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Dynamical systems

On the Anosov character of the Pappus–Schwartz representations

Sur le caractère Anosov des représentations de Pappus–Schwartz

Viviane Pardini Valério

Universidade Federal de Minas Gerais – ICEx – Departamento de Matemática, Av. Antônio Carlos, 6627, Belo Horizonte, Minas Gerais, CEP 31.270-901, Caixa Postal 702, Brazil

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ABSTRACT

In the paper *Pappus's Theorem and The Modular Group* (1993) [4], R.E. Schwartz observed that the classical Pappus theorem gives rise to an action of the modular group on the space of *marked boxes*. He inferred from this a 2-dimensional family of faithful representations of the modular group into the group of projective symmetries. These representations have a dynamical behavior very similar to the one of *Anosov representations*, even if they are never Anosov themselves. In this note, we announce the main result of V. Pardini Valério (2016) [3], which elucidates this Anosov character of the Schwartz representations by proving that their restrictions to the index-2 subgroup are limits of Anosov representations.

RÉSUMÉ

Dans l'article *Pappus's Theorem and The Modular Group* (1993) [4], R.E. Schwartz a mis en évidence le fait que le théorème classique de Pappus définit une action intéressante du groupe modulaire sur l'espace des *boîtes marquées*. Ceci lui a permis de construire une famille à deux paramètres de représentations fidèles du groupe modulaire dans le groupe de symétries projectives. Ces représentations ont un comportement dynamique très similaire à celui des représentations d'Anosov, bien que ne l'étant pas elles-mêmes. Dans cette note, nous annonçons le résultat principal de V. Pardini Valério (2016) [3], qui élucide ce caractère Anosov des représentations de Schwartz, en montrant que leurs restrictions au sous-groupe d'indice 2 sont chacune des limites des représentations d'Anosov.

E-mail address: vivipardini@ufmg.br.

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1. Pappus theorem and marked boxes

Let V be a 3-dimensional vector space and $\mathbb{P}(V)$ the associated projective spaces with V.

Theorem 1.1 *(Pappus). If the points a_1, a_2, a_3 are colinear and the points b_1, b_2, b_3 are colinear in $\mathbb{P}(V)$, then the points $c_3 = a_1b_2 \cap a_2b_1$, $c_2 = a_1b_3 \cap a_3b_1$, $c_1 = a_2b_3 \cap a_3b_2$ are also colinear in $\mathbb{P}(V)$.*

An important fact is that the Pappus Theorem, on certain conditions, can be iterated infinitely many times (see Fig. 1).

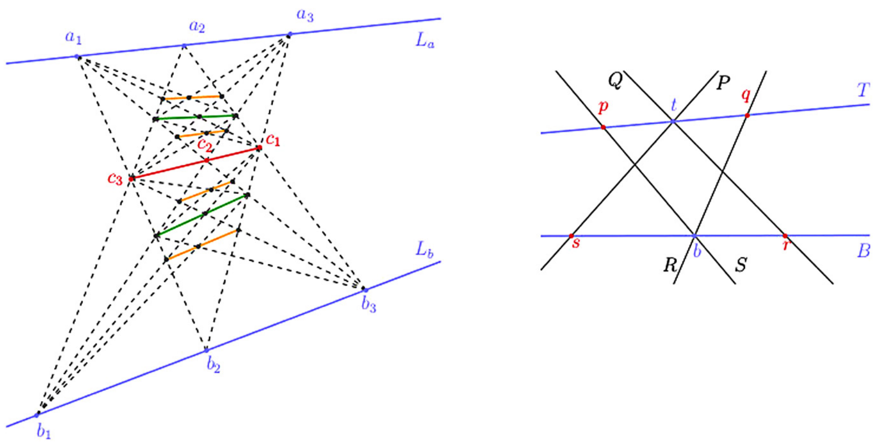

Fig. 1. Iteration of the Pappus Theorem; marked box Θ in $\mathbb{P}(V)$.

A **marked box**[1] Θ is a special pair of 6-tuples having the incidences relatives shown in Fig. 1. If $\Theta = ((p,q,r,s;t,b), (P,Q,R,S;T,B))$, then p, q, r, s, t, $b \in \mathbb{P}(V)$, P, Q, R, S, T, $B \in \mathbb{P}(V^*)$, $T \cap B \notin \{p,q,r,s,t,b\}$, $S = bp$, $R = bq$, $P = ts$, $Q = tr$, $T = pq$ and $B = rs$. Let CM be the set of marked boxes.

