3}\xi_i\widehat{b_i b}(\xi,t)| \le C|\xi|^2\|b\|_{L^2}^2,$$

and

$$|\mathrm{rot}(\widehat{(u\cdot\nabla)\mathrm{rot} b})(\xi,t)| \le |\xi\times\sum_{i=1}^{3}\xi_i\widehat{u(\xi\times b_i)}(\xi,t)| \le C|\xi|^3\|u\|_{L^2}\|b\|_{L^2}.$$

Summing up, we have

$$\begin{aligned}G(\xi,t) \le& C(|\xi|\|u\|_{L^2}\|b\|_{L^2} + |\xi|^2\|b\|_{L^2}^2 + |\xi|^3\|u\|_{L^2}\|b\|_{L^2})\\ \le& C(|\xi| + |\xi|^2 + |\xi|^3)(\|u\|_{L^2}^2 + \|b\|_{L^2}^2).\end{aligned} \tag{18}$$

Combining (17) and (18) together, we obtain (9). This completes the proof. □

Proof of Theorem 4. Testing (2) by u, using (1), we infer that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = \int_{\mathbb{R}^3} (b\cdot\nabla)b\cdot u\mathrm{d}x. \tag{19}$$

Testing (3) by b, using (1), we derive that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\nabla b\|_{L^2}^2 = \int_{\mathbb{R}^3} (b\cdot\nabla)u\cdot b\mathrm{d}x. \tag{20}$$

Define a continuous trilinear form $\mathrm{b}(\cdot,\cdot,\cdot)$ by

$$\mathrm{b}(u,v,w) = \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} u_i \frac{\partial v_j}{\partial x_i} w_j \mathrm{d}x, \quad u,v,w\in H^1(\mathbb{R}^3).$$

We have (see [16])

$$\mathrm{b}(u,v,w) = -\mathrm{b}(u,w,v), \;\; \mathrm{b}(u,v,v) = 0.$$

Hence

$$\int_{\mathbb{R}^3} (b\cdot\nabla)b\cdot u\mathrm{d}x + \int_{\mathbb{R}^3} (b\cdot\nabla)u\cdot b\mathrm{d}x = 0. \tag{21}$$

Combining (19), (20) and (21) together gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) = 0. \tag{22}$$

Then, applying the Plancherel's theorem to (22), we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\\ &+2\int_{\mathbb{R}^3}|\xi|^2(|\hat{u}(\xi,t)|^2 + |\hat{b}(\xi,t)|^2)\mathrm{d}\xi \le 0, \quad \forall t>0.\end{aligned} \tag{23}$$

It then follows from (23) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\\ &+2\int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left(|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi \le 0, \quad \forall t>0.\end{aligned} \tag{24}$$

Set

$$B(t) := \left\{\xi\in\mathbb{R}^3 \,\middle|\, |\xi|^2 \le \frac{g'(t)}{2g(t)-g'(t)}\right\}, \quad B^c(t) := \mathbb{R}^3\setminus B(t),$$

where $g(t)$ is a differentiable function of t satisfying

$$g(0)=1, \;\; g'(t)>0 \;\; \text{and} \;\; 2g(t)>g'(t), \;\; \forall t>0. \tag{25}$$

Multiplying (24) by $g(t)$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\right)\\ &\quad+2g(t)\int_{\mathbb{R}^3}\frac{|\xi|^2}{1+|\xi|^2}\left(|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi\\ &\le g'(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2 + (1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi.\end{aligned} \tag{26}$$

On the basic of the definitions of $B(t)$ and $B^c(t)$, we immediately get $\frac{2g(t)|\xi|^2}{1+|\xi|^2} \geq g'(t),\ \ \forall \xi \in B^c(t)$. Thus,

$$\begin{aligned}
&2g(t)\int_{\mathbb{R}^3} \frac{|\xi|^2}{1+|\xi|^2}\left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi \\
&\geq 2g(t)\int_{B^c(t)} \frac{|\xi|^2}{1+|\xi|^2}\left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi \\
&\geq g'(t)\int_{B^c(t)} \left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi \\
&= g'(t)\int_{\mathbb{R}^3} \left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi \\
&\quad - g'(t)\int_{B(t)} \left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi.
\end{aligned} \tag{27}$$

