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Trace formula for Witt vector rings

Formule de trace pour les anneaux vectoriels de Witt

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ABSTRACT

We commence by giving a generalisation of Pulita exponential series. We then use these series to establish an analog of the trace formula for Witt vector rings.

RÉSUMÉ

On commence par généraliser les séries exponentielles de Pulita. Ensuite, on se sert de cette généralisation pour établir un analogue de la formule de trace de Dwork sur des anneaux de Witt.

1. Introduction

Let p be an odd prime number. Let $\mathbb{F}_q$ be a finite field of q elements of characteristic p. For V an algebraic variety defined over $\mathbb{F}_q$, the zeta function of V is defined by

$$\zeta(V,T) := \exp(\sum_{r\geq 1} N_r \frac{T^r}{r}),$$

where N_r is the number of $\mathbb{F}_{q^r}$-points of V. Dwork, in his paper [4] proved the rationality of this zeta function. In this proof, Dwork formulated N_r in terms of an additive character on $\mathbb{F}_{q^r}$. Then he expressed analytically this additive character by means of a Dwork *splitting function* (cf. [4], §1). At the end, he used a trace formula to express N_r. Our purpose here is to give an analog of Dwork trace formula for Gauss sums on Witt vector ring over $\mathbb{F}_q$ of finite length 2, $\mathbf{W}_2(\mathbb{F}_q)$[1] (cf. the paper [5], Theorem 5.14). Our basic tool will be a generalisation of the exponential series of A. Pulita (cf. [8], Definition 2.6). The starting point of our study was an inspiring paper of R. Blache (cf. [2], Definition 2.2), which treats the *splitting functions* on $\mathbf{W}_\ell(\mathbb{F}_q)$. For the proofs of lemmas, propositions, and theorems, we refer to [1].

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[1] The process used here for the analytic expression of the multiplicative character of $\mathbf{W}_2(\mathbb{F}_q)$ no longer works for $\ell > 2$.

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2. Generalised Pulita exponential series

Dwork gave a transition formula between the analytic representations of the additive characters of different finite fields (of the same characteristic). We will do the same for additive characters on Witt ring $\mathbf{W}_l(\mathbb{F}_q)$ and its extensions $\mathbf{W}_l(\mathbb{F}_{q^r})$ via generalisations of the Pulita series.

We first recall some notations and results as they appear in [1]:

– We define the Artin–Hasse series $AH(x)$ by:

$$AH(x) = \exp\left(\sum_{n=0}^{\infty} p^{-n} x^{p^n}\right).$$

– For any ring A of characteristic 0 such that it is involved in a ring morphism $\mathbb{Z}_p \cap \mathbb{Q} \to A$, and for any element $\boldsymbol{a} = (a_0, a_1, \ldots)$ of the Witt ring $\mathbf{W}(A)$, we denote by $E(\boldsymbol{a})$ the series[2]:

$$E(\boldsymbol{a}) = \prod_{i=0}^{\infty} AH\left(a_i x^{p^i}\right) = \exp\left(\sum_{n=0}^{\infty} \mathrm{fant}_n(\boldsymbol{a})\, p^{-n} x^{p^n}\right),$$

where $\mathrm{fant}_n(\boldsymbol{a}) = \sum_{i=0}^{n} p^i a_i^{p^{n-i}}$.
The map E, called Artin–Hasse morphism, is a morphism from the additive group $\mathbf{W}(A)$ into $\Lambda(A)$, where $\Lambda(A)$ is the multiplicative group of formal power series in the indeterminate x, with coefficients in A, the constant term being 1.

– For a Lubin–Tate series (cf. [7] for notations and results on Lubin–Tate theory), $F(T) = pT + T^p + pT^2G(T)$, where $G(T) \in \mathbb{Z}_p[[T]]$, we define the sequence $(\pi_m)_{m\in\mathbb{N}}$ so that $F(\pi_0) = 0$, $|\pi_0| < 1$ and $\forall m \in \mathbb{N}$, $F(\pi_{m+1}) = \pi_m$.

