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The BMR freeness conjecture for the first two families of the exceptional groups of rank 2

La conjecture de liberté de BMR pour les deux premières familles des groupes exceptionnels de rang 2

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ABSTRACT

The freeness conjecture of Broué, Malle and Rouquier for the generic Hecke algebras associated with complex reflection groups is still open for 14 cases, which cover almost all the exceptional complex reflection groups of rank 2. We prove this conjecture for 9 of these remaining cases, giving a basis similar to the classical case of the finite Coxeter groups.

RÉSUMÉ

La conjecture de liberté de Broué, Malle et Rouquier pour les algèbres de Hecke géneriques associées à des groupes de réflexions complexes est encore ouverte pour 14 cas, qui couvrent la quasi-totalité des groupes de reflexions complexes exceptionnels de rang 2. Nous prouvons cette conjecture pour 9 de ces cas ouverts, en donnant une base similaire à celle du cas classique des groupes de Coxeter finis.

1. Introduction

Between 1994 and 1998, M. Broué, G. Malle, and R. Rouquier generalized in a natural way the definition of the Iwahori–Hecke algebra to arbitrary complex reflection groups. Attempting to also generalize the properties of the Coxeter case, they stated a number of conjectures concerning the Hecke algebras, which had not been proven yet. Even without being proven, those conjectures have been used by a number of papers in the last decades as assumptions, and are still being used in various subjects, such as representation theory of finite reductive groups, Cherednik algebras, and usual braid groups (more details about these conjectures and their applications can be found in [14]).

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One specific example of importance, regarding those yet unsolved conjectures, is the so-called BMR freeness conjecture. In 1998, M. Broué, G. Malle and R. Rouquier conjectured that the generic Hecke algebra H associated with a complex reflection group W is a free module of rank $|W|$ over its ring of definition R. Any complex reflection group can be decomposed as a direct product of the so-called irreducible ones (which means that, considering them as subgroups of the general linear group $GL(V)$, where V is a finite dimensional complex vector space, they act irreducibly on V). Therefore, the proof of the BMR freeness conjecture reduces to the irreducible case. The irreducible complex reflection groups were classified by G. C. Shephard and J. A. Todd (see [16]); they belong either to the infinite family $G(de, e, n)$ depending on 3 positive integer parameters, or to the 34 exceptional groups, which are numbered from 4 to 37 and are known as $G_4, \dots, G_{37}$, in the Shephard and Todd classification.

The BMR freeness conjecture is known to be true for the finite Coxeter groups (see for example [11], lemma 4.4.3.), and also for the infinite series by Ariki and Koike (see [1] and [2]). Considering the exceptional cases, we encounter 6 finite Coxeter groups for which we know the validity of the conjecture: the groups G_{23}, G_{28}, G_{30}, G_{35}, G_{36} and G_{37}. I. Marin's research and his joint work with G. Pfeiffer concluded that the exceptional complex reflection groups for which there is a complete proof for the freeness conjecture are the groups G_4 (this case has also been proved in [3] and independently in [10]), G_{12}, G_{22}, G_{23}, ..., G_{37} (see [12,13,15]). Moreover, in [6] we also proved the cases of G_8 and G_{16}, completing the proof for the validity of the BMR conjecture for the case of the exceptional groups, whose associated complex braid group is an Artin group.

The remaining cases are almost all the exceptional groups of rank 2. Recent work by I. Losev, and the result of P. Etingof and E. Rains on the validity of the weak version of the BMR freeness conjecture for the exceptional groups of rank 2, imply the BMR conjecture for these groups in characteristic zero (for more details, one may refer to [7]). However, this result cannot be used to prove the strong version of the conjecture. Moreover, even in characteristic zero, we cannot provide a basis of the Hecke algebra consisting of braid group elements (see [7], remark 2.4.3).

The exceptional groups of rank 2 are divided into three families: the tetrahedral, octahedral and icosahedral family. The main goal of this note is to explain the proof of the conjecture for the first two families, by providing a basis consisting of braid group elements, which is also similar to the classical case of the finite Coxeter groups.

2. The BMR freeness conjecture

Let W be a complex reflection group of rank n and let B denotes the complex braid group associated with W. Let S denote the set of the distinguished pseudo-reflections of W. For each $s \in S$, we choose a set of e_s indeterminates $u_{s,1}, \dots, u_{s,e_s}$, such that $u_{s,i} = u_{t,i}$ if s and t are conjugate in W. We denote by R the Laurent polynomial ring $\mathbb{Z}[u_{s,i}, u_{s,i}^{-1}]$. The *generic Hecke algebra* H associated with W with parameters $u_{s,1}, \dots, u_{s,e_s}$ is the quotient of the group algebra RB of B by the ideal generated by the elements of the form $(\sigma - u_{s,1})(\sigma - u_{s,2}) \dots (\sigma - u_{s,e_s})$, where s runs over the conjugacy classes of S and σ over the set of braided reflections associated with the pseudo-reflection s.