Combining (26) and (27) together gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi\right) \\
&\leq g'(t)\int_{B(t)} \left(|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2\right)\mathrm{d}\xi.
\end{aligned} \tag{28}$$

By Young's inequality, we have

$$2|\xi|^4 \leq |\xi|^2+|\xi|^6, \quad 2|\xi|^6 \leq |\xi|^4+|\xi|^8.$$

It then follows from the above two inequalities that

$$|\xi|^4+|\xi|^6 \leq |\xi|^2+|\xi|^8.$$

Using the results of Lemma 7 and the above inequality, we derive that

$$\begin{aligned}
&g(t)\int_{\mathbb{R}^3}[|\hat{u}(\xi,t)|^2+(1+|\xi|^2)|\hat{b}(\xi,t)|^2]\mathrm{d}\xi \\
&\leq C + C\int_0^t g'(t)\int_{B(t)} \mathrm{e}^{-2|\xi|^2 t}|\widehat{u_0}(\xi)|^2\mathrm{d}\xi\,\mathrm{d}t \\
&\quad + C\int_0^t g'(t)\int_{B(t)} (1+|\xi|^2)\mathrm{e}^{-\frac{2|\xi|^2 t}{1+|\xi|^2}}|\widehat{b_0}(\xi)|^2\mathrm{d}\xi\,\mathrm{d}t \\
&\quad + C\int_0^t g'(t)\int_{B(t)} (|\xi|^2+|\xi|^8)\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\mathrm{d}\xi\,\mathrm{d}t \\
&\leq C + C\int_0^t g'(t)\int_{B(t)} \mathrm{e}^{-2|\xi|^2 t}|\widehat{u_0}(\xi)|^2\mathrm{d}\xi\,\mathrm{d}t \\
&\quad + C\int_0^t g'(t)\int_{B(t)} (1+|\xi|^2)\mathrm{e}^{-\frac{2|\xi|^2 t}{1+|\xi|^2}}|\widehat{b_0}(\xi)|^2\mathrm{d}\xi\,\mathrm{d}t \\
&\quad + C\int_0^t g'(t)\int_{B(t)} |\xi|^2\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)\mathrm{d}s\right)^2\mathrm{d}\xi\,\mathrm{d}t.
\end{aligned} \tag{29}$$

By the estimates from Lemma 3, we get

$$\begin{aligned}&C\int_0^t g'(s)\int_{B(t)} e^{-2|\xi|^2 t}|\widehat{u}_0(\xi)|^2 d\xi ds\\&\le C\int_0^t g'(s)\|w(s)\|_{L^2}^2 ds \le C\int_0^t g'(s)(1+s)^{-(\frac{3}{2}+r^*)}ds,\end{aligned}\tag{30}$$

and

$$\begin{aligned}&C\int_0^t g'(s)\int_{B(t)}(1+|\xi|^2)e^{-\frac{2|\xi|^2 t}{1+|\xi|^2}}|\widehat{b}_0(\xi)|^2 d\xi dt\\&\le C\int_0^t g'(s)(\|v(s)\|_{L^2}^2+\|\nabla v(s)\|_{L^2}^2)ds \le C\int_0^t g'(s)(1+s)^{-(\frac{3}{2}+r^*)}ds.\end{aligned}\tag{31}$$

For the last term of the right-hand side of (29), after integrating in polar coordinates in $B(t)$, we get

$$\begin{aligned}&C\int_0^t g'(t)\int_{B(t)}|\xi|^2\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)ds\right)^2 d\xi dt\\&\le C\left(\int_0^t g'(s)\rho^5 ds\right)\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)ds\right)^2.\end{aligned}\tag{32}$$

For a fixed r^*, we can choose $g(t)=(1+t)^m$ with $m>\max\{\frac{1}{2},\frac{3}{2}+r^*\}$. It is easy to see that $\rho(t)=(1+t)^{-\frac{1}{2}}$. It then follows from (29)–(32) and from the a priori estimate $\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\le C$ that

$$\begin{aligned}&\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\\&\le C\left((1+t)^{-m}+(1+t)^{-(\frac{3}{2}+r^*)}+(1+t)^{-\frac{1}{2}}+(1+t)^{-\frac{7}{2}}\right)\\&\le C(1+t)^{-\min\{\frac{3}{2}+r^*,\frac{1}{2}\}}.\end{aligned}\tag{33}$$

