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An existence theorem for multidimensional BSDEs with mixed reflections ☆

Un théorème d'existence pour les EDSRs multidimensionnelles avec réflexions mixtes

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ABSTRACT

In this note, we consider the pricing problem for a type of real option, which gives the right to switch investment modes and abandon the investment project before its maturity. The value of this option can be characterized by solutions to multidimensional backward stochastic differential equations (BSDEs) with both normal and oblique reflections, whose coefficients are of linear growth and are left-Lipschitz with respect to (w.r.t) y and Lipschitz w.r.t. z. We provide an existence theorem of minimal solutions for BSDEs in this framework.

RÉSUMÉ

Dans cette note, on considère le problème du *pricing* pour un certain type d'option réelle, donnant les droits de changer le mode d'investissement et d'abandonner le projet d'investissement avant son échéance. La valeur de cette option peut être caractérisée par les solutions des équations différentielles stochastiques rétrogrades (EDSRs) avec deux types de réflexions aux bords, normale et oblique, dont les coefficients sont à croissance linéaire et sont lipschitziens en y à gauche ainsi que lipschitziens en z. Dans ce cadre, on fournit un théorème d'existence des solutions minimales pour les EDSRs.

1. Introduction

The reflected backward stochastic differential equations (RBSDEs for short) were first studied by El Karoui et al. [5] for the one-dimensional case with lower obstacles, and then by Cvitanić and Karatzas [3] with upper and lower obstacles. RBSDEs

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are closely related to the problem of optimal stopping and Dykin games. There is a rich literature on RBSDEs. Readers are referred to [1,6,14,18], among many others. Multidimensional BSDEs with oblique reflections and Lipschitz coefficients have been studied by Hamadène and Jeanblanc [7], Hu and Tang [9,10], and Hamadene and Zhang [8]. They first arise in the problem of optimal switching, see Carmona and Ludkovski [2] and Ludkovski [12], where they presented some applications on energy pricing and investment timing models. See also Porchect, Touzi and Warin [15] and references therein.

Problems with both optimal switching and stopping via the RBSDE approach were first studied by Tang and Zhong [17] and Zhong [19], as far as the author knows. Zhong [19] proved a uniqueness and existence theorem for the solution to RBSDE (1) by imposing Lipschitz conditions on the coefficients. A uniqueness and existence theorem of RBSDE (1) was proved by Tang and Zhong [17] under Lipschitz conditions on the coefficients, but with reflections of $S_t^i \geq Y_t^i \geq h_{ij}\left(t, Y_t^j\right)$. Note that it is required in their paper that $h_{ij}(t, y) \leq y$. Obviously, the lower obstacle in RBSDE (1) is not the case. The superimposed obstacles here make significant sense from the point of view of economics, as illustrated later, and they also bring additional complexity to the proof for the continuity of the first part of the solution.

There is a rigorous financial context for optimal switching and stopping problems. To illustrate this, consider an investment which can be carried out under several modes, for example, under different production scales or techniques. Thus the investment project receives a different cash flow under each mode. The decision-maker can switch among these modes according to the product prices. Each switch needs certain nonnegative cost. Of course, the decision-maker is allowed to stop the project before its maturity when encountering disastrous news. The goal of the investor is to maximize the accumulative yields by adopting optimal switching and stopping decisions. The project is in fact a real option including a switch option and an abandon option. Techniques of evaluating real options have been introduced to capture the value of flexibility or optionality embedded in investments. See, for instance, Dixit and Pindyck [4] and Schwartz and Trigeorgis [16] for the fundamental concepts of this theory.

When there are two or more options in an investment, they are typically not independent. The value of the multiple option is generally not the sum of each option. For instance, there is a project containing a switch option and an abandon option. The project can not switch if it has already been abandoned. Furthermore, the value of the abandon option also depends on which mode the project has been switched to. Pricing such multiple real options is equivalent to solve a multidimensional RBSDE with normal and oblique reflections. Sometimes the coefficients of RBSDEs are not necessarily continuous. This paper provides an existence theorem when the coefficient is discontinuous with respect to (w.r.t. for short) y and Lipschitz w.r.t. z. Compared with [17] and [19], the non-uniqueness of solutions increases the difficulties in checking the last equality in RBSDE (1) after convergence of the approaching sequence. The solution we obtain is the minimal one in a strong order, i.e. it is minimal in each dimension. Furthermore, the main result also holds when y^i, $i = 1, \dots m$ are not only interacted in the obstacles, but also in the generators.

2. Preliminaries

Let $(B_t)_{t\in[0,T]}$ be a standard d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}_T, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ the usual augmented Brownian filtration. $T > 0$ is a fixed time. Let $\Lambda := \{1, \dots, m\}$, $m > 1$, denote all possible modes and $\mathcal{X}$ the indicator function. The process

$$a(s) = \alpha_0 \mathcal{X}_{[\theta_0,\theta_1]}(s) + \sum_{i=1}^{N} \alpha_i \mathcal{X}_{(\theta_i,\theta_{i+1}]}(s), s \in [\theta_0, \theta_N]$$

is called an admissible switching strategy if

(i) $\{\theta_i\}_{i=0}^{\infty}$ is an increasing sequence of stopping times that represent switching times and, for any i, α_i is an $\mathcal{F}_{\theta_i}$-measurable random variable valued in Λ,

(ii) there exists an integer-valued random variable N such that the stopping time $\theta_N \in [0, T]$ and the project will stop at θ_N.

