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Complex analysis

Advances on the coefficients of bi-prestarlike functions

Avancées sur les coefficients des fonctions bi-pré-étoilées

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ABSTRACT

Since 1923, when Löwner proved that the inverse of the Koebe function provides the best upper bound for the coefficients of the inverses of univalent functions, finding sharp bounds for the coefficients of the inverses of subclasses of univalent functions turned out to be a challenge. Coefficient estimates for the inverses of such functions proved to be even more involved under the bi-univalency requirement. In this paper, we use the Faber polynomial expansions to find upper bounds for the coefficients of bi-prestarlike functions and consequently advance some of the previously known estimates.

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RÉSUMÉ

Depuis 1923, lorsque Löwner a montré que l'inverse de la fonction de Koebe fournit la majoration optimale pour les coefficients des inverses des fonctions univalentes, s'est posé le défi de trouver des bornes fines pour les coefficients des inverses de fonctions univalentes dans certaines classes. Ce problème s'est révélé être encore plus intriqué sous la condition de bi-univalence. Utilisant les développements de polynômes de Faber pour les coefficients des fonctions bi-pré-étoilées, nous améliorons dans cette Note quelques estimations déjà connues.

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1. Introduction

Let H be the set of functions that are holomorphic or analytic in the open unit disk $\mathbb{D} := \{z : |z| < 1\}$ and H_0 be the subset of H consisting of functions f that are normalized by $f(0) = f'(0) - 1 = 0$. The class of functions $f \in H_0$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are univalent in $\mathbb{D}$ will be denoted by S. A function is said to be univalent in a simply connected domain if the images of distinct points are distinct there. For $\alpha < 1$, a function $f \in H_0$ is called starlike of order α, denoted by $S^*(\alpha)$, if and only if $\mathrm{Re}\left[zf'(z)/f(z)\right] \geq \alpha$ in $\mathbb{D}$. We note that the set $S^*(\alpha)$ for $0 \leq \alpha < 1$ is a subset of S (e.g., see [5]) and the function

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$$\frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n = z + \sum_{n=2}^{\infty} \left(\frac{\prod_{k=2}^{n} (k-2\alpha)}{(n-1)!} \right) z^n$$

is the well-known extremal function for the class $S^*(\alpha)$.

The convolution or Hadamard product of two power series $\phi(z) = \sum_{n=0}^{\infty} u_n z^n$ analytic in $|z| < r_1$ and $\varphi(z) = \sum_{n=0}^{\infty} v_n z^n$ analytic in $|z| < r_2$ is the power series $(\phi * \varphi)(z) = \phi(z) * \varphi(z) = \sum_{n=0}^{\infty} u_n v_n z^n$, which is analytic in $|z| < r_1 r_2$ [5].

For $\alpha < 1$ and $\beta < 1$, a function $f \in H_0$ is said to be prestarlike of order α and type β, denoted by $R(\alpha; \beta)$, if and only if

$$f(z) * \frac{z}{(1-z)^{2(1-\alpha)}} \in S^*(\beta); \qquad \alpha < 1, \beta < 1, z \in \mathbb{D}.$$

Ruscheweyh [13,14] first studied the class $R(\alpha; \alpha)$ and later Sheil-Small, Silverman and Silvia [17] extended this class to $R(\alpha; \beta)$. We note that $R(1/2; 1/2) \equiv S^*(1/2)$ and $R(0; 0) \equiv K(0)$, the class of functions $f \in S$ that are convex in $\mathbb{D}$. For functions $f \in S$, we have $f \in K(0)$ if and only if $zf' \in S^*(0)$.

The function $g = f^{-1}$, the inverse map of $f \in H_0$, has a Maclaurin series expansion in some disk about the origin [5]. In 1923, Löwner [11] proved that the inverse of the Koebe function $f(z) = z/(1-z)^2$ provides the best upper bounds for the coefficients of the inverses of the functions $f \in S$. Sharp bounds for the coefficients of the inverses of univalent functions have been obtained in a surprisingly straightforward way, whereas the case for the subclasses of univalent functions turned out to be a challenge [5]. Finding coefficient estimates for the inverse of the functions f becomes even more of a challenge when the bi-univalency condition is imposed on the functions f. A function $f \in S$ is said to be *bi-univalent* in $\mathbb{D}$ if its inverse map $g = f^{-1}$ is also univalent in $\mathbb{D}$. The class of bi-univalent analytic functions was first introduced and studied by Lewin [10], who proved that $|a_2| < 1.51$. Brannan and Clunie [4] improved Lewin's result to $|a_2| \le \sqrt{2}$, and later Netanyahu [12] proved that $|a_2| \le 4/3$. Bounds for the first few coefficients of various subclasses of bi-univalent functions were obtained by a number of authors, for example, see Srivastava et al. [18] and the references cited therein. However, not much was known about the bounds of the general coefficients $a_n; n \ge 4$ of subclasses of bi-univalent functions up until the publication of the article [9] by Jahangiri and Hamidi, who used the Faber polynomial series expansions to obtain bounds for the n-th coefficients $a_n; n \ge 3$ of certain subclasses of bi-univalent functions subject to a given gap series condition. In this paper, we further this result to include a larger subclass of bi-univalent functions. The results presented here are new and refine some previously known coefficient estimates.