– For $m \in \mathbb{N}$, we put $K_m = \mathbb{Q}_p(\pi_m)$, $\mathcal{O}_m$ and $\mathfrak{m}_m$, respectively, the ring of integers of K_m and the maximal ideal of $\mathcal{O}_m$.

– We define the Witt vector $\boldsymbol{\varpi}_m \in W(\mathfrak{m}_m)$ as the unique vector verifying $\forall n \in \mathbb{N}$, $\mathrm{fant}_n(\boldsymbol{\varpi}_m) = \pi_{m-n}$.

– Let $\mathcal{O} = \cup_{m\in\mathbb{N}} \mathcal{O}_m$. It is a local ring with maximal ideal, $\mathfrak{m}_{\mathcal{O}} = \cup_{m\in\mathbb{N}} \mathfrak{m}_m$ and residual field $\mathbb{F}_p$. We note by K the field of fractions of the ring $\mathcal{O}$ and by $\mathbb{C}_p^{\circ}$ the ring of integers of $\mathbb{C}_p$. We have $K = \cup_{m\in\mathbb{N}} K_m$.

– Let $\rho\colon A \to B$ be a ring morphism. The application $\mathbf{W}(\rho)$ from $\mathbf{W}(A)$ to $\mathbf{W}(B)$ such that

$$\forall\, \mathbf{a} = (a_n)_{n\in\mathbb{N}} \in \mathbf{W}(A), \qquad \mathbf{W}(\rho)(\mathbf{a}) = (\rho(a_n))_{n\in\mathbb{N}},$$

is a ring morphism.

– According to [3], there exists a continuous automorphism φ of the valued field $\mathbb{C}_p$ such that we have $\varphi(x) = x$ for any x of K and $|\varphi(x) - x^p| < 1$ for any $x \in \mathbb{C}_p$ such that $|x| \le 1$.

– Let $m \ge 0$ be a positive integer. For a Witt vector $\boldsymbol{a}$ on $\mathbb{C}_p$, we note by $\mathbf{a}^{\varphi}$ the vector $\mathbf{W}(\varphi)(a) = (\varphi(a_n))_{n\in\mathbb{N}}$. The Pulita exponential series linked to m and $\boldsymbol{a}$, is defined in [8,1] by

$$\theta_m(\mathbf{a}) = \frac{E(\boldsymbol{\varpi}_m \mathbf{a})(x)}{E(\boldsymbol{\varpi}_m \mathbf{a}^{\varphi})(x^p)}.$$

This series can be expressed in the form $\theta_m(\mathbf{a}) = E\left(\boldsymbol{\varpi}_m \mathbf{a} - V\left(\boldsymbol{\varpi}_m \mathbf{a}^{\varphi}\right)\right)$, where V is the Verschiebung morphism of $\mathbf{W}\left(\mathbb{C}_p\right)$ (cf. [5], Definition 5.25).

– Let m be a positive integer and $\mathbf{a}$ a Witt vector of the ring $\mathbf{W}(\mathbb{C}_p^{\circ})$. The Pulita exponential series $\theta_m(\mathbf{a})$ has a radius of convergence strictly greater than 1.

We propose the following definition as a generalisation of Pulita series.

Definition 2.1. (cf. [1], Notation 2.5) For any Witt vector $\boldsymbol{a}$ on $\mathbb{C}_p$, and for any couple (m, s) of natural numbers such that $s \ge 1$, we note by $\theta_{m,s}(\boldsymbol{a})$ the series

$$\theta_{m,s}(\boldsymbol{a}) = \prod_{i=0}^{s-1} \theta_m(\boldsymbol{a}^{\varphi^i}) \circ x^{p^i}. \tag{1}$$