We make the following notations:

$\mathcal{A}_0^i$: the set of admissible strategies starting at time $\theta_0 = 0$ with initial mode $\alpha_0 = i$;
(X_t): the price process of the product;
$\psi(t, x, a(t))$: the instantaneous yield depending on the price of products;
$\phi(t)$: the cost of stopping the project, for instance, the termination allowance of workers;
$k(i, j)$: the cost when the project switching from state i to state j.

For a given admissible strategy $a(\cdot)$, the dynamic yield follows:

$$Y_t^a = -\sum_{j=1}^{N-1} k\left(\alpha_{j-1}, \alpha_j\right) - \phi(\theta_N)\mathcal{X}_{\{\theta_N<T\}} + \int_t^{\theta_N} \psi(s, X_s, a(s))\,\mathrm{d}s - \int_t^{\theta_N} Z_s^a\,\mathrm{d}B_s, t \in [0, \theta_N].$$

Define $J(a) = Y_0^a$. Then we want to find an optimal decision $a^* \in \mathcal{A}_0^i$ such that

$$J\left(a^*\right) = \max_{a\in\mathcal{A}_0^i} J(a).$$

Under certain conditions, the above optimal switching and stopping problem is equivalent to the following RBSDE:

$$\begin{cases} Y_t^i = \int_t^T \psi(s, X_s, i)\,ds + \int_t^T dK_s^i - \int_t^T Z_s^i\,dB_s, & t \in [0,T], \\ Y_t^i \geq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\}, Y_t^i \geq -\phi(t), & \\ \int_0^T \left(Y_t^i - \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee (-\phi(t))\right) dK_t^i = 0. & \end{cases}$$

Zhong [19] proved that $Y_0^i = \max_{a\in\mathcal{A}_0^i} J(a)$. We now consider the following m-dimensional RBSDE with a more general coefficient defined on $[0,T]$:

$$\begin{cases} Y_t^i = \xi^i + \int_t^T g^i\left(s, Y_s^i, Z_s^i\right) ds + \int_t^T dK_s^i - \int_t^T Z_s^i\,dB_s, & t \in [0,T], \\ Y_t^i \geq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\}, Y_t^i \geq S_t^i, & \\ \int_0^T \left(Y_t^i - \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i\right) dK_t^i = 0. & \end{cases} \tag{1}$$

The function $g = \left(g^1, \ldots, g^m\right)$ with $g^i : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \longmapsto \mathbf{R}$ is called the generator of RBSDE (1) and $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^m)$ is the terminal datum, where $L^2(\mathcal{F}_T; \mathbf{R}^m)$ denotes all the $\mathbf{R}^m$-valued $\mathcal{F}_T$-measurable square-integrable random variables. $k(\cdot,\cdot)$ is a real function defined on $\Lambda \times \Lambda$. The unknown processes $(Y_t)_{t\in[0,T]}$, $(Z_t)_{t\in[0,T]}$ and $(K_t)_{t\in[0,T]}$ are required to be adapted w.r.t. the natural Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Furthermore, $\left(K_t^i\right)$ is an increasing process for each $i \in \Lambda$.

RBSDE (1) evolves in the closure $\bar{Q}$ of the domain Q:

$$Q(t) := \left\{\left(y^1, \ldots, y^m\right) \in \mathbf{R}^m : y^i > \max_{j\neq i}\left\{y^j - k(i,j)\right\} \vee S_t^i, \forall i, j \in \Lambda, j \neq i\right\},$$

which is a nonempty time-depending random set. On the boundary ∂Q, the ith equation is switched to another one. The solution is reflected along some oblique direction $y^i = y^j - k(i,j)$ and normal direction $y = S_t^i$. We denote by $\mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d})$ the space of all $(\mathcal{F}_t)$-progressively measurable $\mathbf{R}^{m\times d}$-valued processes such that $\mathbf{E}\left[\int_0^T |\psi_t|^2\,dt\right] < \infty$ and $\mathcal{S}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ the space of all càdlàg[1] processes in $\mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ such that $\mathbf{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < \infty$. $\mathcal{N}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ is defined as follows:

$$\mathcal{N}_{\mathcal{F}}^2(0,T;\mathbf{R}^m) := \{K = \left(K^1, \ldots, K^m\right) \in \mathcal{S}_{\mathcal{F}}^2(0,T;\mathbf{R}^m) : \text{for each } i \in \Lambda, K^i(0) = 0 \text{ and } t \mapsto K^i(t) \text{ is increasing}\}.$$

We denote by $\mathcal{S}_c^2(0,T;\mathbf{R}^m)$ ($\mathcal{N}_c^2(0,T;\mathbf{R}^m)$) the space of all continuous processes in $\mathcal{S}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ ($\mathcal{N}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ resp.). We make the following assumptions throughout the paper.