2. Faber polynomial coefficients

Consider the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_0$. Then $g = f^{-1}$, the inverse map of $f \in H_0$, may be represented by the Faber polynomial expansion

$$g(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{2.1}$$

where K_{n-1}^{-n} is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$. The first few terms of the coefficients K_{n-1}^{-n} are $K_1^{-2} = -2a_2$, $K_2^{-3} = +3(2a_2^2 - a_3)$ and $K_3^{-4} = -4(5a_2^3 - 5a_2 a_3 + a_4)$.

In general, for $n \ge 1$ and for the real values of p, the coefficients K_n^p are calculated according to (also see [1,2] or [3, p. 349])

$$\begin{aligned} K_n^p &= p\, a_n + \frac{p\,(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!\,3!} D_n^3 + \cdots + \frac{p!}{(p-n)!\,n!} D_n^n \\ &= \frac{p!}{(p-n)!\,n!} a_2^n + \frac{p!}{(p-n+1)!\,(n-2)!} a_2^{n-2} a_3 \\ &+ \frac{p!}{(p-n+2)!\,(n-3)!} a_2^{n-3} a_4 \\ &+ \frac{p!}{(p-n+3)!\,(n-4)!} a_2^{n-4} \left[a_5 + \frac{p-n+3}{2} a_3^2 \right] \\ &+ \frac{p!}{(p-n+4)!\,(n-5)!} a_2^{n-5} \left[a_6 + (p-n+4)\, a_3 a_4 \right] + \sum_{j \ge 6} a_2^{n-j} V_j, \end{aligned}$$

where V_j is a homogeneous polynomial of degree j in the variables $a_3, a_4, \cdots, a_{n-1}$,

$$\frac{p!}{(p-n)!\,n!} = \frac{(p-n+1)\,(p-n+2)\,(p-n+3)\cdots(p)}{n!}, \tag{2.2}$$

and for

$$D_n^m\,(a_2, a_3, \cdots, a_n) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\mu_2} \cdots (a_n)^{\mu_n}}{\mu_2! \cdots \mu_n!},$$

the sum is taken over all nonnegative integers $\mu_2, \cdots, \mu_n$ satisfying

$$\begin{aligned} \mu_2 + \mu_3 + \cdots + \mu_n &= m, \\ 2\,\mu_2 + 3\,\mu_3 + \cdots + n\mu_n &= n. \end{aligned}$$

We note that $D_n^n = a_2^n$, [1] and $D_n^1 = a_n$, [19].

The Faber polynomials introduced by Faber [6] (see also Schur [16]) play an important role in various areas of mathematical sciences, especially in geometric function theory (Gong [7] Chapter III, Schiffer [15], and Todorov [19]). The recent advances in the calculus of Faber polynomials, especially when it involves f^{-1}, the inverse map of f (see [1–3]) beautifully fits our case for the bi-prestarlike functions. As a result, we are able to state and prove the following.

Theorem 2.1. *For $\alpha < 1$ and $\beta < 1$ let $f \in R(\alpha;\beta)$ be bi-prestarlike of order α and type β in $\mathbb{D}$. If $a_k = 0$ for $2 \le k \le n-1$, then*

$$|a_n| \le \frac{2(1-\beta)}{(n-1)C(\alpha,n)}; \qquad n \ge 3.$$

Proof. The Faber polynomial expansion of $z/(1-z)^{2-2\alpha}$ is (e.g., see Airault [1, equation (4)])

$$\frac{z}{(1-z)^{2(1-\alpha)}} = \sum_{n=0}^{\infty} K_n^{2(\alpha-1)}(-1,0,0,\cdots,0)z^{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{(2(\alpha-1))!}{(2(\alpha-1)-n)!\,n!} z^{n+1}.$$

So, by the definition of Hadamard product, we have

$$\begin{aligned} F(z) = z + \sum_{n=2}^{\infty} A_n z^n &= f(z) * \frac{z}{(1-z)^{2(1-\alpha)}} \\ &= \left(z + \sum_{n=2}^{\infty} a_n z^n\right) * \left(z + \sum_{n=2}^{\infty} K_{n-1}^{2(\alpha-1)}(-1,0,0,\cdots,0)z^n\right) \\ &= z + \sum_{n=2}^{\infty}(-1)^{n-1} \frac{(2(\alpha-1))!}{(2(\alpha-1)-(n-1))!(n-1)!} a_n z^n. \end{aligned} \tag{2.3}$$