As the factor $\theta_m(\boldsymbol{a}^{\varphi^i}) \circ x^{p^i}$ that appears in the right member of the identity (1) is the image via the Artin–Hasse morphism of $V^i\left(\boldsymbol{\varpi}_m \boldsymbol{a}^{\varphi^i} - V\left(\boldsymbol{\varpi}_m \boldsymbol{a}^{\varphi^{i+1}}\right)\right)$, we deduce that

$$\theta_{m,s}(\boldsymbol{a}) = E\left(\boldsymbol{\varpi}_m \boldsymbol{a} - V^s\left(\boldsymbol{\varpi}_m \boldsymbol{a}^{\varphi^s}\right)\right).$$

In particular, the application $\theta_{m,s}$ is a morphism from the additive group $\mathbf{W}(\mathbb{C}_p)$ to the multiplicative group $\Lambda(\mathbb{C}_p)$.

[2] This equality is obtained by a simple calculus.

2.1. The morphisms $\bar{\theta}_{m,s}$

Our aim being the study of Gauss sums defined on Witt rings of length $m \geq 1$; $\mathbf{W}_m(\mathbb{F}_{q=p^s})$ and its extensions; we will need at the first time to induce from the morphisms $\theta_{m,s}$ of $\mathbf{W}(\mathbb{C}_p)$ new morphisms $\bar{\theta}_{m,s}$ defined on $\mathbf{W}_{m+1}(\mathbb{C}_p^\circ)$.

Proposition 2.2. *The morphism $\theta_{m,s} : \mathbf{W}(\mathbb{C}_p) \to \Lambda(\mathbb{C}_p)$ induces a unique morphism*

$$\bar{\theta}_{m,s} : \mathbf{W}_{m+1}(\mathbb{C}_p^\circ) \to \Lambda^*(\mathbb{C}_p^\circ) \ .$$

Remark 1. Other properties of the generalised exponential series appear in [1].

3. Analytic expression of the characters of $W_\ell(\mathbb{F}_q)$

Definition 3.1. Let $(A, +, \times)$ be a ring. A group morphism $\psi : (A, +) \to \mathbb{C}_p^\star$ (resp. $\chi : (A^\star, \times) \to \mathbb{C}_p^\star$) is called an additive character (resp. a multiplicative character) of A.

In this section, we will give explicit analytic expressions of an additive and of a multiplicative character of $W_\ell(\mathbb{F}_q)$ via the series $\bar{\theta}_{\ell,s}$. Along the section, we fix a natural integer $s \geq 1$, and we put $q = p^s$.

3.1. Construction of an additive character of $W_\ell(\mathbb{F}_q)$

For $n > 0$ a positive integer, we denote by μ_n the group of roots of unity in $\mathbb{C}_p$ of order dividing n. In the following proposition, we will construct an analytic additive character $\psi_{\ell,s,t}$ of $W_\ell(\mathbb{F}_q)$. For this, we will use $\bar{\theta}_{\ell-1,s}$ and, under a certain condition, we will see that all additive characters of $W_\ell(\mathbb{F}_q)$ are expressed via $\psi_{\ell,s,t}$.

Proposition 3.2. *Let $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^{p^s} = t$. The composition of the morphism $\bar{\theta}_{\ell-1,s}$ followed by the morphism $\mathrm{eval}_t : \Lambda^*(\mathbb{C}_p^\circ) \to \mathbb{C}_p^\times$ of the evaluation at t, induces a unique morphism $\psi_{\ell,s,t}$ from $\mathbf{W}_\ell(\mathbb{F}_q)$ to $\mathbb{C}_p^\times$. Furthermore, if $\mathrm{Tr}_{\mathbb{Q}_p(\mu_{q-1})/\mathbb{Q}_p}(t) \notin p\mathbb{Z}_p$, then:*

1. for any additive character $\psi : \mathbf{W}_\ell(\mathbb{F}_q) \to \mathbb{C}_p^\times$, there exists a unique vector $\boldsymbol{a} \in \mathbf{W}_\ell(\mathbb{F}_q)$ such that we have $\psi(\boldsymbol{y}) = \psi_{\ell,s,t}(\boldsymbol{a}\boldsymbol{y})$ for any vector $\boldsymbol{y} \in \mathbf{W}_\ell(\mathbb{F}_q)$;

