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Dynamical systems

Periodic points in the intersection of attracting immediate basins boundaries

Points périodiques à l'intersection entre les frontières de bassins immédiats attractifs

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A B S T R A C T

We give conditions under which the intersection between two attracting immediate basins boundaries of a rational map contains at least one periodic point.

R É S U M É

Nous donnons des conditions suffisantes pour que l'intersection entre les frontières de deux bassins immédiats attractifs d'une fraction rationnelle contienne au moins un point périodique.

For a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\mathcal{J}(R)$ denotes the Julia set of R, $\mathcal{P}(R)$ the set $\{R^n(c) : R'(c) = 0 \; ; n \geq 1\}$ and $\mathcal{P}_b(R)$ the set of $x \in \mathcal{P}(R)$ that are not in the closure of a connected component of $\hat{\mathbb{C}} \backslash \mathcal{J}(R)$. A point $z \in \hat{\mathbb{C}}$ is said to be *eventually periodic* if there exists a $n \in \mathbb{N}$ such that $R^n(z)$ is periodic. By *sink* we mean a connected component of an attracting immediate basin.

Theorem 1. *Let f be a rational map with two distinct sinks B_1 and B_2 (not necessarily in the same cycle) such that $\partial B_1 \cap \partial B_2 \neq \emptyset$. Assume that B_1 and B_2 are simply connected, and ∂B_1 and ∂B_2 are locally connected.*

1. *If the intersection $\partial B_1 \cap \partial B_2$ contains no critical point with infinite orbit and is disjoint from the ω-limit set of every recurrent critical point, then $\partial B_1 \cap \partial B_2$ contains a periodic point.*
2. *Assume furthermore that each component of $\hat{\mathbb{C}} \backslash \mathcal{J}(f)$ that is eventually mapped to B_1 or to B_2 is simply connected. If $\partial B_1 \cap \partial B_2$ contains no accumulation point of $\mathcal{P}_b(f)$ nor $\mathcal{P}(f) \cap (\partial B_1 \cup \partial B_2)$, then the subset of eventually periodic points in $\partial B_1 \cap \partial B_2$ is non-empty and dense in $\partial B_1 \cap \partial B_2$.*

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As a particular case of part 2 of Theorem 1, if $\sharp\mathcal{P}(f) < +\infty$ then the set of eventually periodic points in $\partial B_1 \cap \partial B_2$ is non-empty and dense in $\partial B_1 \cap \partial B_2$. Nevertheless, the theorem does not require $\mathcal{P}(f)$ to be finite.

Here is an example of a non-empty intersection between two sink boundaries with no periodic point in the intersection. Let us consider $F_\theta(z) = \rho_\theta z^2(z-3)/(1-3z)$, where $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and $\rho_\theta \in S^1$ (S^1 denotes the unit circle in $\mathbb{C}$) is such that $F_\theta : S^1 \to S^1$ has rotation number θ. The map $F_\theta : \mathbb{C} \to \mathbb{C}$ has been studied in [3]. The map F_θ has two attracting fixed points 0 and ∞. The intersection between the boundaries of the corresponding sinks is non-empty and included in S^1. This intersection contains no periodic point since $F_{\theta|S^1}$ is topologically conjugate to $z \mapsto e^{2i\pi\theta}z$. One notes that in this example the intersection contains the point 1, which is a critical point with an infinite orbit.

To prove the theorem, we assume that B_1 and B_2 are fixed, for otherwise we work with an iterate of f. Since there will not be confusion, we will note $\mathcal{J} = \mathcal{J}(f)$, $\mathcal{P} = \mathcal{P}(f)$ and $\mathcal{P}_b = \mathcal{P}_b(f)$.

Proof of part 1. We assume that $\partial B_1 \cap \partial B_2$ does not contain a critical point with finite orbit nor a parabolic point, for otherwise $\partial B_1 \cap \partial B_2$ would contain a periodic point.