The marked box $\Theta = ((p,q,r,s;t,b),(P,Q,R,S;T,B))$ is **convex** if the following two conditions hold: p and q separate t and $T \cap B$ on the line T, and r and s separate b and $T \cap B$ on the line B. The **convex interior** of Θ is the open convex quadrilateral whose vertices, in cyclic order, are p, q, r and s (for more details, see [3, section 2.2]). We denote it by $\mathring{\Theta}$.

1.1. The action of the group of projective symmetries on CM

Let V be a 3-dimensional vector space and V^* its dual vector space. Projective transformations and dualities generate the group $\mathcal{G}$ of projective symmetries of the flag variety $\mathcal{F}$. Projective transformations alone define an index-2 subgroup $\mathcal{H} \cong \mathrm{PGL}(3,\mathbb{R})$ of $\mathcal{G}$.

Given a projective transformation $\mathcal{T}$, and using the notation $\hat{x} = \mathcal{T}(x)$ for every point or line x in $\mathbb{P}(V)$, and for any marked box $\Theta = ((p,q,r,s;t,b),(P,Q,R,S;T,B))$, define (see Fig. 1):

$$\mathcal{T}(\Theta) = ((\hat{p},\hat{q},\hat{r},\hat{s};\hat{t},\hat{b}),(\hat{P},\hat{Q},\hat{R},\hat{S};\hat{T},\hat{B})) \in CM.$$

Similarly, given a duality $\mathcal{D}$, and denoting $x^* = \mathcal{D}(x)$ for $x \in \mathbb{P}(V)$, and $X^* = \mathcal{D}^*(X)$ for X being a projective line, define (pay attention to the maybe surprising Schwartz re-ordering):

$$\mathcal{D}(\Theta) = ((P^*,Q^*,S^*,R^*;T^*,B^*),(q^*,p^*,r^*,s^*;t^*,b^*)) \in CM.$$

1.2. The group of elementary transformations of marked boxes

Let $\Theta = ((p,q,r,s,t,b),(P,Q,R,S,T,B)) \in CM$. Pappus' Theorem gives us two new elements of CM that are images of Θ by two special permutations τ_1 and τ_2 on CM (see Fig. 2). These permutations are defined by

$$\tau_1(\Theta) = ((p,q,QR,PS;t,(qs)(pr)),(P,Q,qs,pr;T,(QR)(PS))),$$

$$\tau_2(\Theta) = ((QR,PS,s,r;(qs)(pr),b),(pr,qs,S,R;(QR)(PS),B)).$$

[1] In this brief note, we abusively do not distinguish overmarked boxes from marked boxes as in [3] and [4].

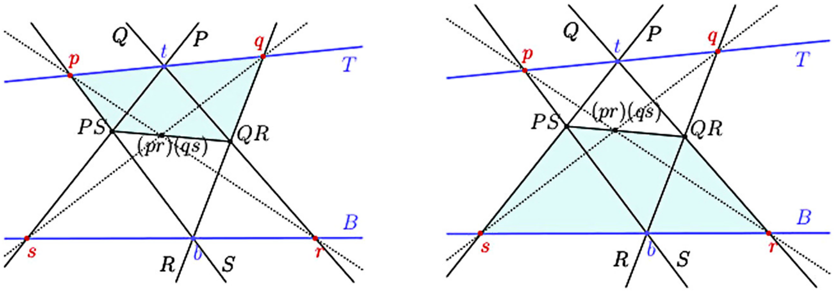

Fig. 2. Permutations τ_1 and τ_2; convex interiors of $\tau_1(\Theta)$ and $\tau_2(\Theta)$ in $\mathbb{P}(V)$ when Θ is convex.

There is also a natural involution, denoted by i, on the set of marked boxes, which gives us another new box (see Fig. 3). This involution is defined by

$$i(\Theta) = ((s, r, p, q; b, t), (R, S, Q, P; B, T)).$$

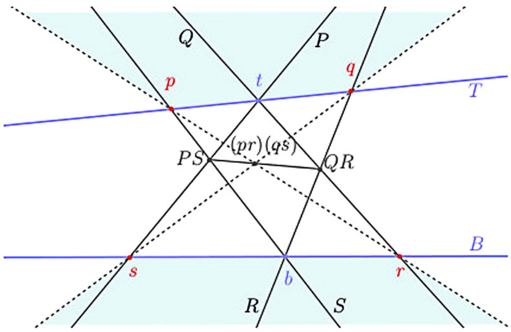

Fig. 3. Permutation i and convex interior of $i(\Theta)$ in $\mathbb{P}(V)$ when Θ is convex.

Let $S(CM)$ be the group of permutations on CM. The elements i, τ_1, τ_2 of $S(CM)$ generate a group $\mathfrak{G}$ that we call **group of elementary transformations of marked boxes**.