Similarly, for the inverse function $g = f^{-1}$ given by (2.1), we obtain

$$\begin{aligned} G(w) = w + \sum_{n=2}^{\infty} B_n w^n &= g(w) * \frac{w}{(1-w)^{2(1-\alpha)}} \\ &= w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)\; K_{n-1}^{2(\alpha-1)}(-1,0,0,\cdots,0)\; w^n \\ &= w + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) \left(\frac{(2(\alpha-1))!}{(2(\alpha-1)-(n-1))!(n-1)!}\right) w^n. \end{aligned} \tag{2.4}$$

By the definition of prestarlike functions, it is required that

$$\mathrm{Re}\frac{zF'(z)}{F(z)} \ge \beta \qquad \text{and} \qquad \mathrm{Re}\frac{wG'(w)}{G(w)} \ge \beta \text{ in } \mathbb{D}.$$

Therefore, there exist two positive real part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$ where $\mathrm{Re}(p(z)) > 0$ and $\mathrm{Re}(q(w)) > 0$ in $\mathbb{D}$ so that

$$\frac{zF'(z)}{F(z)} = \beta + (1-\beta)\,p(z) = 1 + \sum_{n=1}^{\infty}(1-\beta)c_n z^n \tag{2.5}$$

and

$$\frac{wG'(w)}{G(w)} = \beta + (1-\beta)\,q(w) = 1 + \sum_{n=1}^{\infty}(1-\beta)d_n w^n. \tag{2.6}$$

We note that, by the Caratheodory Lemma (e.g., see Duren [5]), $|c_n| \leq 2$ and $|d_n| \leq 2$.

On the other hand, the Faber polynomial coefficients of $zF'(z)/F(z)$ and $wG'(w)/G(w)$ are determined by (e.g., see [2, equation (1.23)] or [3, equation (1.6)])

$$\frac{zF'(z)}{F(z)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \cdots, A_n) z^{n-1}, \tag{2.7}$$

and

$$\frac{wG'(w)}{G(w)} = 1 - \sum_{n=2}^{\infty} F_{n-1}(B_2, B_3, \cdots, B_n) w^{n-1}, \tag{2.8}$$

where F_{n-1} is a Faber polynomial of degree $n-1$.

A simple calculation reveals the first few terms of $F_{n-1}(A_2, A_3, \cdots, A_n)$ to be $F_1 = -A_2$, $F_2 = A_2^2 - 2A_3$, $F_3 = -A_2^3 + 3A_2A_3 - 3A_4$ and $F_4 = A_2^4 - 4A_2^2A_3 + 4A_2A_4 + 2A_3^2 - 4A_5$. In general,

$$F_{n-1}(A_2, A_3, \cdots, A_n) = \sum_{i_1+2i_2+\cdots+(n-1)i_{n-1}=n-1} [\Gamma(i_1, i_2, \cdots, i_{n-1})](A_2^{i_1} A_3^{i_2} \cdots A_n^{i_{n-1}})$$

where

$$\Gamma(i_1, i_2, \cdots, i_{n-1}) := (-1)^{(n-1)+2i_1+\cdots+ni_{n-1}} \frac{(i_1+i_2+\cdots+i_{n-1}-1)!(n-1)}{(i_1!)(i_2!)\cdots(i_{n-1}!)}.$$

A similar formula can be verified for $F_{n-1}(B_2, B_3, \cdots, B_n)$.

Comparing the coefficients of (2.5) and (2.7) yields

$$-F_{n-1}(A_2, A_3, \cdots, A_n) = (1-\beta)\,c_{n-1}; \qquad n \geq 2. \tag{2.9}$$

Similarly, from (2.6) and (2.8) we get

$$-F_{n-1}(B_2, B_3, \cdots, B_n) = (1-\beta)\,d_{n-1}; \qquad n \geq 2. \tag{2.10}$$

For $a_k = 0$, $(2 \leq k \leq n-1)$, the equations (2.9) and (2.10) yield $(n-1)A_n = (1-\beta)c_{n-1}$ and $(n-1)B_n = (1-\beta)d_{n-1}$ or equivalently,

$$(1-\beta)\,c_{n-1} = (n-1)\left((-1)^{n-1}\frac{(2(\alpha-1))!}{(2\,(\alpha-1)-(n-1))!\,(n-1)!}\right)a_n$$

and

$$(1-\beta)\,d_{n-1} = (n-1)\left((-1)^{n}\frac{(2\,(\alpha-1))!}{(2\,(\alpha-1)-(n-1))!\,(n-1)!}\right)a_n.$$