Using this first preliminary decay, we bootstrap to find sharper estimates for (32). Assume that $\min\{\frac{3}{2}+r^*,\frac{1}{2}\}=\frac{3}{2}+r^*$, for $g(t)=(1+t)^{-m}$ with $m>\max\{\frac{3}{2}+r^*,\frac{3}{2}\}$, we get $\rho(t)=C(1+t)^{-\frac{1}{2}}$ and

$$\begin{aligned}&C\int_0^t g'(t)\int_{B(t)}(|\xi|^2)\left(\int_0^t(\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2)ds\right)^2 d\xi dt\\&\le C\left(\int_0^t g'(s)\rho^5 ds\right)\left(\int_0^t(1+s)^{-(\frac{3}{2}+r^*)}ds\right)^2\\&\le C\int_0^t g'(s)\left((1+s)^{-(\frac{7}{2}+2r^*)}\right)ds.\end{aligned}\tag{34}$$

It then follows from (29)–(31) and (34) that

$$\begin{aligned}\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2&\le C\left((1+t)^{-m}+(1+t)^{-(\frac{3}{2}+r^*)}+(1+t)^{-\frac{7}{2}-2r^*}\right)\\&\le C(1+t)^{-(\frac{3}{2}+r^*)},\end{aligned}\tag{35}$$

the decay is still the slower one, there is no improvement for the decay rate. Suppose that $\min\{\frac{3}{2}+r^*,\frac{1}{2}\}=\frac{1}{2}$, we have

$$C\left(\int_0^t g'(s)\rho^5 ds\right)\left(\int_0^t (1+s)^{-(\frac{3}{2}+r^*)}ds\right)^2 \le C\int_0^t g'(s)\left((1+s)^{-\frac{3}{2}}\right)ds. \tag{36}$$

From (29)–(31) and (36), we derive that

$$\begin{aligned}&\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\\ \le& C\left((1+t)^{-m}+(1+t)^{-(\frac{3}{2}+r^*)}+(1+t)^{-\frac{3}{2}}+(1+t)^{-\frac{9}{2}}\right)\\ \le& C(1+t)^{-\min\{\frac{3}{2}+r^*,\frac{3}{2}\}}.\end{aligned} \tag{37}$$

We bootstrap once again. Suppose that $\min\{\frac{3}{2}+r^*,\frac{3}{2}\}=\frac{3}{2}+r^*$ and $r^*\neq-\frac{1}{2}$. Choose $m>\max\{\frac{3}{2}+r^*,\frac{5}{2}\}$. By the same computations as before, we obtain that the decay is the same to (32). There is no improvement for the decay rate. On the other hand, if $r^*=-\frac{1}{2}$, we choose $g(t)=(1+t)^m$ and $m>\frac{5}{2}$. Hence

$$\begin{aligned}\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2 &\le C\left((1+t)^{-m}+(1+t)^{-1}+(1+t)^{-\frac{5}{2}}\ln^2(1+t)\right)\\ &\le C(1+t)^{-1}=C(1+t)^{-(\frac{3}{2}+r^*)}.\end{aligned} \tag{38}$$

If $\min\{\frac{3}{2}+r^*,\frac{3}{2}\}=\frac{3}{2}$, we easily obtain

$$\int_0^t (\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2)dt \le C.$$

Suppose that $g(t)=(1+t)^m$ and $m>\frac{5}{2}$. Then

$$\begin{aligned}&\|u\|_{L^2}^2+\|b\|_{L^2}^2+\|\nabla b\|_{L^2}^2\\ \le& C\left((1+t)^{-m}+(1+t)^{-(\frac{3}{2}+r^*)}+(1+t)^{-\frac{5}{2}}\ln^2(1+t)\right)\\ \le& C(1+t)^{-\min\{(\frac{3}{2}+r^*),\frac{5}{2}\}}.\end{aligned} \tag{39}$$

Hence, we complete the proof. □

Acknowledgements

The author is indebted to the referee for careful reading of the paper and helpful suggestions.

This work is partially supported by the Natural Science Foundation of China (Grant No. 11401258) and China Postdoctoral Science Foundation (Grant No. 2015M581689).

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