(**H1**) For each $i \in \Lambda$, $\forall t$, $\forall (y,z)$,

$$|g^i(t,y,z)| \leq L(1 + |y| + |z|), L \geqslant 0.$$

(**H2**) For each $i \in \Lambda$, $\forall t$, $\forall (y,y')$, $\forall (z,z')$ such that $y \geq y'$, we have

$$g^i(t,y,z) - g^i\left(t,y',z'\right) \geq -L\left(\left(y - y'\right) + \left|z - z'\right|\right)$$

and $g^i(t,\cdot,z)$ is left-continuous.

(**H3**) $k(i,j) > 0, k(i,i) = 0,\ i,j \in \Lambda, i \neq j;\ k(i,j) + k(j,l) > k(i,l),\ i,j,l \in \Lambda,\ i \neq j,\ j \neq l.$

(**H4**) For each $i \in \Lambda$, $t \mapsto S_t^i$ is continuous, $\left(S_t^i\right)^+ \in \mathcal{S}_{\mathcal{F}}^2$ and $S_t^i \geq \max_{j\neq i}\left\{S_t^j - k(i,j)\right\}$.

Remark 2.1. Condition (H2) implies that g^i is in fact Lipschitz-continuous w.r.t. z. Taking $y' = y$, we obtain that $g^i(t,y,z) - g^i\left(t,y,z'\right) \geq -L\left|z - z'\right|$, then interchanging the position of z, z', we have $g^i(t,y,z) - g^i\left(t,y,z'\right) \leq L|z - z'|$, therefore g^i is Lipschitz-continuous w.r.t. z. Condition (H2) has been used by Jia [11] to obtain solutions for standard BSDEs.

Remark 2.2. Condition (H3) implies that there is no sequence $i_2 \in \Lambda\backslash i_1, \ldots, i_k \in \Lambda\backslash i_{k-1}, i_1 \in \Lambda\backslash i_k$ and $\left(y_{i_1}, \ldots, y_{i_k}\right)$, $k \in \Lambda$, such that $y_{i_1} = y_{i_2} - k(i_1,i_2), y_{i_2} = y_{i_3} - k(i_2,i_3), \ldots, y_{i_{k-1}} = y_{i_k} - k\left(i_{k-1},i_k\right), y_{i_k} = y_{i_1} - k(i_k,i_1)$, which means that it is not free to make a circle of instantaneous switchings.

[1] The French abbreviation for right continuous and left limited, or RCLL for short.

Remark 2.3. For normal reflections, the condition $S_t^i \geq \max_{j\neq i}\left\{S_t^j - k(i,j)\right\}$ is necessary. If all modes stop at normal obstacles at the same time, they must evolve in the domain of $\bar{Q}$. A simple example is that all normal obstacles are the same.

The following comparison theorem for RBSDEs is a variant of Bahlali et al. [1] Theorem 2.1 and Hamadène [6] Theorem 1.5.

Lemma 2.1 *(Comparison Theorem). Consider the following two (1-dimensional) RBSDEs with lower barrier $\left(S_t^i\right)$, i=1,2:*

$$\begin{cases} Y_t^i = \xi^i + \int_t^T g^i\left(s, Y_s^i, Z_s^i\right) \mathrm{d}s + \int_t^T \mathrm{d}K_s^i - \int_t^T Z_s^i \,\mathrm{d}B_s, & t \in [0,T], \\ Y_t^i \geq S_t^i, \\ \int_0^T \left(Y_{t^-}^i - S_{t^-}^i\right) \mathrm{d}K_t^i = 0. \end{cases} \tag{2}$$

For each i, assume $\xi^i \in L^2(\mathcal{F}_T;\mathbf{R})$, one of $g^i : \Omega \times [0,T] \times \mathbf{R} \times \mathbf{R}^d \longmapsto \mathbf{R}$ satisfies (a) $g^i(t,0,0) \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R})$, (b) for each $i = 1,2, \forall t, \forall (y,y'), \forall (z,z'), |g^i(t,y,z) - g^i\left(t,y',z'\right)| \leq L\left(|y-y'| + |z-z'|\right), L \geqslant 0$. The barrier $\left(S_t^i\right)$ is an adapted càdlàg process such that $\mathbf{E}[\sup_{0\leq t\leq T} |\left(S_t^i\right)^+|^2] < \infty$.

If $\xi^1 \geq \xi^2$, P-a.s., $S_t^1 \geq S_t^2$, $t \in [0,T]$, P-a.s., g^2 satisfies (a), (b) and $g^1(t,Y_t^1,Z_t^1) \geq g^2(t,Y_t^1,Z_t^1)$ (or g^1 satisfies (a), (b) and $g^1(t,Y_t^2,Z_t^2) \geq g^2(t,Y_t^2,Z_t^2)$), $t \in [0,T]$, P-a.s., then $Y_t^1 \geq Y_t^2$, $\forall t \in [0,T]$, P-a.s.

In equation (2), the lower barrier S_t^i needs not to be continuous w.r.t. t. The existence and uniqueness of solutions to RBSDE (2) with continuous barriers are given in El Karoui et al. [5] and to the RCLL barrier in Hamadène [6] and Peng and Xu [14], among many others. The following lemma comes from Peng [13].