In [4], it is proved that the action of $\mathfrak{G}$ on CM is free. In particular, $\mathfrak{G}$ is isomorphic to the modular group $\mathrm{PSL}(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

2. Schwartz representations

2.1. Construction of Schwartz representations

Remark 1. If Θ is a convex marked box, $\tau_1(\Theta)$ and $\tau_2(\Theta)$ are two new marked boxes with convex interiors contained in the convex interior of Θ. On the other hand, the marked box $i(\Theta)$ does not have convex interior contained in the interior of Θ. Arising this, Schwartz observed that the convexity of Θ ensures the nesting property of the marked boxes of $\mathfrak{G}$-orbit of Θ (for more details, see [3, section 2.5.2]); thus combinatorics of $\mathfrak{G}$-orbit of Θ are nicely described by the Farey graph and its associated $\mathrm{PSL}(2,\mathbb{Z})$-invariant triangulation $\mathcal{L}_o$ of $\mathbb{H}^2$: the oriented leaves (geodesics) of $\mathcal{L}_o$ can be labeled by elements of the $\mathfrak{G}$-orbit, giving rise to an action of $\mathfrak{G} \cong \mathrm{PSL}(2,\mathbb{Z})$ commuting with the action of $\mathrm{PSL}(2,\mathbb{Z})$ by isometries.

Theorem 2.1 *(**Schwartz representation theorem**). Let Θ be a convex marked box. Then, there is a faithful representation $\rho_\Theta : \mathrm{PSL}(2,\mathbb{Z}) \to \mathcal{G}$ which takes isometries of $\mathrm{PSL}(2,\mathbb{Z})$ to projective symmetries of $\mathcal{G}$ respecting the labeling of $\mathcal{L}_o$; i.e., such that for every Farey geodesic e and every $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$, we have:*

$$\Theta(\gamma e) = \rho_\Theta(\gamma)(\Theta(e)) \quad (\rho_\Theta\textit{-equivariant property}).$$

Proof. The proof follows basically from the fact that the actions of $\mathrm{PSL}(2,\mathbb{Z})$ and $\mathfrak{G}$ on $\mathcal{L}_o$ commute with each other (Remark 1), even if the actions of $\mathfrak{G}$ and $\mathcal{G}$ on CM commute with each other (see [4, Theorem 2.4] and, for more details, [3, Lemma 3.1, Theorem 3.2]). Already the fact that $\rho_\Theta : \mathrm{PSL}(2,\mathbb{Z}) \to \mathcal{G}$ is a faithful representation follows from the fact that the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathcal{L}_o$ is free. $\square$

2.2. The Schwartz map

Two Farey geodesics have the same tail in $\partial\mathbb{H}^2$ if and only if their labels are marked boxes with the same top point. Therefore, it defines a map $\varphi : \mathbb{Q} \cup \{\infty\} \to \mathbb{P}(V)$ that can be extended to an injective ρ_Θ-equivariant continuous map $\varphi_o : \partial\mathbb{H}^2 \to \mathbb{P}(V)$ (see [4, Theorem 3.2]). Similarly, there is an injective ρ_Θ-equivariant continuous map $\varphi_o^* : \partial\mathbb{H}^2 \to \mathbb{P}(V^*)$. The maps φ_o and φ_o^* combine to form a ρ_Θ-equivariant map:

$$\Phi := (\varphi_o, \varphi_o^*) : \partial\mathbb{H}^2 \to \mathcal{F} \subset \mathbb{P}(V) \times \mathbb{P}(V^*),$$

where $\mathcal{F}$ is the flag variety. We call the composition of Φ with the canonical projection of $\partial\mathrm{PSL}(2,\mathbb{Z})$ into $(\partial\mathbb{H}^2)$ the **Schwartz map**, where $\partial\mathrm{PSL}(2,\mathbb{Z})$ is the Gromov boundary.

3. Anosov representations

The Anosov representation theory was introduced by François Labourie in [2] for representations of closed surface groups. It does not apply directly to the modular group $\mathrm{PSL}(2,\mathbb{Z})$. However $\mathrm{PSL}(2,\mathbb{Z})$ is Gromov-hyperbolic. Hence we use here a formulation inspired from [1], in the simple case of convex cocompact subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

3.1. Definition of Anosov representations

Given $x \in \mathbb{P}(V)$, let $Q_x(V)$ be the space of norms on tangent space $T_x\mathbb{P}(V)$ at x. Similarly, given $X \in \mathbb{P}(V^*)$, let $Q_X(V^*)$ be the space of norms on tangent space $T_X\mathbb{P}(V^*)$ at X. We denote by $Q(V)$ the bundle of base $\mathbb{P}(V)$ with fiber $Q_x(V)$ on $x \in \mathbb{P}(V)$. Similarly, we denote by $Q(V^*)$ the bundle of base $\mathbb{P}(V^*)$ with fiber $Q_X(V^*)$ on $X \in \mathbb{P}(V^*)$. Let $\Omega(\phi^t)$ be the nonwandering set of the geodesic flow ϕ^t on $T^1(\Gamma \setminus \mathbb{H}^2)$.