2. The image of $\psi_{\ell,s,t}$ is exactly μ_{p^ℓ}.

Remark 2. In the case where $s = 1$, A. Pulita confirmed in [8] (Theorem 2.7) and S. Kedlaya proved in [6] (Chapter 9), that the image of $\mathbf{1} \in \mathbf{W}_\ell(\mathbb{F}_p)$ is a primitive p^l-th root of unity. Furthermore, Pulita attempted to characterise $\xi_{\ell-1}$, this p^ℓ-th root of unity. He described it as being the unique p^ℓ-th root of unity which verifies $|t\pi_{\ell-1} - (\xi_{\ell-1} - 1)| < |\pi_{\ell-1}|$. Unfortunately, it is not true. We have proved [1] that the set of p^ℓ-th roots of unity that satisfy the above inequality is of cardinal $p^{\ell-1}$, which means that this root of unity is unique only when $\ell = 1$, a case previously treated by Dwork [4].

3.2. Transition formula

We will now explicit a relation between the additive characters $\psi_{\ell,sr,t}$, $\psi_{\ell,s,t}$ of both $\mathbf{W}_\ell(\mathbb{F}_{q^r})$ and $\mathbf{W}_\ell(\mathbb{F}_q)$, respectively. To do so, we shall use the trace application $\mathrm{Tr}_{\mathbf{W}_\ell(\mathbb{F}_{q^r})/\mathbf{W}_\ell(\mathbb{F}_q)} : \mathbf{W}_\ell(\mathbb{F}_{q^r}) \to \mathbf{W}_\ell(\mathbb{F}_q)$, whose definition requires some preliminaries:

- we consider $\widetilde{\varphi}$ the Frobenius automorphism $\widetilde{x} \mapsto \widetilde{x}^p$ of the residue field $\widetilde{\mathbb{C}_p} = \mathbb{C}_p^\circ/C_p^\vee$, where $\mathbb{C}_p^\vee$ is the maximal ideal of $\mathbb{C}_p^\circ$;
- an automorphism $\widetilde{\varphi}_r$ is induced on the field $\mathbb{F}_{q^r}$ by the automorphism $\widetilde{\varphi}$, which associates y^p with an element y of $\mathbb{F}_{q^r}$.

Then we define the application trace by the following definition.

Definition 3.3.

$$\forall \boldsymbol{y} \in \mathbf{W}_\ell(\mathbb{F}_{q^r}) \qquad \mathrm{Tr}_{\mathbf{W}_\ell(\mathbb{F}_{q^r})/\mathbf{W}_\ell(\mathbb{F}_q)}(\boldsymbol{y}) = \sum_{i=0}^{r-1} \mathbf{W}_\ell(\widetilde{\varphi}_r^{is})(\boldsymbol{y}) \ . \tag{2}$$

We establish the transition formula via the following proposition

Proposition 3.4. *Let $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$. We have the transition formula*

$$\psi_{\ell,sr,t} = \psi_{\ell,s,t} \circ \mathrm{Tr}_{\mathbf{W}_\ell(\mathbb{F}_{q^r})/\mathbf{W}_\ell(\mathbb{F}_q)} \ .$$