Lemma 2.2 *(Monotonic Limit Theorem). Consider the following family of semi-martingales:*

$$^n y_t = {}^n y(0) + \int_0^t {}^n g(s)\,\mathrm{d}s - \int_0^t \mathrm{d}\left({}^n K_s\right) + \int_0^t {}^n z_s \,\mathrm{d}B_s, n = 1,2,\ldots. \tag{3}$$

Here for each n, $\left({}^n g, {}^n z\right) \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^d)$, $\left({}^n K_s\right)$ is a continuous and increasing process with $\mathbf{E}[|{}^n K_T|^2] < \infty$. We assume further that

(i) $\left({}^n g\right)$ and $\left({}^n z\right)$ are bounded in $\mathcal{M}_{\mathcal{F}}^2(0,T)$: $\mathbf{E}\left[\int_0^T \left(|{}^n g_t|^2 + |{}^n z_t|^2\right)\mathrm{d}t\right] < C$;

(ii) $\left({}^n y_t\right)$ converges increasingly to (y_t) with $\mathbf{E}[\sup_{0\leq t\leq T} |y_t|^2] < \infty$.

Then $\lim_{n\to\infty} \mathbf{E}\left[\int_0^T |{}^n y_t - y_t|^2 \,\mathrm{d}t\right] = 0$ and (y_t) has the form:

$$y_t = y(0) + \int_0^t g(s)\,\mathrm{d}s - \int_0^t \mathrm{d}K_s + \int_0^t z_s \,\mathrm{d}B_s, \tag{4}$$

where (g_s, z_s) is the weak limit of $\left({}^n g_s, {}^n z_s\right)$ in $\mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^d)$ and (K_s) is the weak limit of $\left({}^n K_s\right)$ in $L^2(\mathcal{F}_T;\mathbf{R})$. (K_s) is an RCLL increasing process. Furthermore, for any $p \in [0,2)$, $\lim_{n\to\infty} \mathbf{E}\left[\int_0^T |{}^n z_t - z_t|^p \,\mathrm{d}t\right] = 0$.

3. Main result

We now show the existence of the minimal solution to RBSDE (1) under assumptions (H1)–(H4). For each $i \in \Lambda$, let $({}^0Y_t^i, {}^0Z_t^i, {}^0K_s^i)$ denote the solution to the following RBSDE:

$$\begin{cases} {}^0Y_t^i = \xi^i - \int_t^T L\left(1 + |{}^0Y_s^i| + |{}^0Z_s^i|\right)\mathrm{d}s + \int_t^T \mathrm{d}\left({}^0K_s^i\right) - \int_t^T {}^0Z_s^i \,\mathrm{d}B_s, & t \in [0,T], \\ {}^0Y_t^i \geq S_t^i, \\ \int_0^T \left({}^0Y_t^i - S_t^i\right)\mathrm{d}\left({}^0K_t^i\right) = 0. \end{cases} \tag{5}$$

Let $(\hat{y}_t, \hat{z}_t, \hat{k}_t)$ denote the solution to the following RBSDE:

$$\begin{cases} \hat{y}_t = \sum_{i=1}^m \left|\xi^i\right| + \int_t^T L\left(1 + |\hat{y}_s| + |\hat{z}_s|\right)\mathrm{d}s + \int_t^T \mathrm{d}(\hat{k}_s) - \int_t^T \hat{z}_s \,\mathrm{d}B_s, & t \in [0,T], \\ \hat{y}_t \geq \sum_{i=1}^m \left|S_t^i\right|, \\ \int_0^T \left(\hat{y}_t - \sum_{i=1}^m \left|S_t^i\right|\right)\mathrm{d}(\hat{k}_t) = 0. \end{cases} \tag{6}$$

Consider the following sequence of m-dimensional RBSDEs parameterized by $n = 1, 2, \ldots,$

$$\begin{cases} {}^nY_t^i = \xi^i + \int_t^T \left(g^i\left(s, {}^{(n-1)}Y_s^i, {}^{(n-1)}Z_s^i\right) - L\left(\left({}^nY_s^i - {}^{(n-1)}Y_s^i\right) + \left|{}^nZ_s^i - {}^{(n-1)}Z_s^i\right|\right)\right) \mathrm{d}s, & t \in [0,T], \\ \qquad\qquad + \int_t^T \mathrm{d}\left({}^nK_s^i\right) - \int_t^T {}^nZ_s^i \,\mathrm{d}B_s, \\ {}^nY_t^i \geq \max_{j\neq i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\}, {}^nY_t^i \geq S_t^i, \\ \int_0^T \left({}^nY_t^i - \max_{j\neq i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\} \vee S_t^i\right) \mathrm{d}\left({}^nK_t^i\right) = 0, \end{cases} \tag{7}$$

where $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^m)$ evolves in $\bar{Q}$ and $\xi^i \geq S_T^i$ for each $i \in \Lambda$. For the sequence $\left\{\left({}^nY_t\right)\right\}_{n=1}^\infty$, we have the following result.

Lemma 3.1. *Under conditions (H1)–(H4), the following properties hold true:*

(i) for any $n = 1, 2, \ldots,$ there is a unique solution $\left({}^nY_t, {}^nZ_t, {}^nK_t\right) \in \mathcal{S}_c^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d}) \times \mathcal{N}_c^2(0,T;\mathbf{R}^m)$ to RBSDE (7);

(ii) for each $i \in \Lambda$, for any $n = 1, 2, \ldots,$ ${}^0Y_t^i \leq {}^nY_t^i \leq {}^{(n+1)}Y_t^i \leq \hat{y}_t$, $\forall t \in [0,T]$, P-a.s.