A point $x \in \partial B_i$ is said to be *multiple* if it belongs to the impression of at least two prime ends in B_i. Using the expansion of f on ∂B_i, it is easy to show that a multiple point of ∂B_i in $\partial B_1 \cap \partial B_2$ is eventually periodic. Thus we assume that $\partial B_1 \cap \partial B_2$ contains no multiple point of ∂B_1 nor ∂B_2.

In this context we show, using Theorem 3, that $f_{|\partial B_1 \cap \partial B_2}$ is *distance-expanding* with respect to the spherical metric, that is there exist $\lambda > 1$, $\eta > 0$ and $N \geq 0$ such that for any $x, y \in \partial B_1 \cap \partial B_2$, if $d(x, y) \leq \eta$ then $d(f^N(x), f^N(y)) \geq \lambda d(x, y)$. Then we find a periodic point in $\partial B_1 \cap \partial B_2$ using the Theorem 4 dealing with periodic points for distance-expanding maps.

Lemma 2. *The restriction $f_{|\partial B_1 \cap \partial B_2}$ is distance-expanding with respect to the spherical metric.*

Proof. By Theorem 3 below, there exists an integer $N \geq 0$ such that $\min_{x \in \partial B_1 \cap \partial B_2} ||(f^N)'(x)|| > 1$. By continuity of the map $x \mapsto ||(f^N)'(x)||$, there exist $\lambda > 1$ and a neighborhood U of $\partial B_1 \cap \partial B_2$ such that $\min_{x \in U} ||(f^N)'(x)|| \geq \lambda$. By compactness of $\partial B_1 \cap \partial B_2$, there exists $\eta > 0$ such that if $d(x, y) \leq \eta$, then the geodesic Γ between $f^N(x)$ and $f^N(y)$ lifts to a path γ from x to y with $\gamma \subset U$. Thus we get $d(f^N(x), f^N(y)) = \text{length}(\Gamma) \geq \lambda.\text{length}(\gamma) \geq \lambda d(x, y)$. $\square$

Theorem 3. ([2]) *Let g be a rational map of degree at least 2, and $\Lambda \subset \mathcal{J}(g)$ be a compact forward invariant set containing no critical point nor parabolic point. If Λ is disjoint from the ω-limit set of every recurrent critical point, then there exists $N \in \mathbb{N}$ such that $\min_{z \in \Lambda} ||(g^n)'(z)|| > 1$ for every $n \geq N$.*

Theorem 4. ([5], chapter 4) *Let (X, ρ) be a compact metric space. If $T : X \to X$ is continuous, open and distance-expanding, then there exists $\alpha > 0$ such that the following holds: if there exist $x \in X$ and $L \geq 1$ such that $\rho(x, T^L(x)) \leq \alpha$, then X contains a periodic point.*

Lemma 5. *The restriction $f_{|\partial B_1 \cap \partial B_2}$ is open.*

Proof. Let $O \subset \partial B_1 \cap \partial B_2$ and assume $f(O)$ is not open.

There exists a sequence $(y_n)_{n \geq 0} \subset (\partial B_1 \cap \partial B_2) \backslash f(O)$ converging to some $y \in f(O)$. Let $x \in O$ be such that $f(x) = y$. Since $\partial B_1 \cap \partial B_2$ contains no critical point, there exist a neighborhood U of x and a neighborhood V of y such that $f : U \to V$ is a homeomorphism. Thus for n large enough $y_n \in V$, the point $x_n = f^{-1}(y_n) \cap U$ is well defined and $x_n \to x$.

We show now that $x_n \in (\partial B_1 \cap \partial B_2) \backslash O$ so that O is not open. It is clear that $x_n \notin O$ since $f(x_n) \notin f(O)$. For any n, there exists a Fatou component B_i^n such that $f(B_i^n) = B_i$ and $x_n \in \partial B_1^n \cap \partial B_2^n$. The following assertion finishes the proof of the lemma.