3.3. Splitting functions

We will generalise below the definition of the splitting function given in [2] and [4] from the primary field $\mathbb{F}_p$ to the finite ring $\mathbf{W}_\ell(\mathbb{F}_q)$. But we first remind the definition of *Teichmüller*. The sub-ring $\mathbb{Z}_p\left[\mu_{q-1}\right]$ of $\mathbb{C}_p$ is the ring of integers of $\mathbb{Q}_p\left(\mu_{q-1}\right)$, the non-ramified extension of $\mathbb{Q}_p$ of p-adic numbers, so that its residual field is of order q. There is at least one (in fact exactly s) ring reduction morphisms of $\mathbb{Z}_p\left[\mu_{q-1}\right]$ onto the field $\mathbb{F}_q$. Let $x \mapsto \widetilde{x}$ one of them. This allows us to introduce a Teichmüller character of the field $\mathbb{F}_q$ as follows: for $x \in \mathbb{F}_q$, we note by $\mathrm{Teich}(x)$ the unique element of $\mathbb{Z}_p\left[\mu_{q-1}\right]$ such that $\mathrm{Teich}(x)^q = \mathrm{Teich}(x)$ and $\widetilde{\mathrm{Teich}(x)} = x$.

The existence and the unicity of the element $\mathrm{Teich}(x)$ results immediately from the Hensel lemma. Then the application

$$\mathrm{Teich} : \mathbb{F}_q \to \mathbb{C}_p$$

is a generator of the group of the multiplicative characters of $\mathbb{F}_q$. This application is called *Teichmüller character.*

Definition 3.5. Let ℓ be an integer $\geq 1, q = p^s$ and $\Omega(x_1, \ldots, x_\ell)$ a power series in ℓ indeterminates over $\mathcal{O}$, which converges on an open polydisc of the form $D(0, r_1) \times \cdots \times D(0, r_\ell)$, where $r_1, \ldots, r_\ell > 1$. We say that Ω is an *s-splitting function of level* ℓ if it verifies the following conditions:

1. the map $\psi : \mathbf{W}_\ell(\mathbb{F}_q) \to \mathbb{C}_p^\star$, which associates with the Witt vector $\boldsymbol{y} = (y_0, \ldots, y_{\ell-1})$ the image $\Omega(\mathrm{Teich}(y_0), \mathrm{Teich}(y_1), \ldots, \mathrm{Teich}(y_{\ell-1}))$, is an additive character of order p^ℓ;
2. for any integer $r \geq 1$, the additive character of $\mathbf{W}_\ell(\mathbb{F}_{q^r})$ obtained by composing ψ with the application trace $\mathrm{Tr}_{\mathbf{W}_\ell(\mathbb{F}_q^r)/\mathbf{W}_\ell(\mathbb{F}_q)}$ is expressed as follows

$$\psi\left(\mathrm{Tr}_{\mathbf{W}_\ell(\mathbb{F}_q^r)/\mathbf{W}_\ell(\mathbb{F}_q)}(\boldsymbol{y})\right) = \prod_{i=0}^{r-1} \Omega\left(\mathrm{Teich}(y_0)^{q^i}, \ldots, \mathrm{Teich}(y_{\ell-1})^{q^i}\right) \tag{3}$$

for any Witt vector $\boldsymbol{y} = (y_0, \ldots, y_{\ell-1}) \in \mathbf{W}_\ell(\mathbb{F}_{q^r})$.

According to the previous definition and to Lemmas 2.22, 2.23 [1], we will represent analytically the additive character $\psi_{\ell,s,t}$ (Proposition 3.2).

Definition 3.6. We denote by $\Omega_{\ell,s,t}$, the power series with ℓ indeterminates $x_0, x_1, \ldots, x_{\ell-1}$ over the ring $\mathcal{O}[\mu_{q-1}]$ defined by

$$\Omega_{\ell,s,t}(x_0, x_1, \ldots, x_{\ell-1}) = \prod_{j=0}^{\ell-1} \theta_{\ell-j-1,s}(\mathbf{1}) \circ (t^{p^j} x_j). \tag{4}$$

Proposition 3.7. *Let $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$. Then, for any vector $\boldsymbol{y} = (y_i)_{0 \leq i < \ell}$ with length ℓ on $\mathbb{F}_q$, we have*

$$\psi_{\ell,s,t}(\boldsymbol{y}) = \Omega_{\ell,s,t}(\mathrm{Teich}(y_0), \ldots, \mathrm{Teich}(y_{\ell-1}))\ .$$