Definition 3.1. Let Γ be a convex cocompact discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with limit set Λ_Γ. A homomorphism $\rho : \Gamma \to \mathcal{H} \cong \mathrm{PGL}(3,\mathbb{R})$ is an **Anosov representation** if there are

(i) a Γ-equivariant map

$$\Phi = (\varphi, \varphi^*) : \Lambda_\Gamma \to \mathcal{F} \subset \mathbb{P}(V) \times \mathbb{P}(V^*).$$

(ii) two maps $\nu_+ : \Omega(\phi^t) \to Q(V)$ and $\nu_- : \Omega(\phi^t) \subset\to Q(V^*)$ such that, for every nonwandering geodesic $c : \mathbb{R} \to \mathbb{H}^2$ with extremities $c_-, c_+ \in \Lambda_\Gamma$ we have that
- for all $v \in T_{\varphi(c_+)}\mathbb{P}(V)$ the size of v for the norm $\nu_+(c(t), c'(t))$, increases exponentially with t;
- for all $v \in T_{\varphi^*(c_-)}\mathbb{P}(V^*)$ the size of v for the norm $\nu_-(c(t), c'(t))$, decreases exponentially with t.

The group Γ of this definition is a Gromov-hyperbolic group. Since it is convex cocompact, its Gromov boundary $\partial\Gamma$ is Γ-equivariantly homeomorphic to its limit set Λ_Γ.

In the sequel, we will consider Anosov representations of a finite index subgroup of $\mathrm{PSL}(2,\mathbb{Z})$, which is not convex cocompact. But we replace simply $\mathrm{PSL}(2,\mathbb{Z})$ by a convex cocompact discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ obtained by "opening the cusps", thus we build an example on a 3-fold symmetric 3-punctured sphere having geodesic boundaries of small length.

3.2. Schwartz representations are not Anosov

The Schwartz representation ρ_Θ preserves a topological circle in the flag variety, on which it is topologically conjugated to the usual action of $\mathrm{PSL}(2,\mathbb{Z})$ on the conformal boundary of the hyperbolic plane. This property is very similar to the one associated with Anosov representations of surface groups into $\mathrm{PGL}(3,\mathbb{R})$. However, ρ_Θ cannot be Anosov since the Gromov boundary of $\mathrm{PSL}(2,\mathbb{Z})$ is a Cantor set and not a circle. Thus the Schwartz maps φ and φ^* cease to be injective, contradicting a property of Anosov representations.

4. A new family of representations

In order to show that Schwartz representations are limits of Anosov representations, we define a new group of transformations of CM.

4.1. A new group of transformations of CM

Let $\Theta = ((p, q, r, s; t, b), (P, Q, R, S; T, B))$ be a convex marked box. Let us consider the unique affine chart in $P(V)$ such that Θ is seen as the "special square" where $p = (-1, 1), q = (1, 1), r = (1, -1)$ and $s = (-1, -1)$. Let λ and μ be real numbers. Let $\sigma_{(\lambda,\mu)} : CM \to CM$ be a new transformation of marked boxes such that the image of Θ is given by applying the matrix $\Sigma_{(\lambda,\mu)} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}$ to this special square in $\mathbb{P}(V)$. This new transformation has some interesting properties:

(1) it commutes with elements of $\mathcal{H}$ (projective transformations), but **it does not** commute with elements of $\mathcal{G} \setminus \mathcal{H}$ (dualities) acting on CM.
(2) considering the particular case where $\mu = 2\lambda$ and let $\sigma_\lambda := \sigma_{(\lambda,2\lambda)}$, then the relation $i\sigma_\lambda = \sigma_\lambda^{-1} i$ holds.

Let us define three more new transformations on CM as follows:

$$i^\lambda := \sigma_\lambda i \qquad \tau_1^\lambda := \sigma_\lambda \tau_1 \qquad \tau_2^\lambda := \sigma_\lambda \tau_2.$$

The semigroup $\mathfrak{G}^\lambda$ of $S(CM)$, generated by i^λ, τ_1^λ and τ_2^λ, is also an isomorphic group to the modular group ($\mathrm{PSL}(2,\mathbb{Z}) \cong \mathfrak{G} \cong \mathfrak{G}^\lambda$) and, for $\lambda = 0$, of course $\mathfrak{G}^\lambda = \mathfrak{G}$.