Assertion 6. *For n large enough $B_i^n = B_i$, $i \in \{1, 2\}$.*

Proof. Otherwise, for some $i_0 \in \{1, 2\}$ there exists a Fatou component B such that $B \neq B_{i_0}$, $f(B) = B_{i_0}$ and $x \in \partial B$. The boundary ∂B has finitely many connected components, thus each one of them is locally connected. Let $\tilde{B}$ be either B or B_{i_0}. There exists a connected component $U_{\tilde{B}}$ of $U \cap \tilde{B}$ such that $x \in \partial U_{\tilde{B}}$. Since $f(x)$ is simple in ∂B_{i_0}, there exists a unique connected component $V_{B_{i_0}}$ of $V \cap B_{i_0}$ such that $f(x) \in \partial V_{B_{i_0}}$. Hence $f(U_{\tilde{B}}) = V_{B_{i_0}}$. Since $B \neq B_{i_0}$, we have $U_{B_{i_0}} \cap U_B = \emptyset$ and $f(U_{B_{i_0}}) = f(U_B)$, which contradicts the injectivity of $f_{|U}$. $\square$

Now we apply Theorem 4 to finish the proof of part 1. Let w be an accumulation point of the orbit of some $z \in \partial B_1 \cap \partial B_2$. There exist $P > Q \geq 0$ such that $f^P(z), f^Q(z) \in B(w; \alpha/2)$, where α is the constant in Theorem 4. Hence $d(f^Q(z), f^P(z)) = d(f^Q(z), f^{P-Q}(f^Q(z))) \leq \alpha$, and we get a periodic point in $\partial B_1 \cap \partial B_2$.

The proof of part 2 uses ideas and techniques developed by K. Pilgrim in his thesis ([4], chapter 5). In case where f is hyperbolic and $\sharp\mathcal{P} < +\infty$, part 2 is a corollary of his work.

We assume that $\sharp\mathcal{P} > 2$, for otherwise f is conjugate to $z \mapsto z^d$ for some $d \in \mathbb{Z}$, and the conclusion follows. Up to make a quasi-conformal deformation, we also assume that all the critical points in $\bigcup_{j\geq 0} f^{-j}(B_1 \cup B_2)$ have a finite orbit (see [1], theorem VI 5.1; this is why we assume that each component of $\hat{\mathbb{C}}\backslash\mathcal{J}$ that is eventually mapped to B_1 or to B_2 is simply connected).

Let $d_k \geq 2$ be the degree of $f_{|B_k}$ and let $\phi_k : \mathbb{D} \to B_k$ be an isomorphism conjugating f with z^{d_k}. For $t \in \mathbb{R}$, set $R_k(t) := \phi_k(\{re^{2i\pi t} : 0 \leq r < 1\})$. Since ∂B_k is locally connected, ϕ_k extends continuously to $\overline{\phi_k} : \overline{\mathbb{D}} \to \overline{B_k}$.

Denote χ the set of *chords*, that is the set of $\overline{R_1(t)} \cup \overline{R_2(t')}$ such that $\overline{R_1(t)} \cap \overline{R_2(t')} \neq \emptyset$. If $\alpha \in \chi$ is periodic, then the point $\alpha \cap \mathcal{J} \in \partial B_1 \cap \partial B_2$ is periodic. For any chord α and any set $X \subset \hat{\mathbb{C}}$, $[\alpha]_X$ will denote the isotopy class of α rel X. For any distinct $\alpha, \beta \in \chi$, the complement $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$ has at least two connected components and at most three, with points of $\mathcal{J}$ in each of them. For any $m \geq 0$, $[\alpha]_{f^{-m}(\mathcal{P})} = [\beta]_{f^{-m}(\mathcal{P})}$ if and only if one connected component of $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$ contains all but two points of $f^{-m}(\mathcal{P})$ (these two points being the extremities of the chords).

