



Ordinary differential equations/Partial differential equations

Uniqueness and Lagrangianity for solutions with lack of integrability of the continuity equation



Unicité et propriété lagrangienne des solutions manquant d'intégrabilité de l'équation de continuité

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ABSTRACT

We deal with the uniqueness of distributional solutions to the continuity equation with a Sobolev vector field and with the property of being a Lagrangian solution, i.e. transported by a flow of the associated ordinary differential equation. We work in a framework of lack of local integrability of the solution, in which the classical DiPerna–Lions theory of uniqueness and Lagrangianity of distributional solutions does not apply due to the insufficient integrability of the commutator. We introduce a general principle to prove that a solution is Lagrangian: we rely on a disintegration along the unique flow and on a new directional Lipschitz extension lemma, used to construct a large class of test functions in the Lagrangian distributional formulation of the continuity equation.

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R É S U M É

On étudie l'unicité des solutions au sens des distributions de l'équation de continuité avec des champs de vecteurs Sobolev et la propriété d'être une solution lagrangienne, c'est-à-dire une solution transportée par le flot de l'équation différentielle ordinaire associée au champ de vecteurs. On travaille dans un cadre où les solutions considérées manquent d'intégrabilité locale et où on ne peut pas appliquer la théorie classique de DiPerna–Lions d'unicité des solutions au sens des distributions et de la propriété d'être lagrangienne, parce que l'on n'a pas assez d'intégrabilité pour le commutateur. On introduit un principe général pour démontrer la propriété d'être une solution lagrangienne : notre technique se base sur une désintégration le long du flot unique et sur un lemme d'extension lipschitzienne directionnelle, qui nous permet de construire une vaste famille de fonctions tests pour la formulation lagrangienne au sens des distributions de l'équation de continuité.

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1. Introduction and statement of the main result

In this note, we deal with the uniqueness of distributional solutions to the continuity equation with a Sobolev vector field and with the property of being a Lagrangian solution, i.e. transported by a flow of the associated ordinary differential equation.

Let us first recall the by now classical DiPerna–Lions theory [5]. We fix $1 \leq p \leq \infty$ and $T > 0$ and we consider a vector field

$$\mathbf{b} \in L^1\left([0, T]; W_{\text{loc}}^{1,p}(\mathbb{R}^n; \mathbb{R}^n)\right), \quad \text{div } \mathbf{b} \in L^1\left([0, T]; L^\infty(\mathbb{R}^n)\right),$$

$$\frac{|\mathbf{b}(t, \mathbf{x})|}{1 + |\mathbf{x}|} \in L^1\left([0, T]; L^1(\mathbb{R}^n)\right) + L^1\left([0, T]; L^\infty(\mathbb{R}^n)\right).$$
(1)

Given an initial datum u_0 , we consider distributional solutions to the Cauchy problem for the continuity equation

$$\begin{cases} \partial_t u + \text{div}(\mathbf{b}u) = 0 \\ u(t = 0, \mathbf{x}) = u_0(\mathbf{x}) \end{cases} \quad \text{in } \mathcal{D}'([0, T] \times \mathbb{R}^n),$$
(2)

defined as usual by a formal “integration by parts” after testing the equation with Lipschitz test functions. Given a vector field \mathbf{b} as in (1), the DiPerna–Lions theory [5] guarantees the uniqueness of distributional solutions

$$u \in L^\infty([0, T]; L^q(\mathbb{R}^n))$$
(3)

to the problem (2), where q is the conjugate exponent of p , that is, $1/p + 1/q = 1$. If $u_0 \in L^q(\mathbb{R}^n)$, the existence of solutions in this class can be proved by an easy approximation procedure. Moreover, such a unique solution is transported by the unique regular Lagrangian flow associated with \mathbf{b} (see Definition 2.1). We remark that the theory of [5] has been extended to vector fields with bounded variation by Ambrosio [1].