Proof. We first prove the case where $n = 1$. The obstacle can be rewritten as ${}^nY_t^i \geq \max_{j\neq i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\} \vee S_t^i$. Since

$$\mathbf{E}[\sup_{0\leq t\leq T}\left|\left(\max_{j\neq i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\} \vee S_t^i\right)^+\right|^2] \leq C\left(\sum_{i=1}^m \mathbf{E}[\sup_{0\leq t\leq T}|Y_t^i|^2] + \mathbf{E}[\sup_{0\leq t\leq T}\left|\left(S_t^i\right)^+\right|^2]\right) < \infty,$$

by [5], there is a unique solution $\left({}^1Y_t, {}^1Z_t, {}^1K_t\right) \in \mathcal{S}_c^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d}) \times \mathcal{N}_c^2(0,T;\mathbf{R}^m)$ to RBSDE (7).

By condition (H1) and Lemma 2.1, we deduce that ${}^0Y_t^i \leq {}^1Y_t^i$, for each $i \in \Lambda$. By Lemma 2.1 again, ${}^0Y_t^i \leq \hat{y}_t$. Note that the following triple

$$\left(\hat{Y}_t^i, \hat{Z}_t^i, \hat{K}_t^i\right) := \left(\hat{y}_t, \hat{z}_t, \hat{k}_t\right)$$

satisfies the following RBSDE:

$$\begin{cases} \hat{Y}_t^i = \sum_{i=1}^m |\xi^i| + \int_t^T L\left(1 + \left|\hat{Y}_s^i\right| + \left|\hat{Z}_s^i\right|\right) \mathrm{d}s + \int_t^T \mathrm{d}\hat{K}_s^i - \int_t^T \hat{Z}_s^i \,\mathrm{d}B_s, \quad t \in [0,T], \\ \hat{Y}_t^i \geq \max_{j\neq i}\left\{\hat{y}_t - k(i,j)\right\}, \hat{Y}_t^i \geq \sum_{i=1}^m \left|S_t^i\right|, \\ \int_0^T \left(\hat{Y}_t^i - \max_{j\neq i}\left\{\hat{y}_t - k(i,j)\right\} \vee \sum_{i=1}^m \left|S_t^i\right|\right) \mathrm{d}\hat{K}_t^i = 0. \end{cases}$$

It follows

$$\begin{aligned} &L(1 + \left|{}^1Y_s^i\right| + \left|{}^1Z_s^i\right|) - \left[g^i\left(s, {}^0Y_s^i, {}^0Z_s^i\right) - L\left(\left({}^1Y_s^i - {}^0Y_s^i\right) + \left|{}^1Z_s^i - {}^0Z_s^i\right|\right)\right] \\ &\qquad \geq L(1 + \left|{}^1Y_s^i\right| + \left|{}^1Z_s^i\right|) - L(1 + \left|{}^0Y_s^i\right| + \left|{}^0Z_s^i\right|) + L\left(\left({}^1Y_s^i - {}^0Y_s^i\right) + \left|{}^1Z_s^i - {}^0Z_s^i\right|\right) \\ &\qquad \geq 0, \end{aligned}$$

and Lemma 2.1 that ${}^1Y_t^i \leq \hat{Y}_t^i$, $t \in [0,T]$, P-a.s., for each $i \in \Lambda$.

RBSDE (7) also has a unique solution when $n = 2$. By the left-Lipschitz condition (H2), we have

$$\begin{aligned} &\left[g^i\left(s, {}^1Y_s^i, {}^1Z_s^i\right) - L\left(\left({}^1Y_s^i - {}^1Y_s^i\right) + \left|{}^1Z_s^i - {}^1Z_s^i\right|\right)\right] - \left[g^i\left(s, {}^0Y_s^i, {}^0Z_s^i\right) - L\left(\left({}^1Y_s^i - {}^0Y_s^i\right) + \left|{}^1Z_s^i - {}^0Z_s^i\right|\right)\right] \\ &\geq g^i\left(s, {}^1Y_s^i, {}^1Z_s^i\right) - g^i\left(s, {}^0Y_s^i, {}^0Z_s^i\right) + L\left(\left({}^1Y_s^i - {}^0Y_s^i\right) + \left|{}^1Z_s^i - {}^0Z_s^i\right|\right) \\ &\geq 0. \end{aligned}$$

As a consequence of Lemma 2.1, we have ${}^1Y_t^i \leq {}^2Y_t^i$, $t \in [0,T]$, P-a.s., for each $i \in \Lambda$. Similarly to the case of "$n = 1$", it is easy to prove that ${}^2Y_t^i \leq \hat{y}_t$, $\forall t \in [0,T]$, P-a.s.

Assuming ${}^0Y_t^i \leq {}^{(n-1)}Y_t^i \leq {}^nY_t^i \leq \hat{y}_t$, by induction and repeating the above procedure, we can obtain the desired results. □

Now we present an existence theorem for RBSDE (1).