Set $GO(\mathcal{P}) := \bigcup_{z\in\mathcal{P}} \bigcup_{m\in\mathbb{Z}} f^m(z)$. From the hypothesis of part 2, the set $GO(\mathcal{P}) \cap (\partial B_1 \cap \partial B_2)$ is finite. Thus if $\alpha \in \chi$ is such that $\alpha \cap \mathcal{J} \in GO(\mathcal{P})$, then the point $\alpha \cap \mathcal{J}$ is eventually periodic. We denote $\chi(S^2, GO(\mathcal{P}))$ the set $\{\alpha \in \chi : \alpha \cap \mathcal{J} \notin GO(\mathcal{P})\}$. For any $n \geq 0$ we denote $\chi(S^2, f^{-n}(\mathcal{P}))$ the set $\{\alpha \in \chi : \alpha \cap \mathcal{J} \notin f^{-n}(\mathcal{P})\}$.

The proof of part 2 is as follows. We equip χ with the Hausdorff distance d_H, so that it is a compact metric space. Pick $\alpha \in \chi(S^2, GO(\mathcal{P}))$. If the sequence $([f^j(\alpha)]_{f^{-1}(\mathcal{P})})_{j\geq 0}$ is eventually cyclical then α is eventually periodic (Lemma 10). Otherwise, noting that $([f^j(\alpha)]_{\mathcal{P}})_{j\geq 0}$ contains twice the same element (Lemma 11), we build a sequence $(\beta_n)_{n\geq 0} \subset \chi(S^2, GO(\mathcal{P}))$ by a series of adjustments (Lemma 7) such that β_n converges (Lemma 8) to a chord β with the following property: either $\beta \cap \mathcal{J} \in GO(\mathcal{P})$, or $([f^j(\beta)]_{f^{-1}(\mathcal{P})})_{j\geq 0}$ is eventually cyclical. This proves the existence of a periodic point in $\partial B_1 \cap \partial B_2$. The density part will follow from the fact that we can build β as close as we want to α.

Let $\alpha \in \chi(S^2, f^{-m}(\mathcal{P}))$ (resp. $\chi(S^2, GO(\mathcal{P}))$). A *lift of* α is the closure of a connected component of $f^{-1}(\mathring{\alpha})$. If a lift of α is a chord, then it belongs to $\chi(S^2, f^{-(m+1)}(\mathcal{P}))$ (resp. $\chi(S^2, GO(\mathcal{P}))$).

Lemma 7. *Let $N \geq 1$, $\alpha \in \chi(S^2, GO(\mathcal{P}))$ and $\beta_N \in \chi(S^2, \mathcal{P})$ be such that $\beta_N \in [f^N(\alpha)]_{\mathcal{P}}$. There exists a unique chord β_0 isotopic to α rel $f^{-1}(\mathcal{P})$, such that $f^N(\beta_0) = \beta_N$ and $[f^i(\beta_0)]_{f^{-1}(\mathcal{P})} = [f^i(\alpha)]_{f^{-1}(\mathcal{P})}$ for every $0 \leq i \leq N-1$. Furthermore, $\beta_0 \in [\alpha]_{f^{-N}(\mathcal{P})}$.*

Proof. Since $f^{N-1}(\alpha)$ is a lift of $f^N(\alpha)$, there exists a unique lift β_{N-1} of β_N such that $\beta_{N-1} \in [f^{N-1}(\alpha)]_{f^{-1}(\mathcal{P})}$. In particular, $\beta_{N-1} \in [f^{N-1}(\alpha)]_{\mathcal{P}}$. For each $1 \leq i \leq N$, we construct inductively a unique $\beta_{N-i} \in [f^{N-i}(\alpha)]_{f^{-1}(\mathcal{P})}$ such that $f^i(\beta_{N-i}) = \beta_N$ and $f^k(\beta_{N-i}) \in [f^k(f^{N-i}(\alpha))]_{f^{-1}(\mathcal{P})}$ for any $0 \leq k \leq i-1$. Note that $\beta_{N-i} \in [f^{N-i}(\alpha)]_{f^{-i}(\mathcal{P})}$. □