The need for considering solutions in the class (3) follows from the strategy of proof in [5], which consists in showing the renormalization property for distributional solutions. To this aim, the authors prove the convergence to zero of a suitable commutator, which can be rewritten as an integral expression involving essentially the product of $D\mathbf{b}$ and u . However, distributional solutions to the Cauchy problem (2) can be defined as long as the product $\mathbf{b}u \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^n)$. Therefore, the theory in [5] leaves open the question of whether uniqueness holds for solutions with less integrability than (3). Ideally, the “extreme” case would be that of $\mathbf{b} \in L^\infty \cap W^{1,1}$ and $u \in L^1$, both locally in space.

Our main result in this direction is the following:

Theorem 1.1. *Let \mathbf{b} be a vector field as in (1), with $1 < p \leq \infty$. Assume in addition that $\mathbf{b}(t, \cdot)$ is continuous for \mathcal{L}^1 -a.e. $t \in [0, T]$, with modulus of continuity on compact sets which is uniform in time. Then, given an initial datum $u_0 \in L^1_{\text{loc}}(\mathbb{R}^n)$, the Cauchy problem for (2) has a unique solution*

$$u \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^n).$$

Such unique solution is Lagrangian and renormalized.

Remark 1.2. The continuity assumption on the vector field in Theorem 1.1 is satisfied, for example, when $\mathbf{b} \in L^\infty([0, T]; W_{\text{loc}}^{1,p}(\mathbb{R}^n))$ with $p > n$.

Remark 1.3. Theorem 1.1 can be easily extended to the case where a source term or a linear term of zero order is present in the continuity equation, under suitable integrability conditions on the coefficients. In particular, we can also deal with the transport equation

$$\partial_t u + \mathbf{b} \cdot \nabla u = 0$$

instead of the continuity equation (2).

Let us describe in a few words the strategy of the proof of Theorem 1.1. Given a distributional solution $u \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^n)$ of the Cauchy problem (2), we aim at proving that it is transported by the regular Lagrangian flow \mathbf{X} associated with \mathbf{b} . To this aim, we change variable using the flow in the distributional formulation of (2). However, due to the lack of Lipschitz regularity of the flow with respect to the space variable, we do not obtain yet the Lagrangian formulation in the distributional sense: after the change of variable, we do not obtain the full class of test functions.

Nevertheless, some regularity of the flow “on large sets” is in fact available (see Theorem 2.2). This guarantees that the test function we obtain is Lipschitz on a “large flow tube”, although with a possibly large Lipschitz constant. We need to extend this function to a globally Lipschitz test function. The key remark is that, in order to estimate the error resulting from this extension, only the Lipschitz constant along the characteristics is relevant, not the global Lipschitz constant. We then

implement a “directional extension lemma” (Lemma 4.1), stating that we can construct an extension that is both globally Lipschitz and directionally Lipschitz along the flow, and the directional Lipschitz constant can be estimated quantitatively. This allows us to conclude the proof.

After presenting in §2 some background material, in §3 we give a complete proof of Theorem 1.1, under the additional Assumption 3.3 on the existence of a directional Lipschitz extension. In §4, we sketch a proof of the validity of Assumption 3.3 under the continuity assumptions on the vector field in Theorem 1.1. A complete proof is deferred to the follow-up paper [2].

2. Some preliminaries

In the non-smooth context, the suitable notion of flow of a vector field is that of regular Lagrangian flow, introduced in the following form in [1]:

Definition 2.1. We say that a map $\mathbf{X} : [0, T]^2 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a regular Lagrangian flow associated with the vector field \mathbf{b} if

- (1) for $(t, s) \in [0, T]^2$ we have $C^{-1}\mathcal{L}^n \leq \mathbf{X}(t, s, \cdot)_\# \mathcal{L}^n \leq C\mathcal{L}^n$.
- (2) for \mathcal{L}^n -a.e. $\mathbf{x} \in \mathbb{R}^n$ the map $\mathbf{X}(\cdot, s, \mathbf{x})$ satisfies the ordinary differential equation

$$\begin{cases} \partial_t \mathbf{X}(t, s, \mathbf{x}) = \mathbf{b}(t, \mathbf{X}(t, s, \mathbf{x})) \\ \mathbf{X}(s, s, \mathbf{x}) = \mathbf{x} \end{cases} \quad \text{in } \mathcal{D}'([0, T]). \quad (4)$$