Theorem 3.1. *Let (H1)–(H4) hold and $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^m)$ take values in $\bar{Q}$. Then there is a triple $(Y_t, Z_t, K_t)_{t\in[0,T]} \in \mathcal{S}_c^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d}) \times \mathcal{N}_c^2(0,T;\mathbf{R}^m)$ that solves RBSDE (1). Moreover, the limit (Y_t) of $\left\{\left({}^nY_t\right)\right\}_{n=1}^\infty$ is the minimal solution, i.e. for any other solution $(y_t, z_t, k_t)_{t\in[0,T]}$ to (1), we have, for each $i \in \Lambda$, $Y_t^i \leq y_t^i$, $t \in [0,T]$, P-a.s.*

Proof. By Lemma 3.1, for each $i \in \Lambda$, we have $\sup_n \mathbf{E}[\sup_{0\le t\le T} |{}^nY_t^i|^2] \le \mathbf{E}[\sup_{0\le t\le T} |{}^0Y_t^i|^2] + \mathbf{E}[\sup_{0\le t\le T} |\hat{y}_t|^2] < \infty$. Applying Itô's formula to $\left|{}^nY_t^i\right|^2$ and taking expectation, we have

$$\mathbf{E}\left[|{}^nY^i(0)|^2\right] + \mathbf{E}\left[\int_0^T |{}^nZ_t^i|^2\,\mathrm{d}t\right] = \mathbf{E}\left[|\xi^i|^2\right] + 2\mathbf{E}[\int_0^T {}^nY_t^i(g^i\left(s, {}^{(n-1)}Y_s^i, {}^{(n-1)}Z_s^i\right) - L(\left({}^nY_s^i - {}^{(n-1)}Y_s^i\right)$$

$$+ \left|{}^nZ_s^i - {}^{(n-1)}Z_s^i\right|))\,\mathrm{d}t + \int_0^T {}^nY_t^i\mathrm{d}\left({}^nK_t^i\right)]$$

$$\le C_1 + \frac{1}{16}\mathbf{E}[\int_0^T \left(|{}^{(n-1)}Z_t^i|^2 + |{}^nZ_t^i|^2\right)\mathrm{d}t] + \beta\mathbf{E}[|{}^nK_T^i|^2],$$

where $C_1 = \mathbf{E}\left[|\xi^i|^2\right] + T + \left(88L^2 + \frac{1}{\beta}\right)\left(\mathbf{E}[\sup_{0\le t\le T} |{}^0Y_t^i|^2] + \mathbf{E}[\sup_{0\le t\le T} |\hat{y}_t|^2]\right)$, $\beta > 0$. On the other hand,

$$-{}^nK_t^i = {}^nY_t^i - {}^nY_0^i + \int_0^t \left(g^i\left(s, {}^{(n-1)}Y_s^i, {}^{(n-1)}Z_s^i\right) - L\left(\left({}^nY_s^i - {}^{(n-1)}Y_s^i\right) + \left|{}^nZ_s^i - {}^{(n-1)}Z_s^i\right|\right)\right)\mathrm{d}s - \int_0^t {}^nZ_s^i\,\mathrm{d}B_s.$$

Thus

$$\mathbf{E}[|{}^nK_T^i|^2] \le C_2 + \left(8L^2 + 2\right)\mathbf{E}[\int_0^T \left(|{}^{(n-1)}Z_t^i|^2 + |{}^nZ_t^i|^2\right)\mathrm{d}t],$$

where $C_2 = 8\left(\mathbf{E}|\xi^i|^2 + |{}^0Y_0^i|^2 + |\hat{y}_0|^2\right) + 16L^2\left(\mathbf{E}[\sup_{0\le t\le T} |{}^0Y_t^i|^2] + \mathbf{E}[\sup_{0\le t\le T} |\hat{y}_t|^2]\right)$. Take $\beta = \frac{1}{16(8L^2+2)}$. Consequently, we have:

$$\mathbf{E}\left[\int_0^T |{}^nZ_t^i|^2\,\mathrm{d}t\right] \le \frac{1}{8}\mathbf{E}[\int_0^T \left(|{}^{(n-1)}Z_t^i|^2 + |{}^nZ_t^i|^2\right)\mathrm{d}t] + C_1 + \frac{C_2}{16(8L^2+2)}.$$

Hence

$$\mathbf{E}\left[\int_0^T |{}^nZ_t^i|^2\,\mathrm{d}t\right] \le \frac{1}{7}\mathbf{E}[\int_0^T |{}^{(n-1)}Z_t^i|^2\,\mathrm{d}t] + \frac{1}{7}(C_1 + \frac{C_2}{16(8L^2+2)}),$$

which yields that $\sup_n \mathbf{E}[\int_0^T |{}^nZ_t^i|^2\,\mathrm{d}t] < \infty$ and $\sup_n \mathbf{E}[|{}^nK_T^i|^2] < \infty$.