Proposition 3.8. *Let $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$. If $r \geq 1$ is a natural integer, then we have the identity*

$$\Omega_{\ell,sr,t}(x_0, \ldots, x_{\ell-1}) = \prod_{i=0}^{r-1} \Omega_{\ell,s,t}(x_0^{q^i}, \ldots, x_{\ell-1}^{q^i})\ .$$

Theorem 3. *Let $\ell, s \geq 1$ be two integers, $q = p^s$ and $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$, $\mathrm{Tr}_{\mathbb{Q}_p(\mu_{q-1})/\mathbb{Q}_p}(t) \notin p\mathbb{Z}_p$. The series $\Omega_{\ell,s,t}$ is an s-splitting function of level ℓ.*

3.4. An analytic expression of a multiplicative character of $W_2(\mathbb{F}_q)$

We will restrict our study of multiplicative characters on $W_\ell(\mathbb{F}_q)$ to the case $\ell = 2$. We note by $W_2(\mathbb{F}_q)^\star$ the group of invertible elements of the ring of Witt vectors defined over $\mathbb{F}_q$ of length 2. A Witt vector $\boldsymbol{z} = (z_0, z_1)$ is an element of $\mathbf{W}_\ell(\mathbb{F}_q)^\star$ if and only if $z_0 \neq 0$.

Proposition 3.9. *Let $t \in \mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$, $\mathrm{Tr}_{\mathbb{Q}_p(\mu_{q-1})/\mathbb{Q}_p}(t) \notin p\mathbb{Z}_p$, and χ a multiplicative character of $W_2(\mathbb{F}_q)$. There are then an integer m in $[0..q-2]$ and an element b of $\mathbb{F}_q$ such that, for any Witt vector $\boldsymbol{z} = (z_0, z_1)$ in $\mathbf{W}_2(\mathbb{F}_q)^\star$, we have:*

$$\chi(\boldsymbol{z}) = \mathrm{Teich}(z_0)^m\, \Omega_{1,s,t}(\mathrm{Teich}(b)\mathrm{Teich}(z_1)\mathrm{Teich}(z_0)^{p(q-2)}).$$

4. Trace formula

Substituting $\mathbf{W}_2(\mathbb{F}_q)^\star$ for $\mathbb{F}_q^\star$, we will establish an analog for the Dwork trace formula [4], which is an identity involving a Gauss sum on $\mathbb{F}_q^\star$ and a trace of a certain operator.

4.1. Analytic expression of Gauss sums

Definition 4.1. Let ψ be a non-trivial additive character of $\mathbb{W}_2(\mathbb{F}_q)$ and χ be a multiplicative character of $\mathbb{W}_2(\mathbb{F}_q)^\star$. The *Gauss sum* denoted $g(\psi,\chi)$ and associated with χ and ψ is defined as follows

$$g(\psi,\chi) = -\sum_{\mathbf{z}=(z_0,z_1)\in\mathbb{W}_2(\mathbb{F}_q)^\star} \psi(\mathbf{z})\chi(\mathbf{z}) = -\sum_{z_0\in\mathbb{F}_q^\star}\sum_{z_1\in\mathbb{F}_q} \psi_{2,s,t}(z_0,z_1)\chi(z_0,z_1)\,.$$

We will restrict the study to the case of an additive character ψ with order p^2.

To use Proposition 3.2, we choose an element $t\in\mathbb{Z}_p[\mu_{q-1}]$ such that $t^{p^s}=t$ and $\mathrm{Tr}_{\mathbb{Q}_p(\mu_{q-1})/\mathbb{Q}_p}(t)\notin p\mathbb{Z}_p$, then we can express an additive character ψ in terms of the character $\psi_{2,s,t}$. Thereafter, we reduce to the case of the Gauss sums $g(\psi_{2,s,t},\chi)$.