Lemma 8. *For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that: $\forall \alpha, \beta \in \chi(S^2, f^{-N}(\mathcal{P}))$, if $[\alpha]_{f^{-N}(\mathcal{P})} = [\beta]_{f^{-N}(\mathcal{P})}$ then $d_H(\alpha, \beta) \leq \varepsilon$. As a consequence, if $[\alpha]_{f^{-n}(\mathcal{P})} = [\beta]_{f^{-n}(\mathcal{P})}$ for every $n \in \mathbb{N}$ then $\alpha = \beta$.*

It follows from the following assertion:

Assertion 9. *For every $\varepsilon > 0$ there exists $\eta > 0$ such that : for any $\alpha, \beta \in \chi(S^2, GO(\mathcal{P}))$, if $d_H(\alpha, \beta) > \varepsilon$ then in at least two connected components of $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$ lie an open ball centered at a point of $\mathcal{J}$ and with radius η.*

Proof. By contradiction, assume that there exists $\varepsilon > 0$, a sequence $(\eta_n)_{n\geq 0} \subset \mathbb{R}_+^*$ tending to 0, and a sequence $((\alpha_n, \beta_n))_{n\geq 0} \subset \chi(S^2, GO(\mathcal{P}))^2$ such that, for any $n \geq 0$: $d_H(\alpha_n, \beta_n) \geq \varepsilon$ and there does not exist two connected components of $\hat{\mathbb{C}}\backslash(\alpha_n \cup \beta_n)$ in which lies an open ball centered at a point of $\mathcal{J}$ and with radius η_n. By compactness of $\chi(S^2, GO(\mathcal{P}))^2$, we choose an accumulation point (α, β) of $((\alpha_n, \beta_n))_{n\geq 0}$ and up to extraction $(\alpha_n, \beta_n) \to (\alpha, \beta)$. We have $d_H(\alpha, \beta) \geq \varepsilon$. If $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$ has three connected components, then we note U_1 one of the two connected components that are Jordan domains and we note U_2 the connected component that is not a Jordan domain. If $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$ has two connected components, then we note them U_1 and U_2. In any case, there exist $\eta > 0$ and $z_i \in \mathcal{J} \cap U_i$ such that $B(z_i; \eta) \subset U_i$, $i \in \{1, 2\}$. For n large enough, $B(z_1; \eta/2)$ and $B(z_2; \eta/2)$ are included in two distinct connected components of $\hat{\mathbb{C}}\backslash(\alpha_n \cup \beta_n)$. This is a contradiction as soon as $\eta_n < \eta/2$. □

Proof of Lemma 8. Let $\varepsilon > 0$ and η as in the assertion. Since $\sharp\mathcal{P} > 2$, there exists $N \geq 0$ such that each one of the two balls of the assertion contains a point of $f^{-N}(\mathcal{P})$. Hence there is a point of $f^{-N}(\mathcal{P})$ in at least two connected components of $\hat{\mathbb{C}}\backslash(\alpha \cup \beta)$, thus $[\alpha]_{f^{-N}(\mathcal{P})} \neq [\beta]_{f^{-N}(\mathcal{P})}$. □

Lemma 10. *For any $\alpha \in \chi(S^2, GO(\mathcal{P}))$, if the sequence $([f^n(\alpha)]_{f^{-1}(\mathcal{P})})_{n=0}^{\infty}$ is cyclical, then α is periodic.*

Proof. Assume that there exists $Q \geq 1$ such that $[f^{n+Q}(\alpha)]_{f^{-1}(\mathcal{P})} = [f^n(\alpha)]_{f^{-1}(\mathcal{P})}$ for any $n \geq 0$. In particular, $f^{n+Q}(\alpha) \in [f^n(\alpha)]_{\mathcal{P}}$. By Lemma 7, there exists a unique chord $\beta_n \in [\alpha]_{f^{-1}(\mathcal{P})}$ such that $f^n(\beta_n) = f^{n+Q}(\alpha)$ and for all $0 \leq i \leq n$,