By the Monotonic Limit Theorem (Lemma 2.2), for each $i \in \Lambda$, there is a triple $(Y_t^i, Z_t^i, K_t^i)_{t\in[0,T]} \in \mathcal{S}^2_{\mathcal{F}}(0,T;R) \times\mathcal{M}^2_{\mathcal{F}}(0,T;R^d) \times \mathcal{N}^2_{\mathcal{F}}(0,T;R)$ such that $\lim_{n\to\infty} \mathbf{E}\left[\int_0^T |{}^nY_t^i - Y_t^i|^2\,\mathrm{d}t\right] = 0$ and $\left(Y_t^i\right)$ has the form:

$$Y_t^i = \xi^i + \int_t^T g_s^i\,\mathrm{d}s + \int_t^T \mathrm{d}K_s^i - \int_t^T Z_s^i\,\mathrm{d}B_s,$$

where $\left(g_s^i, Z_s^i\right)$ is the weak limit of $\left({}^ng_s^i, {}^nZ_t^i\right)$ in $\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbf{R}) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbf{R}^d)$ and $\left(K_s^i\right)$ is the weak limit of $\left({}^nK_s^i\right)$ in $L^2(\mathcal{F}_T;\mathbf{R})$. (K_s) is an RCLL increasing process. Furthermore, for any $p \in [0,2)$, $\lim_{n\to\infty} \mathbf{E}\left[\int_0^T |{}^nZ_t^i - Z_t^i|^p\,\mathrm{d}t\right] = 0$ and $g_s^i = g^i\left(s, Y_s^i, Z_s^i\right)$ by the left-continuity of $g^i(s,\cdot,z)$ and Lipschitz-continuity of $g^i(s,y,\cdot)$.

Passing to the limit on both sides of ${}^nY_t^i \ge \max_{j\ne i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\}$ and ${}^nY_t^i \ge S_t^i$, we obtain $Y_t^i \ge \max_{j\ne i}\left\{Y_t^j - k(i,j)\right\}$ and $Y_t^i \ge S_t^i$. Thus (Y_t^i, Z_t^i, K_t^i) satisfies

$$\begin{cases} Y_t^i = \xi^i + \int_t^T g^i\left(s, Y_s^i, Z_s^i\right)\mathrm{d}s + \int_t^T \mathrm{d}K_s^i - \int_t^T Z_s^i\,\mathrm{d}B_s, & t \in [0,T], \\ Y_t^i \ge \max_{j\ne i}\left\{Y_t^j - k(i,j)\right\}, Y_t^i \ge S_t^i. \end{cases}$$

Let $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)$ be the limit of sequence $({}^n\tilde{Y}_t, {}^n\tilde{Z}_t, {}^n\tilde{K}_t)$, which satisfies

$$\begin{cases} {}^n\tilde{Y}_t = \xi^i + \int_t^T \left(g^i\left(s, {}^{(n-1)}\tilde{Y}_s^i, {}^{(n-1)}\tilde{Z}_s^i\right) - L\left(\left({}^n\tilde{Y}_s^i - {}^{(n-1)}\tilde{Y}_s^i\right) + \left|{}^n\tilde{Z}_s^i - {}^{(n-1)}\tilde{Z}_s^i\right|\right)\right)\mathrm{d}s & t \in [0,T], \\ \qquad + \int_t^T \mathrm{d}\left({}^n\tilde{K}_s^i\right) - \int_t^T {}^n\tilde{Z}_s^i \,\mathrm{d}B_s, & \\ {}^n\tilde{Y}_t^i \geq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i, & \\ \int_0^T \left({}^n\tilde{Y}_{t-}^i - \max_{j\neq i}\left\{Y_{t-}^j - k(i,j)\right\} \vee S_t^i\right)\mathrm{d}\left({}^n\tilde{K}_t^i\right) = 0, & \end{cases} \tag{8}$$

where we set $({}^0\tilde{Y}_t^i, {}^0\tilde{Z}_t^i, {}^0\tilde{K}_t^i) := \left({}^0Y_t^i, {}^0Z_t^i, {}^0K_t^i\right)$. Note that equations (8) are a collection of 1-dimensional BSDEs with normal reflections. The convergence of the sequence $({}^n\tilde{Y}_t, {}^n\tilde{Z}_t, {}^n\tilde{K}_t)$ to $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)$ can be obtained similarly by Zheng and Zhou [18] under conditions *(H1)–(H4)* and $(\tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)$ is the solution to the following RBSDE:

$$\begin{cases} \tilde{Y}_t^i = \xi^i + \int_t^T g^i\left(s, \tilde{Y}_s^i, \tilde{Z}_s^i\right)\mathrm{d}s + \int_t^T \mathrm{d}\tilde{K}_s^i - \int_t^T \tilde{Z}_s^i \,\mathrm{d}B_s, \qquad t \in [0,T], \\ \tilde{Y}_t^i \geq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i, \\ \int_0^T \left(\tilde{Y}_{t-}^i - \max_{j\neq i}\left\{Y_{t-}^j - k(i,j)\right\} \vee S_t^i,\right)\mathrm{d}\tilde{K}_t^i = 0. \end{cases} \tag{9}$$

Since ${}^{(n-1)}Y_t^i \leq Y_t^i$, we have $\max_{j\neq i}\left\{{}^{(n-1)}Y_t^j - k(i,j)\right\} \vee S_t^i \leq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i$. By Lemma 2.1, we have ${}^nY_t^i \leq \tilde{Y}_t^i$ and passing to the limit as $n \to \infty$, we obtain that $Y_t^i \leq \tilde{Y}_t^i$, $t \in [0,T]$, P-a.s., for each $i \in \Lambda$. On the other hand, we define

$$L_t^i = \begin{cases} Y_t^i & \text{if } \mathrm{d}K_t^i \neq 0, \\ \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i & \text{otherwise.} \end{cases}$$