We notice that

$$\mathbf{x} \mapsto \mathbf{X}(0, t, \mathbf{x}) \quad \text{is the inverse of} \quad \mathbf{x} \mapsto \mathbf{X}(t, 0, \mathbf{x}).$$

For later use we set

$$\rho(t, \cdot)_\# \mathcal{L}^n := \mathbf{X}(t, 0, \cdot)_\# \mathcal{L}^n, \quad R(t, \mathbf{y}) := \rho(t, \mathbf{X}(t, 0, \mathbf{y})) \quad (5)$$

and observe that by Definition 2.1(i) we have

$$C^{-1} \leq \rho(t, \mathbf{x}) \leq C, \quad C^{-1} \leq R(t, \mathbf{y}) \leq C. \quad (6)$$

The theory in [5,1] guarantees that, given a vector field \mathbf{b} as in (1), there exists a unique regular Lagrangian flow associated with it. Moreover, in [4] the following regularity of the regular Lagrangian flow has been proved.

Theorem 2.2. Let \mathbf{b} be a vector field as in (1) and let \mathbf{X} be the associated regular Lagrangian flow. Assume that $1 < p \leq \infty$. Then, for all $R > 0$ and $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset B_R(\mathbf{0})$ such that

- (1) $\mathbf{X}(t, s, \cdot)$ is Lipschitz continuous on K_ε , uniformly w.r.t. $t, s \in [0, T]$.
- (2) $\mathcal{L}^n(B_R(\mathbf{0}) \setminus K_\varepsilon) \leq \varepsilon$.

The restriction to the case $p > 1$ in Theorem 2.2 and therefore in Theorem 1.1 is due to the use of some harmonic analysis estimates in its proof.

We finally introduce the following concept of directional Lipschitz continuity:

Definition 2.3. Let ϕ be defined on a Borel set $B \subset [0, T] \times \mathbb{R}^n$ and let $\mathbf{Z}(t, \mathbf{y}) : [0, T] \times A \rightarrow \mathbb{R}^n$ be a Borel map, where $A \subset \mathbb{R}^n$ is a Borel set. We say that the function ϕ is (L, \mathbf{Z}) -directionally Lipschitz continuous if for all $t, t' \in [0, T]$ and for all $\mathbf{y} \in A$ such that $\mathbf{Z}(t, \mathbf{y}), \mathbf{Z}(t', \mathbf{y}) \in B$ there holds

$$|\phi(t, \mathbf{Z}(t, \mathbf{y})) - \phi(t', \mathbf{Z}(t', \mathbf{y}))| \leq L|t - t'|.$$

We focus in this paper only on directional Lipschitz continuity in the specific case $\mathbf{Z}(t, \mathbf{y}) = \mathbf{X}(t, 0, \mathbf{y})$, where \mathbf{X} is a regular Lagrangian flow.

3. Proof of Theorem 1.1: disintegration along the regular Lagrangian flow

In this section, we give a complete proof of Theorem 1.1, under the additional Assumption 3.3 on the existence of a directional Lipschitz extension that we introduce in Step 2 here below. A proof of Assumption 3.3 is sketched in §4 below, and a full proof deferred to [2].

Step 0. By the linearity of the continuity equation (2), it is enough to prove that $u_0 \equiv 0$ implies $u \equiv 0$. We do this by showing that every distributional solution u of (2) satisfies a Lagrangian formulation. In this context this amounts to the fact that the function

$$U(t, \mathbf{y}) := u(t, \mathbf{X}(t, 0, \mathbf{y})) \tag{7}$$

solves in distributional sense the equation $\partial_t[U/R] = 0$, where R is defined in (5), with initial datum $U(t = 0, \mathbf{y}) \equiv 0$, that is