We have the following conjecture due to M. Broué, G. Malle, and R. Rouquier (see [4]). This conjecture is known to be true in the real case, i.e. for the finite Coxeter groups (see, for example, [11], lemma 4.4.3).

Conjecture 2.1 *(The BMR freeness conjecture). The generic Hecke algebra H is a free module over R of rank $|W|$.*

The next proposition (theorem 4.24 in [4] or proposition 2.4(1) in [13]) states that in order to prove the validity of the BMR conjecture, it is enough to find a spanning set of H over R of $|W|$ elements.

Proposition 2.2. *If H is generated as R-module by $|W|$ elements, then it is a free module over R of rank $|W|$.*

We know the validity of the conjecture for the groups G_{23}, G_{28}, G_{30}, G_{35}, G_{36} and G_{37}, since these groups are finite Coxeter groups. The next theorem summarizes the results found in [1,2,6,12,13,15].

Theorem 2.3. *The BMR freeness conjecture holds for the infinite family $G(de, e, n)$ and for the exceptional groups G_4, G_8, G_{12}, G_{16}, G_{22}, G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33} and G_{34}.*

It remains to prove the conjecture for the exceptional groups G_5, G_6, G_7, G_9, G_{10}, G_{11}, G_{13}, G_{14}, G_{15}, G_{17}, G_{18}, G_{19}, G_{20} and G_{21}. These groups cover almost all the exceptional groups of rank 2.

Let W be an exceptional irreducible complex reflection group of rank 2. We know that these groups fall into 3 families, according to whether the group $\overline{W} := W/Z(W)$ is the tetrahedral, octahedral or icosahedral group. In each family, there is a maximal group of order $|\overline{W}|^2$, and all the other groups are its subgroups; these are the groups G_7, G_{11}, G_{19}. The next proposition rephrases proposition 3.2.11 in [5].

Proposition 2.4. *Let W be an exceptional group of rank 2, whose associated Hecke algebra H is torsion-free as R-module. If the BMR freeness conjecture holds for the maximal group in the family of W, then it holds for W, as well.*

Therefore, if H is torsion free, then we only have to prove the validity of the conjecture for the cases of G_7, G_{11} and G_{19}. Unfortunately, this torsion-free assumption does not appear to be easy to check a priori. In the following sections, we describe another method of proving the BMR freeness conjecture for the first two families, without using this assumption.

3. Deformed Coxeter group algebras

Let W be an exceptional group of rank 2. The group $\overline{W}$ is the group of even elements in a finite Coxeter group C of rank 3 (of type A_3, B_3 and H_3 for the tetrahedral, octahedral and icosahedral family, respectively), with Coxeter system y_1, y_2, y_3 and Coxeter matrix (m_{ij}). We set $\tilde{\mathbb{Z}} := \mathbb{Z}\left[e^{\frac{2\pi i}{m_{ij}}}\right]$. In §2 of [9], P. Etingof and E. Rains defined an $\tilde{\mathbb{Z}}$-algebra, which they call $A(C)$, presented as follows:

generators: $Y_1, Y_2, Y_3, t_{ij,k}$, where $i, j \in \{1, 2, 3\}$, $i \neq j$ and $k \in \mathbb{Z}/m_{ij}\mathbb{Z}$.

relations: $Y_i^2 = 1$, $t_{ij,k}^{-1} = t_{ji,-k}$, $\prod_{k=1}^{m_{ij}} (Y_iY_j - t_{ij,k}) = 0$, $t_{ij,k}Y_r = Y_r t_{ij,k}$, $t_{ij,k}t_{i'j',k'} = t_{i'j',k'}t_{ij,k}$.

This construction of $A(C)$ is more general and can be done also for any Coxeter group, not necessarily finite. Let $R^C = \tilde{\mathbb{Z}}\left[t_{ij,k}^{\pm}\right] = \tilde{\mathbb{Z}}\left[t_{ij,k}\right]$. The subalgebra $A_+(C)$ generated by Y_iY_j, $i \neq j$ becomes an R^C algebra and can be presented as follows:

Generators: $A_{ij} := Y_iY_j$, where $i, j \in \{1, 2, 3\}$, $i \neq j$;

Relations: $A_{ij}^{-1} = A_{ji}$, $\prod_{k=1}^{m_{ij}} (A_{ij} - t_{ij,k}) = 0$, $A_{ij}A_{jl}A_{li} = 1$, for $\#\{i, j, l\} = 3$.

The next lemma can be found in §3.2 of [5].