4.2. New representations

Given a convex marked box Θ and a real number λ, again let us consider the Farey lamination $\mathcal{L}_0$ of $\mathbb{H}^2$ introduced in Remark 1; and the new group $\mathfrak{G}^\lambda$ of transformations of CM. In order to circumvent the inconvenient of $\mathfrak{G}^\lambda$ not commuting with dualities acting on CM, we restrict to the unique index 2 subgroup $\mathrm{PSL}(2,\mathbb{Z})_o$ of $\mathrm{PSL}(2,\mathbb{Z})$, isomorphic to $\mathbb{Z}_3 * \mathbb{Z}_3$. The main Theorem announced in this note is:

Theorem 4.1. *Let Θ be a convex marked box and let $\lambda \in \mathbb{R}$. There is a representation $\rho_\Theta^\lambda : \mathrm{PSL}(2,\mathbb{Z})_o \to \mathcal{H} \triangleleft \mathcal{G}$ such that for every leaf e of $\mathcal{L}_0$ and every $\gamma \in \mathrm{PSL}(2,\mathbb{Z})_o$ we have:*

$$[\Theta](\gamma e) = \rho_\Theta^\lambda(\gamma)([\Theta](e)).$$

Moreover, if λ is negative, then ρ_Θ^λ is Anosov.

The key point of the our construction is: if $\lambda \leq 0$, then for any convex marked box Θ, we have $\tau_1^\lambda(\overset{\circ}{\Theta}) \subsetneq \overset{\circ}{\Theta}$, $\tau_2^\lambda(\overset{\circ}{\Theta}) \subsetneq \overset{\circ}{\Theta}$, and $i^\lambda(\overset{\circ}{\Theta}) \cup \overset{\circ}{\Theta} = \emptyset$ in $\mathbb{P}(V)$. Furthermore, if λ is negative, then we have the same properties, but now for the **closures** of the interiors of the marked boxes. The Anosov character of the representations ρ_Θ^λ, for $\lambda < 0$, is a consequence of this stronger property.

Remark 2. When the marked box Θ is symmetric, i.e. when $t = (0,1)$ and $b = (0,-1)$ on the special affine chart, the Schwartz representation, restricted to the index 2 subgroup $\mathrm{PSL}(2,\mathbb{Z})_o$, is the one arising by the inclusion $\mathrm{PSL}(2,\mathbb{Z})_o \subset \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PGL}(3,\mathbb{Z})$ where the last inclusion is reducible, i.e. is such that $\mathrm{PSL}(2,\mathbb{R})$ preserves a splitting of V as a sum of a line and a plane. The representation ρ_Θ^λ, for $\lambda < 0$, corresponds to the deformation of $\mathrm{PSL}(2,\mathbb{Z})_o$ inside $\mathrm{PSL}(2,\mathbb{R})$ consisting in opening up the cusp.

5. Conclusion

In summary, since the space of marked boxes up to projective transformations is 2-dimensional, we have defined a 3-dimensional family of representations $\rho_\Theta^\lambda : \mathrm{PSL}(2,\mathbb{Z})_o \to \mathrm{PGL}(3,\mathbb{R})$ where λ is a real parameter. When λ vanishes, ρ_Θ^λ is the restriction of the Schwartz representation ρ_Θ to $\mathrm{PSL}(2,\mathbb{Z})_o$, and when λ is negative, ρ_Θ^λ is Anosov. In particular, the Schwartz representations are limits of the Anosov representations in the space of all representations of $\mathrm{PSL}(2,\mathbb{Z})_o$ into $\mathrm{PGL}(3,\mathbb{R})$.

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References

[1] O. Guichard, A. Wienhard, Anosov representations: domains of discontinuity and applications, Invent. Math. 190 (2) (2012) 357–438.
[2] F. Labourie, Anosov flows, surface groups and curves in projective space, Invent. Math. 165 (1) (2006) 51–114.
[3] V. Pardini Valério, Teorema de Pappus, Representções de Schwartz e Representações Anosov, PhD thesis, UFMG, Brazil, January 2016, http://www.mat.ufmg.br/intranet-atual/pgmat/TesesDissertacoes/uploaded/Tese68.pdf.
[4] R.E. Schwartz, Pappus's theorem and the modular group, Publ. Math. Inst. Hautes Études Sci. 78 (1993) 187–206.