$$\int_0^T \int_{\mathbb{R}^n} \frac{U(t, \mathbf{y})}{R(t, \mathbf{y})} \partial_t \Psi(t, \mathbf{y}) \, dt \, d\mathbf{y} = 0 \quad \text{for all test functions } \Psi(t, \mathbf{y}) = \Psi_1(t) \Psi_2(\mathbf{y}), \tag{8}$$

where $\Psi_1(t) \in \text{Lip}_c([0, T])$ and $\Psi_2 \in L_c^\infty(\mathbb{R}^n)$, the spaces of Lipschitz functions with compact support, and of essentially bounded functions with compact support, respectively. Notice that the validity of (8) implies that $U \equiv 0$, and thus with (7) we obtain $u \equiv 0$. Since $\text{Lip}_c(\mathbb{R}^n)$ is dense in $L_c^\infty(\mathbb{R}^n)$ with respect to the weak star topology of $L^\infty(\mathbb{R}^n)$, we reduced the proof of Theorem 1.1 to the proof of the following claim:

Claim 3.1. *The Lagrangian formulation (8) holds for every $\Psi \in \text{Lip}_c([0, T] \times \mathbb{R}^n)$.*

We fix

$$\text{a function } \Psi \text{ as in Claim 3.1 and we set } L = \text{Lip}(\Psi). \tag{9}$$

We prove in the next steps that Claim 3.1 holds.

Step 1. Fix $\varepsilon > 0$ and consider a compact set of the form $[0, T] \times B_R(\mathbf{0})$ that contains the support of the function Ψ fixed in (9). We use Theorem 2.2 to find a compact subset $K_\varepsilon \subset B_R(\mathbf{0})$ on which the regular Lagrangian flow $\mathbf{X}(t, s, \cdot)$ is uniformly Lipschitz continuous.

Lemma 3.2. *On the compact flow tube $\{\mathbf{X}(t, 0, K_\varepsilon)\}_{t \in [0, T]}$ starting from K_ε , the function*

$$\psi(t, \mathbf{x}) := \Psi(t, \mathbf{X}(0, t, \mathbf{x})) \quad \text{for } \mathbf{x} \in \mathbf{X}(t, 0, K_\varepsilon) \tag{10}$$

is Lipschitz continuous and (L, \mathbf{X}) -directionally Lipschitz continuous, with L as in (9).

Proof. We start by proving the (L, \mathbf{X}) -directional Lipschitz continuity. Let

$$\mathbf{x} = \mathbf{X}(t, 0, \mathbf{y}) \text{ and } \mathbf{x}' = \mathbf{X}(t', 0, \mathbf{y}'), \quad \text{so that by (10) } \psi(t, \mathbf{x}) = \Psi(t, \mathbf{y}) \text{ and } \psi(t', \mathbf{x}') = \Psi(t', \mathbf{y}'),$$

and thus by (9) we get

$$|\psi(t, \mathbf{x}) - \psi(t', \mathbf{x}')| = |\Psi(t, \mathbf{y}) - \Psi(t', \mathbf{y}')| \leq \text{Lip}(\Psi) |t - t'| = L |t - t'|.$$

We now prove the Lipschitz continuity of ψ on $\{\mathbf{X}(t, 0, K_\varepsilon)\}_{t \in [0, T]}$. Given $\mathbf{x}, \mathbf{x}^* \in \mathbf{X}(t, 0, K_\varepsilon)$ one has

$$|\psi(t, \mathbf{x}) - \psi(t, \mathbf{x}^*)| = |\Psi(t, \mathbf{X}(0, t, \mathbf{x})) - \Psi(t, \mathbf{X}(0, t, \mathbf{x}^*))| \leq \text{Lip}(\Psi) \cdot \text{Lip}(\mathbf{X}(0, t, \cdot)|_{K_\varepsilon}) |\mathbf{x} - \mathbf{x}^*|.$$