Lemma 3.1. *Let C be a finite Coxeter group of type either A_3, B_3 or H_3. We can present the R^C algebra $A_+(C)$ as follows:*

$$\left\langle A_{13}, A_{32}, A_{21} \;\middle|\; \begin{array}{c} (A_{13} - t_{13,1})(A_{13} - t_{13,2}) = 0 \\ (A_{32} - t_{32,1})(A_{32} - t_{32,2})(A_{32} - t_{32,3}) = 0, \qquad A_{13}A_{32}A_{21} = 1 \\ (A_{21} - t_{21,1})(A_{21} - t_{21,2}) \ldots (A_{21} - t_{21,m}) = 0 \end{array} \right\rangle,$$

where m is 3, 4 or 5 for each family, respectively.

If w is a word in letters y_i we let T_w denote the corresponding element of $A(C)$. For every $x \in \overline{W}$, let us choose a reduced word w_x that represents x in $\overline{W}$. We notice that T_{w_x} is an element in $A_+(C)$, since w_x is reduced and $\overline{W}$ is the group of even elements in C. The following theorem is theorem 2.3(ii) in [9].

Theorem 3.2. *The algebra $A_+(C)$ is generated as R^C-module by the elements T_{w_x}, $x \in \overline{W}$.*

4. The connection between the algebras H and $A_+(C)$

Let W be an exceptional group of rank 2 with associated complex braid group B and generic Hecke algebra H, defined over R. Following the notations of §2.2 of [13], we set $R_{\tilde{\mathbb{Z}}} := R \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ and $H_{\tilde{\mathbb{Z}}} := H \otimes_R R_{\tilde{\mathbb{Z}}}$. We denote by $\tilde{u}_{s,i}$ the images of $u_{s,i}$ inside $R_{\tilde{\mathbb{Z}}}$. By definition, $H_{\tilde{\mathbb{Z}}}$ is the quotient of the group algebra $R_{\tilde{\mathbb{Z}}}B$ of B by the ideal generated by $P_s(\sigma)$, where s runs over the conjugacy classes of distinguished reflections, σ over the set of braided reflections associated with s and $P_s[X]$ are the monic polynomials $(X - \tilde{u}_{s,1}) \ldots (X - \tilde{u}_{s,e_s})$ inside $R_{\tilde{\mathbb{Z}}}[X]$. If s and t are conjugate in W, the polynomials $P_s(X)$ and $P_t(X)$ coincide.

Let $Z(B)$ denote the center of B and let $z \in Z(B)$. We set $\bar{B} := B/\langle z \rangle$ and $R_{\tilde{\mathbb{Z}}}^+ := R_{\tilde{\mathbb{Z}}}\left[x, x^{-1}\right]$. Let f be a set-theoretic section of the natural projection $\pi : B \to \bar{B}$. For every $b \in B$, we denote by $\bar{b}$ the image of b under π. The following proposition rephrases proposition 2.10 in [13].

Proposition 4.1. *$H_{\tilde{\mathbb{Z}}}$ inherits a structure of $R_{\tilde{\mathbb{Z}}}^+$-module. Moreover, there is an isomorphism Φ_f between the $R_{\tilde{\mathbb{Z}}}^+$-modules $H_{\tilde{\mathbb{Z}}}$ and $R_{\tilde{\mathbb{Z}}}^+\bar{B}/Q_s(\sigma)$, where $Q_s(X) = x^{c_\sigma \deg P_s} \cdot P_s(x^{-c_\sigma} \cdot X) \in R_k^+[X]$, the $c_\sigma \in \mathbb{Z}$ being defined by $f(\bar{\sigma}) = z^{c_\sigma}\sigma$.*

We know that for W and B, we have a Coxeter-like and an Artin-like presentation, respectively. We call these presentations *the BMR presentations*, due to M. Broué, G. Malle and R. Rouquier. In 2006, P. Etingof and E. Rains gave different presentations of W and B, based on the BMR presentations associated with the maximal groups (see §6.1 of [8]). We call these presentations *the ER presentations*.

The next propositions (propositions 3.2.5 and 3.2.6 in [5]) together with Proposition 4.1 relate the algebras $A_+(C)$ and $H_{\tilde{\mathbb{Z}}}$.

Proposition 4.2. *Let W be an exceptional group of rank 2, apart from G_{13} and G_{15}. There is a ring morphism $\theta : R^C \twoheadrightarrow R^+_{\tilde{\mathbb{Z}}}$ inducing $\Psi : A_+(C) \otimes_\theta R^+_{\tilde{\mathbb{Z}}} \twoheadrightarrow R^+_{\tilde{\mathbb{Z}}} \bar{B}/Q_s(\bar{\sigma})$ through $A_{13} \mapsto \bar{\alpha}$, $A_{32} \mapsto \bar{\beta}$, $A_{21} \mapsto \bar{\gamma}$, where α, β and γ are the generators of B in ER presentation.*