Then (Y_t^i, Z_t^i, K_t^i) satisfies

$$\begin{cases} Y_t^i = \xi^i + \int_t^T g^i\left(s, Y_s^i, Z_s^i\right)\mathrm{d}s + \int_t^T \mathrm{d}K_s^i - \int_t^T Z_s^i \,\mathrm{d}B_s, \qquad t \in [0,T], \\ Y_t^i \geq L_t^i, \\ \int_t^T \left(Y_{t-}^i - L_{t-}^i\right)\mathrm{d}K_t^i = 0. \end{cases}$$

Observing that $L_t^i \geq \max_{j\neq i}\left\{Y_t^j - k(i,j)\right\} \vee S_t^i$, by Lemma 2.1, we have $Y_t^i \geq {}^n\tilde{Y}_t^i$ and passing to the limit, we get that $Y_t^i \geq \tilde{Y}_t^i$, $t \in [0,T]$, P-a.s., for each $i \in \Lambda$. Consequently $Y_t^i = \tilde{Y}_t^i$, which implies further that $Z_t^i = \tilde{Z}_t^i$ and $K_t^i = \tilde{K}_t^i$.

We now show that $t \mapsto Y_t^i$ is continuous, and consequently $t \mapsto K_t^i$ is continuous. Let ΔY_t^i (ΔK_t^i) denote the jump value of Y_t^i (K_t^i resp.) at time t. We consider the continuity along a path (ω_t) excluding a trivial set. For some $i_1 \in \Lambda$, if $Y_t^{i_1}$ is not continuous at t, then $\Delta Y_t^{i_1} = -\Delta K_t^{i_1} < 0$, which implies that

$$Y_{t-}^{i_1} = \max_{j\neq i_1}\left\{Y_{t-}^j - k(i_1,j)\right\} \vee S_t^{i_1}.$$

Let $i_2 \in \Lambda/i_1$ be the optimal index. Since $\Delta Y_t^{i_1} < 0$, we have

$$\left(Y_{t-}^{i_2} - k(i_1,i_2)\right) \vee S_t^{i_1} = Y_{t-}^{i_1} > Y_t^{i_1} \geq \max_{j\neq i_1}\left\{Y_t^j - k(i_1,j)\right\} \vee S_t^{i_1} \geq \left(Y_t^{i_2} - k(i_1,i_2)\right) \vee S_t^{i_1}.$$

From the above (strict) inequality, it is obviously impossible that $S_t^{i_1} \geq \left(Y_{t-}^{i_2} - k(i_1,i_2)\right)$. Thus

$$Y_{t-}^{i_1} = \max_{j\neq i_1}\left\{Y_{t-}^j - k(i_1,j)\right\} = \left(Y_{t-}^{i_2} - k(i_1,i_2)\right) > \left(Y_t^{i_2} - k(i_1,i_2)\right).$$

Hence $\Delta Y_t^{i_2} < 0$. Repeating the above procedure, we obtain that for some $i_n \in \Lambda/i_{n-1}$, $\Delta Y_t^{i_n} < 0$. Since i_n only takes values $1, \ldots, m$, without loss of generality, we can assume that $i_{n+1} = i_1$, for some $n > 1$. Then we derive a loop:

$$Y_{t-}^{i_1} = Y_{t-}^{i_2} - k(i_1,i_2), \ldots, Y_{t-}^{i_{n-1}} = Y_{t-}^{i_n} - k(i_{n-1},i_n), Y_{t-}^{i_n} = Y_{t-}^{i_1} - k(i_n,i_1),$$

which contradicts with Remark 2.2. Hence for all $i \in \Lambda$, $t \mapsto Y_t^i$ is continuous.

For any solution $(y_t, z_t, k_t)_{t\in[0,T]}$ of (1), applying the comparison theorem repeatedly, we have ${}^nY_t^i \leq y_t^i$ and hence $Y_t^i \leq y_t^i$, $t \in [0,T]$, P-a.s., for each $i \in \Lambda$. Thus the solution $(Y_t)_{t\in[0,T]}$ constructed above is the minimal solution. The proof is complete. □

Remark 3.1. If condition (H1) is replaced by

(**H1′**) There exist a constant $L \geq 0$ and a nonnegative process $(h_t) \in \mathcal{M}^2_{\mathcal{F}}(0, T; \mathbf{R}^+)$ such that for each $i \in \Lambda$, $\forall t$, $\forall (y, z)$,

$$|g^i(t, y, z)| \leq h_t + L(|y| + |z|),$$

then Theorem 3.1 still holds.

Remark 3.2. $k(i, j)$ can depend on t, i.e. if $k(i, j)$ is replaced by $k_t(i, j)$ with $t \mapsto k_t(i, j)$ being continuous, then the proof is not altered.

Remark 3.3. Results in this paper can be also generalized to the case that $Y^i, i = 1, \ldots m$ are interacted in the generators, if we assume the following condition:

(**H2′**) For each $i \in \Lambda$, $\forall t \in [0, T]$, $\forall y \in \mathbf{R}^m$, g_i does not depend on $(z_j)_{j \neq i}$, g_i is non-decreasing in $(y_j)_{j \neq i}$ and left-Lipschitz w.r.t. y_i, and left-continuous w.r.t. y, Lipschitz w.r.t. z_i.

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