When comparing two points $\mathbf{x} \in \mathbf{X}(t, 0, \mathbf{y})$ and $\mathbf{x}' \in \mathbf{X}(t', 0, \mathbf{y}')$, for some $\mathbf{y}, \mathbf{y}' \in K_\varepsilon$, we simply define $\mathbf{x}^* = \mathbf{X}(t, 0, \mathbf{y}')$ and we estimate

$$\begin{aligned} |\psi(t, \mathbf{x}) - \psi(t', \mathbf{x}')| &\leq |\psi(t, \mathbf{x}) - \psi(t, \mathbf{x}^*)| + |\psi(t, \mathbf{x}^*) - \psi(t', \mathbf{x}')| \\ &\leq C(\text{Lip}(\Psi), \text{Lip}(\mathbf{X}(0, t, \cdot)|_{K_\varepsilon}), \|\mathbf{b}\|_\infty) \cdot (|\mathbf{x} - \mathbf{x}^*| + |t - t'|) \\ &\leq C(\text{Lip}(\Psi), \text{Lip}(\mathbf{X}(0, t, \cdot)|_{K_\varepsilon}), \|\mathbf{b}\|_\infty) \cdot (|t - t'| + |\mathbf{x} - \mathbf{x}'|), \end{aligned}$$

where in the last inequality we applied

$$|\mathbf{x} - \mathbf{x}^*| \leq |\mathbf{x} - \mathbf{x}'| + |\mathbf{x}' - \mathbf{x}^*| \quad \text{and} \quad |\mathbf{x}^* - \mathbf{x}'| \leq \|\mathbf{b}\|_\infty |t' - t|.$$

This concludes the proof of the lemma. \square

Step 2. We can proceed with the proof under the following assumption.

Assumption 3.3. Given $\varepsilon > 0$ let ψ be as in (10). We assume that there exists $\psi_\varepsilon : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ which is an extension of ψ and in addition is

- (1) Lipschitz continuous, and
- (2) (L', \mathbf{X}) -directionally Lipschitz continuous, where $L' > 0$ does not depend on ε .

In fact, we are able to prove that [Assumption 3.3](#) holds when the vector field \mathbf{b} satisfies the continuity condition assumed in [Theorem 1.1](#). In Section 4, we give a sketch of the proof of this fact, and we defer a complete proof to a next paper.

Step 3. We now derive some consequences of [Assumption 3.3](#) in the (t, \mathbf{y}) -variables. We define

$$\Psi_\varepsilon(t, \mathbf{y}) := \psi_\varepsilon(t, \mathbf{X}(t, 0, \mathbf{y})) \quad \text{for } t \in [0, T] \text{ and } \mathbf{x} \in \mathbb{R}^n \tag{11}$$

and we observe that

- (1) $\Psi_\varepsilon(\cdot, \mathbf{y})$ is L' -Lipschitz continuous for all \mathbf{y} . This follows from [Assumption 3.3\(ii\)](#) and from the definition of directional Lipschitz continuity ([Definition 2.3](#));
- (2) $\Psi_\varepsilon(t, \mathbf{y}) \equiv \Psi(t, \mathbf{y})$ for every $\mathbf{y} \in K_\varepsilon$ and every $t \in [0, T]$.

In particular, we can test $\partial_t[U/R](t, \mathbf{y})$ against $\Psi_\varepsilon(t, \mathbf{y})$: by the definitions in (7) and (11) we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \frac{U(t, \mathbf{y})}{R(t, \mathbf{y})} \partial_t \Psi_\varepsilon(t, \mathbf{y}) \, dt \, d\mathbf{y} &= \int_0^T \int_{\mathbb{R}^n} \frac{u(t, \mathbf{X}(t, 0, \mathbf{y}))}{\rho(t, \mathbf{X}(t, 0, \mathbf{y}))} \frac{d}{dt} \psi_\varepsilon(t, \mathbf{X}(t, 0, \mathbf{y})) \, dt \, d\mathbf{y} \\ &= \int_0^T \int_{\mathbb{R}^n} \frac{u(t, \mathbf{X}(t, 0, \mathbf{y}))}{\rho(t, \mathbf{X}(t, 0, \mathbf{y}))} [(\partial_t \psi_\varepsilon)(t, \mathbf{X}(t, 0, \mathbf{y})) + \mathbf{b}(t, \mathbf{X}(t, 0, \mathbf{y})) \cdot (\nabla \psi_\varepsilon)(t, \mathbf{X}(t, 0, \mathbf{y}))] \, dt \, d\mathbf{y}. \end{aligned}$$