$[f^i(\beta_n)]_{f^{-1}(\mathcal{P})} = [f^i(\alpha)]_{f^{-1}(\mathcal{P})}$. This chord is $f^Q(\alpha)$. Thanks to Lemma 7, we also have $f^Q(\alpha) \in [\alpha]_{f^{-n}(\mathcal{P})}$. Since this is true for any $n \geq 0$, we conclude by Lemma 8 that $f^Q(\alpha) = \alpha$. □

Lemma 11. *For any $\alpha \in \chi(S^2, GO(\mathcal{P}))$, there exist $M, N \in \mathbb{N}$ distinct such that $[f^M(\alpha)]_{\mathcal{P}} = [f^N(\alpha)]_{\mathcal{P}}$.*

Proof. Assume that for any $m, n \geq 0$ distinct, we have $[f^m(\alpha)]_{\mathcal{P}} \neq [f^n(\alpha)]_{\mathcal{P}}$. Let us show that the set $\mathcal{P}_b$ or the set $\mathcal{P} \cap (\partial B_1 \cup \partial B_2)$ accumulate on $\partial B_1 \cap \partial B_2$, which contradicts the hypothesis of part 2 of Theorem 1.

Since (χ, d_H) is compact, up to extraction the sequence $(f^n(\alpha))_{n\geq 0}$ accumulates on a chord β. Since for any k, k' distinct at least two connected components of $\hat{\mathbb{C}} - (f^k(\alpha) \cup f^{k'}(\alpha))$ contain a point of $\mathcal{P}$, one can construct a non-stationary sequence $(z_n)_{n\geq 0} \subset \mathcal{P}$, which accumulates on a point $z \in \beta$.

Assume that $(z_n)_{n\geq 0} \cap (\mathcal{P}\backslash\mathcal{P}_b)$ is infinite and up to extraction that $(z_n)_{n\geq 0} \subset \mathcal{P}\backslash\mathcal{P}_b$. There exist finitely many distinct connected components $V_1, \ldots, V_N$ of $\hat{\mathbb{C}}\backslash\mathcal{J}$, which are distinct from B_1 and B_2 and such that $(z_n)_{n\geq 0} \subset \overline{B_1} \cup \overline{B_2} \cup \overline{V_1} \cup \ldots \cup \overline{V_N}$. Each V_j is included in $\hat{\mathbb{C}}\backslash\chi$, but by construction there is an infinite subset of $(z_n)_{n\geq 0}$ whose elements are pairwise separated by chords, thus there is an infinite subset of $(z_n)_{n\geq 0}$ included in $\overline{B_1} \cup \overline{B_2}$. Since we assume that the extremities of the chords are the only points of $\mathcal{P}$ in $B_1 \cup B_2$, we conclude that there is an infinite subset of $(z_n)_{n\geq 0}$ included in $\mathcal{P} \cap (\partial B_1 \cup \partial B_2)$.

Hence, up to extraction, we have $(z_n)_{n\geq 0} \subset \mathcal{P}_b$ or $(z_n)_{n\geq 0} \subset \mathcal{P} \cap (\partial B_1 \cup \partial B_2)$. In particular, $(z_n)_{n\geq 0} \subset \mathcal{J}$, and $z = \lim_{n\to\infty} z_n = \beta \cap \mathcal{J} \in \partial B_1 \cap \partial B_2$. □

Proof of part 2. Let α be a chord. We have three cases.

Case 1: $\alpha \cap \mathcal{J} \in GO(\mathcal{P})$. Thus $\alpha \cap \mathcal{J}$ is eventually periodic, as explained before.