Let us define

$$\widehat{H}(x_0,x_1) = -\Omega_{2,s,t}(x_0,x_1)x_0^m\Omega_{1,s,t}(\mathrm{Teich}(b)x_1x_0^{p(q-2)}). \tag{5}$$

Thus we have an analytic expression of $g(\psi_{2,s,t},\chi)$ given bellow:

Proposition 4.2. *There is an integer $m\in[0..q-2]$ and an element $b\in\mathbb{F}_q$ such that, for any vector $(z_0,z_1)\in\mathbf{W}_2(\mathbb{F}_q)^\star$, we have*

$$g(\psi_{2,s,t},\chi) = \sum_{z_0\in\mathbb{F}_q^\star}\sum_{z_1\in\mathbb{F}_q}\widehat{H}(\mathrm{Teich}(z_0),\mathrm{Teich}(z_1)), \tag{6}$$

4.2. Operators

We remind that $q=p^s$.

4.2.1. The space of functions

For any real $\beta>0$, we denote by E_β the vector space of two variables power series, with coefficients in $K_1[\mu_{q-1}]$, defined by:

$$E_\beta = \{G(x_0,x_1) = \sum_{(n_0,n_1)\in\mathbb{N}^2} a_{n_0,n_1}x_0^{n_0}x_1^{n_1},\, a_{n_0,n_1}\in K_1[\mu_{q-1}] \text{ and } \lim_{n_0,n_1\to\infty}|a_{n_0,n_1}|\beta^{n_0+n_1}=0\}. \tag{7}$$

We define a norm on E_β

$$\|G\|_\beta = \sup\{|a_{n_0,n_1}|\beta^{n_0+n_1},\,(n_0,n_1)\in\mathbb{N}^2\}\,,$$

which makes of E_β a Banach space.

Definition 4.3. Let E and F be two Banach spaces, and $f: E\to F$ a linear map. We say that f is completely continuous if it is the limit of a sequence of continuous linear maps of finite rank.

Proposition 4.4. *Let β,β' be two real numbers such that $0<\beta<\beta'$. The canonical injection $i_{\beta',\beta}: E_{\beta'}\to E_\beta$ is completely continuous and its image is dense.*

4.2.2. Dwork operator

For $G=\sum_{(n_0,n_1)\in\mathbb{N}^2} a_{n_0,n_1}x_0^{n_0}x_1^{n_1}$ a two-variable power series, and coefficients in $\mathbb{C}_p$, we denote by $\mathrm{Dw}_q(G)$ the power series $\mathrm{Dw}_q(G)=\sum_{(n_0,n_1)\in\mathbb{N}^2} a_{qn_0,qn_1}x_0^{n_0}x_1^{n_1}$.

Definition 4.5. For any real $\beta>0$, we call Dw_q the Dwork operator from E_β to E_{β^q}.

Proposition 4.6. *The Dwork operator $\mathrm{Dw}_q: E_\beta\to E_{\beta^q}$ is continuous for any $\beta>0$.*

4.2.3. Operators of multiplication

Proposition 4.7. *For any real $\beta>0$, let $H=\sum b_{n_0,n_1}x_0^{n_0}x_1^{n_1}$ be a series of E_β. The multiplication operator $\mathrm{mult}_H: E_\beta\to E_\beta$ that sends G onto GH is continuous.*

4.2.4. Alpha operator

Definition 4.8. Let $\beta > 1$, and $H(x_0, x_1)$ a series of $E_{\beta^{\frac{1}{q}}}$. We denote by α the following operator on E_β

$$\alpha = \mathrm{Dw}_q \circ \mathrm{mult}_H \circ i_{\beta, \beta^{\frac{1}{q}}} ,$$

where $\mathrm{Dw}_q : E_{\beta^{\frac{1}{q}}} \to E_\beta$ is the Dwork operator, $\mathrm{mult}_H : E_{\beta^{\frac{1}{q}}} \to E_{\beta^{\frac{1}{q}}}$ is the multiplication by the H operator, and $i_{\beta,\beta^{\frac{1}{q}}} : E_\beta \to E_{\beta^{\frac{1}{q}}}$ is the canonical injection. This operator α is called the α-operator associated with H.