We now apply the change of variable $\mathbf{x} = \mathbf{X}(t, 0, \mathbf{y})$, obtaining

$$\int_0^T \int_{\mathbb{R}^n} \frac{U(t, \mathbf{y})}{R(t, \mathbf{y})} \partial_t \Psi_\varepsilon(t, \mathbf{y}) \, dt \, d\mathbf{y} = \int_0^T \int_{\mathbb{R}^n} u(t, \mathbf{x}) [\partial_t \psi_\varepsilon(t, \mathbf{x}) + \mathbf{b}(t, \mathbf{x}) \cdot \nabla \psi_\varepsilon(t, \mathbf{x})] \, dt \, d\mathbf{x} = 0, \tag{12}$$

because u is a distributional solution to (2) with zero initial datum. We stress that the first equality in (12) follows by the definition of push-forward measure, because the results in [5] establish that the regular Lagrangian flow \mathbf{X} satisfies the absolute continuity estimate in [Definition 2.1\(1\)](#). This is a very important brick in this disintegration strategy, and in other settings it requires to be proved ad hoc, see for instance [3].

Step 4. We conclude the proof of [Claim 3.1](#), thus establishing [Theorem 1.1](#) under [Assumption 3.3](#). The main observation is that equation (12) gives the validity of [Claim 3.1](#) with the test function Ψ replaced by the approximation Ψ_ε defined in (11). Therefore, we simply estimate the integral containing Ψ with the integral containing Ψ_ε plus an error, and we only need to show that the error converges to zero as $\varepsilon \downarrow 0$. Indeed, we compute as follows:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \frac{U}{R} \partial_t \Psi \, dt \, d\mathbf{y} &= \int_0^T \int_{\mathbb{R}^n} \frac{U}{R} \partial_t \Psi_\varepsilon \, dt \, d\mathbf{y} + \int_0^T \int_{\mathbb{R}^n} \frac{U}{R} \partial_t [\Psi - \Psi_\varepsilon] \, dt \, d\mathbf{y} \\ &= \int_0^T \int_{K_\varepsilon} \frac{U}{R} \partial_t [\Psi - \Psi_\varepsilon] \, dt \, d\mathbf{y} + \int_0^T \int_{B_R(\mathbf{0}) \setminus K_\varepsilon} \frac{U}{R} \partial_t [\Psi - \Psi_\varepsilon] \, dt \, d\mathbf{y}, \end{aligned}$$

where K_ε is as in Step 1, and by construction $\Psi_\varepsilon \equiv \Psi$ on $[0, T] \times K_\varepsilon$. Since $\Psi(\cdot, \mathbf{y})$ is L -Lipschitz continuous by definition (9) and each $\Psi_\varepsilon(\cdot, \mathbf{y})$ is L' -Lipschitz continuous by Step 3(1), we finally get

$$\left| \int_0^T \int_{\mathbb{R}^n} \frac{U}{R} \partial_t \Psi \, dt \, d\mathbf{y} \right| \leq C(L + L') \int_0^T \int_{B_R(\mathbf{0}) \setminus K_\varepsilon} |U| \, dt \, d\mathbf{y} \xrightarrow{\varepsilon \downarrow 0} 0,$$

using (6) and the fact that the function U in (7) belongs to $L^1_{\text{loc}}([0, T] \times \mathbb{R}^n)$. This concludes the proof of [Theorem 1.1](#) under [Assumption 3.3](#).

4. Idea of the proof of [Assumption 3.3](#): directional Lipschitz extension lemma

We finally briefly sketch the strategy of proof of the following lemma. A full proof in a more general context is deferred to [2].

Lemma 4.1. *Let \mathbf{b} be a vector field as in (1), with $1 < p \leq \infty$. Assume in addition that $\mathbf{b}(t, \cdot)$ is continuous for \mathcal{L}^1 -a.e. $t \in [0, T]$, with modulus of continuity on compact sets which is uniform in time. Then [Assumption 3.3](#) holds.*

In the above lemma, one can as well require that $\min \psi \leq \psi_\varepsilon \leq \max \psi$.