We set $\tilde{R}^C := \tilde{\mathbb{Z}}\left[t_{13,1}, t_{13,2}, t_{32,1}, t_{32,2}, t_{32,3}, \sqrt{t_{21,1}}, \sqrt{t_{21,3}},\right]$ and let $\phi : R^C \to \tilde{R}^C$, defined by $t_{21,1} \mapsto \sqrt{t_{21,1}}$, $t_{21,2} \mapsto -\sqrt{t_{21,1}}$, $t_{21,3} \mapsto \sqrt{t_{21,3}}$ and $t_{21,4} \mapsto -\sqrt{t_{21,3}}$. Let $\tilde{A}_+(C)$ denote the $\tilde{R}^C$ algebra $A_+(C) \otimes_\phi \tilde{R}^C$.

Proposition 4.3. *Let W be the exceptional group G_{13} or G_{15}. There is a ring morphism $\theta : \tilde{R}^C \twoheadrightarrow R^+_{\tilde{\mathbb{Z}}}$ inducing $\Psi : \tilde{A}_+(C) \otimes_\theta R^+_{\tilde{\mathbb{Z}}} \twoheadrightarrow R^+_{\tilde{\mathbb{Z}}} \bar{B}/Q_s(\bar{s})$ through $\tilde{A}_{13} \mapsto \bar{\alpha}$, $\tilde{A}_{32} \mapsto \bar{\beta}$, $\tilde{A}_{21} \mapsto \bar{\gamma}$, where α, β and γ are the generators of B in ER presentation.*

For every exceptional group of rank 2, we call the surjection Ψ as described in Propositions 4.2 and 4.3 the *ER surjection* associated with W.

5. Finding the basis

Let W be an exceptional group belonging to the tetrahedral or octahedral family. For every $x \in \overline{W}$, we fix a reduced word w_x in letters y_1, y_2 and y_3 that represents x in $\overline{W}$. In chapter 4, §4.1 of [5], we explain how one can obtain, from the reduced word w_x, a word $\tilde{w}_x$ (not necessarily reduced) that also represents x in $\bar{W}$ and that corresponds to a well-defined element in $A_+(C)$, which we denote by $T_{\tilde{w}_x}$. In particular, we have $T_{\tilde{w}_1} = T_{w_1} = 1_{A_+(C)}$.

By the definition of the ER-surjection, the element $\Psi(T_{\tilde{w}_x})$ is a product of $\bar{\alpha}$, $\bar{\beta}$, and $\bar{\gamma}$. We use the group isomorphism ϕ_2 that we describe in table B.1 of Appendix B in [5] to write the elements α, β and γ in BMR presentation, and we set $\sigma_\alpha := \phi_2(\alpha)$, $\sigma_\beta := \phi_2(\beta)$ and $\sigma_\gamma := \phi_2(\gamma)$. Therefore, we can also consider the element $\Psi(T_{\tilde{w}_x})$ as being a product of $\bar{\sigma}_\alpha$, $\bar{\sigma}_\beta$ and $\bar{\sigma}_\gamma$. We denote this element by $\bar{v}_x$.

For every $x \in \overline{W}$, let $x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$ be the corresponding factorization of $\bar{v}_x$ into a product of $\bar{\sigma}_\alpha$, $\bar{\sigma}_\beta$ and $\bar{\sigma}_\gamma$ (meaning that $x_i \in \{\bar{\sigma}_\alpha, \bar{\sigma}_\beta, \bar{\sigma}_\gamma\}$ and $m_i \in \mathbb{Z}$). Let $f_0 : \bar{B} \to B$ be a set theoretical section such that $f_0(x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}) = f_0(x_1)^{m_1} f_0(x_2)^{m_2} \ldots f_0(x_r)^{m_r}$, $f_0(\bar{\sigma}_\alpha) = \sigma_\alpha$, $f_0(\bar{\sigma}_\beta) = \sigma_\beta$ and $f_0(\bar{\sigma}_\gamma) = \sigma_\gamma$ and, hence, we obtain an isomorphism Φ_{f_0} between the $R^+_{\tilde{\mathbb{Z}}}$-modules $R^+_{\tilde{\mathbb{Z}}} \bar{B}/Q_s(\bar{\sigma})$ and $H_{\tilde{\mathbb{Z}}}$ (see Proposition 4.1). We set $v_x := \Phi_{f_0}(\bar{v}_x)$.

The main result of this note is Theorem 5.1. Notice that the second part of this theorem follows directly from Proposition 2.2. The first part has been proven in chapter 4 of [5] by using a case-by-case analysis.

Theorem 5.1. $H_W = \sum\limits_{x \in \overline{W}} \sum\limits_{k=0}^{|Z(W)|-1} R z^k v_x$ *and, therefore, the BMR freeness conjecture holds for all the groups belonging to the tetrahedral and octahedral families.*

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