Proposition 4.9. *Let $\beta > 1$, and H an element of $E_{\beta^{\frac{1}{q}}}$; the α-operator associated with H on E_β is a completely continuous endomorphism on E_β.*

Proposition 4.10. *The family $(x_0^{n_0} x_1^{n_1})_{(n_0,n_1)\in\mathbb{N}^2}$ is an orthogonal Schauder basis [9] of the space E_β. In this basis, the matrix of the operator α associated with the series $H(x_0, x_1) = \sum_{(n_0,n_1)\in\mathbb{N}^2} b_{(m_0,m_1)} x_0^{m_0} x_1^{m_1} \in E_{\beta^{\frac{1}{q}}}$ is given by*

$$\alpha(x_0^{n_0} x_1^{n_1}) = \sum_{(m_0,m_1)\in\mathbb{N}^2, qm_0\geq n_0, qm_1 \geq n_1} b_{qm_0-n_0, qm_1-n_1} x_0^{m_0} x_1^{m_1} .$$

Proposition 4.11. *Let $\beta > 1$, and $H(x_0, x_1) = \sum_{(m_0,m_1)\in\mathbb{N}^2} b_{m_0,m_1} x_0^{m_0} x_1^{m_1}$ be an element of $E_{\beta^{\frac{1}{q}}}$; the operator α associated with H is a nuclear operator with trace $\mathrm{Tr}(\alpha)$ equal to*

$$\mathrm{Tr}(\alpha) = \sum_{(n_0,n_1)\in\mathbb{N}^2} b_{(q-1)n_0,(q-1)n_1} .$$

4.3. Trace formula for $W_2(\mathbb{F}_q)$

Proposition 4.12. *Let an integer $s \geq 1$, $q = p^s$ and t be an element of $\mathbb{Z}_p[\mu_{q-1}]$ such that $t^q = t$. For any multiplicative character χ of $\mathbf{W}_2(\mathbb{F}_q)$, there exists a real $\beta_0 > 1$ such that, for all $\beta \in]1, \beta_0[$, we have the equality*

$$g(\psi_{2,s,t}, \chi) = (q-1)^2 \mathrm{Tr}(\alpha) ,$$

where α is the α-operator on E_β associated with the series $\widehat{H}$ defined by the identity (5).

References

[1] B. Benzaghou, S. Mokhfi, Trace formula for Witt vector rings, Arxiv, https://arxiv.org/pdf/1605.07676v1.pdf, 2016.
[2] R. Blache, Stickelberger theorem for p-adic Gauss sums, Acta Arith. 118 (1) (2005) 11–26.
[3] G. Christol, Modules différentiels et équations différentielles p-adiques, Queen's Papers in Pure and Applied Mathematics, vol. 66, Queen's University, Kingston, ON, Canada, 1983, vi+218 pp.
[4] B. Dwork, On the rationality of the zeta function of an algebraic variety, Amer. J. Math. 82 (1960) 631–648.
[5] M. Hazewinkel, Witt vectors. Part I, section 4H, in: Handbook of Algebra, vol. 6, Elsevier/North-Holland, Amsterdam, 2009, pp. 319–472.
[6] K.S. Kedlaya, Convergence polygons for connections on nonarchimedean curves, Arxiv, http://arxiv.org/pdf/1505.01890v2.pdf, 2015.
[7] J. Lubin, J. Tate, Formal complex multiplication in local fields, Ann. of Math. (2) 81 (1965) 380–387.
[8] A. Pulita, Rank one solvable p-adic differential equations and finite Abelian characters via Lubin–Tate groups, Math. Ann. 337 (3) (2007) 489–555.
[9] W.H. Schikof, Ultrametric Calculus, Cambridge Studies in Advanced Mathematics, vol. 4, Springer-Verlag, New York, 1973.