Case 2: $\alpha \in \chi(S^2, GO(\mathcal{P}))$ and $([f^n(\alpha)]_{f^{-1}(\mathcal{P})})_{n=0}^{\infty}$ is eventually cyclical. Then α is eventually periodic by Lemma 10, and the point $\alpha \cap \mathcal{J}$ is eventually periodic.

Case 3: $\alpha \in \chi(S^2, GO(\mathcal{P}))$ and $([f^n(\alpha)]_{f^{-1}(\mathcal{P})})_{n=0}^{\infty}$ is not eventually cyclical. Let us build from α a chord β fitting case 1 or 2.

By Lemma 11 there exist $N \geq 0$ and $Q \geq 1$ such that $[f^{N+Q}(\alpha)]_{\mathcal{P}} = [f^N(\alpha)]_{\mathcal{P}}$. Set $\beta_0 := f^N(\alpha)$. By Lemma 7, there exists a chord $\beta_1 \in [\beta_0]_{f^{-Q}(\mathcal{P})}$ such that $[f^i(\beta_1)]_{f^{-1}(\mathcal{P})} = [f^i(\beta_0)]_{f^{-1}(\mathcal{P})}$ for any $0 \leq i \leq Q-1$, and $f^Q(\beta_1) = \beta_0$. Thus $[f^{Q+i}(\beta_1)]_{f^{-1}(\mathcal{P})} = [f^i(\beta_1)]_{f^{-1}(\mathcal{P})}$ for any $0 \leq i \leq Q-1$, and $\beta_1 \in [f^{2Q}(\beta_1)]_{\mathcal{P}}$. We build inductively a sequence of chords $(\beta_q)_{q=0}^{\infty}$ such that:

(i) $\beta_{q+n} \in [\beta_q]_{f^{-2^q Q}(\mathcal{P})}$ for any $n \geq 0$, and
(ii) $[f^{jQ+i}(\beta_q)]_{f^{-1}(\mathcal{P})} = [f^i(\beta_q)]_{f^{-1}(\mathcal{P})}$ for any $0 \leq j \leq 2^q - 1$ and $0 \leq i \leq Q-1$.

Assertion 12. *The sequence $(\beta_q)_{q=0}^{\infty}$ converges to a chord β whose point $\beta \cap \mathcal{J}$ is eventually periodic.*

Proof. The convergence follows from (i) and Lemma 8. The limit β is a chord since χ is compact. If $\beta \notin \chi(S^2, GO(\mathcal{P}))$ then β fits case 1. If $\beta \in \chi(S^2, GO(\mathcal{P}))$, then we get for the limit $[f^{jQ+i}(\beta)]_{f^{-1}(\mathcal{P})} = [f^i(\beta)]_{f^{-1}(\mathcal{P})}$ for every $j \in \mathbb{N}$ and $0 \leq i \leq Q-1$. Hence $([f^i(\beta)]_{f^{-1}(\mathcal{P})})_{i=0}^{\infty}$ is cyclical, and β fits case 2. □

Thus there exists an eventually periodic point in $\partial B_1 \cap \partial B_2$.

To finish the proof, let us explain now the density. The chord β is in $\chi(S^2, f^{-Q}(\mathcal{P}))$. Applying Lemma 7 to $\beta \in [f^N(\alpha)]_{f^{-1}(\mathcal{P})}$, we obtain a chord $\gamma \in [\alpha]_{f^{-(N+Q)}(\mathcal{P})}$ such that $\gamma \cap \mathcal{J}$ is eventually periodic. Using Lemma 11, we can have N as large as we want. Thus we can build a sequence $(\gamma_n)_{n\geq 0} \subset \chi$ converging to α, such that every $\gamma_n \cap \mathcal{J}$ is eventually periodic. Since ∂B_1 and ∂B_2 are locally connected, the sequence $(\gamma_n \cap \mathcal{J})_{n\geq 0}$ converge to $\alpha \cap \mathcal{J}$. □

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