We start by noticing that a function ψ is (L, \mathbf{X}) -directionally Lipschitz continuous according to Definition 2.3 if and only if ψ is L -Lipschitz continuous for the following degenerate distance d_0 :

$$d_0((t, \mathbf{x}), (t', \mathbf{x}')) := \begin{cases} |t - t'| & \text{if there exists } \mathbf{y} \text{ with } \mathbf{x} = \mathbf{X}(t, 0, \mathbf{y}) \text{ and } \mathbf{x}' = \mathbf{X}(t', 0, \mathbf{y}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, we denote by d_1 the usual Euclidean distance in $[0, T] \times \mathbb{R}^n$.

Consider a Lipschitz continuous function ψ defined on the compact flow tube $\{\mathbf{X}(t, 0, K_\varepsilon)\}_{t \in [0, T]}$ of Assumption 3.3. We remind that we assume that ψ is (L, \mathbf{X}) -directionally Lipschitz continuous, and that we need to extend ψ to $[0, T] \times \mathbb{R}^n$ in such a way that the extension is

- (1) (L, \mathbf{X}) -directionally Lipschitz continuous, i.e. L' -Lipschitz continuous for d_0 , with L' depending on L , and
- (2) Lipschitz continuous for the Euclidean distance, i.e. Lipschitz continuous for d_1 .

In other words, we need to prove a Lipschitz extension theorem with respect to two non-equivalent distances at the same time: to the best of our knowledge, this is a new and non trivial task. Notice that for our purposes, we need that the Lipschitz constant for d_0 only depends on L , while we do not need a quantitative control on the Lipschitz constant for d_1 .

We now give a rough idea of the proof of Lemma 4.1. For $0 < \lambda < 1$, we introduce a family of distances d_λ , each of them equivalent to the Euclidean distance d_1 . The distance d_λ penalizes with a factor λ^{-1} displacements which are not along the flow. Moreover, the distances d_λ converge to the degenerate distance d_0 , i.e. $d_\lambda \uparrow d_0$ as $\lambda \downarrow 0$. In particular, a function which is L' -Lipschitz continuous for d_λ is also L' -Lipschitz continuous for d_0 .

The key point in the proof of Lemma 4.1 is the fact that, when $\lambda \downarrow 0$, the Lipschitz constant of ψ for d_λ converges to the Lipschitz constant L of ψ for d_0 :

$$L_\lambda := \text{Lip}(\psi; d_\lambda) \xrightarrow{\lambda \downarrow 0} \text{Lip}(\psi; d_0) = L. \quad (13)$$

Using this property, we choose $\bar{\lambda}$ small enough so that $L_{\bar{\lambda}}$ is close to L . We extend ψ by using McShane extension theorem for the distance $d_{\bar{\lambda}}$. In this way, we get an extension that is

- (1) $(L_{\bar{\lambda}}, \mathbf{X})$ -directionally Lipschitz continuous, and $L_{\bar{\lambda}}$ is close to L , and
- (2) Lipschitz continuous for the Euclidean distance d_1 , since $d_{\bar{\lambda}}$ is equivalent to d_1 .

In the above procedure, we are currently able to prove (13) only assuming that the vector field \mathbf{b} is continuous for \mathcal{L}^1 -a.e. $t \in [0, T]$, with modulus of continuity on compact sets which is uniform in time.

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References

- [1] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, *Invent. Math.* 158 (2) (2004) 227–260.
- [2] L. Caravenna, G. Crippa, work in progress.
- [3] L. Caravenna, S. Daneri, A disintegration of the Lebesgue measure on the faces of a convex function, *J. Funct. Anal.* 258 (2010) 3604–3661.
- [4] G. Crippa, C. De Lellis, Estimates and regularity results for the DiPerna–Lions flow, *J. Reine Angew. Math.* 616 (2008) 15–46.
- [5] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, *Invent. Math.* 98 (3) (1989) 511–547.