An application of the factorial formula (2.2) gives

$$-(1-\beta)\,c_{n-1} = (-1)^n\frac{2\,(\alpha-1)\,(2(\alpha-1)-1)\cdots(2(\alpha-1)-(n-2))}{(n-2)!}a_n,$$

and

$$(1-\beta)\,d_{n-1} = (-1)^n\frac{2\,(\alpha-1)\,(2(\alpha-1)-1)\cdots(2(\alpha-1)-(n-2))}{(n-2)!}a_n.$$

Add the above two equations to obtain

$$(1-\beta)\,(d_{n-1}-c_{n-1}) = 2\,(-1)^n\left(\frac{2\,(\alpha-1)\,(2\alpha-3)\,(2\alpha-4)\cdots(2\alpha-n)}{(n-2)!}\right)a_n.$$

Solving for a_n and taking the absolute values, we obtain

$$|a_n| \leq \frac{(n-2)!\,(1-\beta)}{(1-\alpha)\,(3-2\alpha)\,(4-2\alpha)...(n-2\alpha)} = \frac{2\,(1-\beta)}{(n-1)\,C(\alpha,n)};\ n \geq 3. \qquad \square$$

Remark 2.2. We note that the bound

$$|a_n| \le \frac{2\,(1-\beta)}{(n-1)\,C(\alpha,n)} < \frac{(n-2)!}{(3-2\alpha)\,[(2)(3)\cdots(n-2)]} = \frac{1}{3-2\alpha}$$

is better than the bound obtained by Ruscheweyh [14, p. 100, Theorem 2.45] for $\alpha=\beta$ and in general

$$|a_n| \le \frac{2\,(1-\beta)}{(n-1)\,C(\alpha,n)} < \frac{C(\beta,n)}{C(\alpha,n)}$$

is better than the bound obtained by Sheil-Small, Silverman, and Silvia [17, p. 188, Theorem 7].

The following theorem gives the estimate for the specific coefficient body (a_2,a_3) of bi-prestarlike functions.

Theorem 2.3. *For $\alpha<1$ and $\beta<1$ let $f\in R(\alpha;\beta)$ be bi-prestarlike of order α and type β in $\mathbb{D}$. Then*

$$(i).\ |a_2| \le \sqrt{\frac{1-\beta}{1-\alpha}},$$

$$(ii).\ \left|a_3 - a_2^2\right| \le \frac{(1-\beta)-(1-\alpha)\,|a_2|^2}{(1-\alpha)\,(3-2\alpha)};\ \alpha\le\beta.$$

Proof. (i). From (2.9) and (2.10), respectively, we have

$$-F_2(A_2,A_3) = -A_2^2 + 2A_3 = (1-\beta)\,c_2,$$

and

$$-F_2(B_2,B_3) = -B_2^2 + 2B_3 = (1-\beta)\,d_2.$$

Note that A_n and B_n are determined by equations (2.3), (2.4), (2.7) and (2.8), which upon a simple calculation, yield

$$-4\,(\alpha-1)^2 a_2^2 + 2\,(\alpha-1)\,[2\,(\alpha-1)-1]\,a_3 = (1-\beta)\,c_2, \tag{2.11}$$

and

$$-4\,(\alpha-1)^2 a_2^2 + 2\,(\alpha-1)\,[2\,(\alpha-1)-1]\,(2\,a_2^2 - a_3) = (1-\beta)\,d_2. \tag{2.12}$$

Adding (2.11) and (2.12), we obtain $4\,(1-\alpha)\,a_2^2 = (1-\beta)\,(c_2+d_2)$. Solving for a_2 and taking the absolute values we obtain the required estimate (i) for $|a_2|$.

(ii). Divide (2.11) by $2\,(1-\alpha)\,(3-2\alpha)$ to obtain

$$a_3 - \frac{2\,(1-\alpha)}{3-2\alpha}a_2^2 = \frac{1-\beta}{2\,(1-\alpha)\,(3-2\alpha)}c_2. \tag{2.13}$$

Rewrite (2.13) as

$$a_3 - a_2^2 = \frac{1-\beta}{2\,(1-\alpha)\,(3-2\alpha)}\,c_2 - \frac{1}{3-2\alpha}\,a_2^2$$

and take the absolute values to obtain

$$\left|a_3 - a_2^2\right| = \frac{1-\beta}{2\,(1-\alpha)\,(3-2\alpha)}\left|c_2 - \frac{2\,(1-\alpha)}{1-\beta}\,a_2^2\right|.$$

Substituting $a_2 = \frac{1-\beta}{2(1-\alpha)}c_1$ in the right hand side of the above equation yields

$$\left|a_3 - a_2^2\right| = \frac{1-\beta}{2\,(1-\alpha)\,(3-2\alpha)}\left|c_2 - \frac{1-\beta}{2\,(1-\alpha)}c_1^2\right|.$$

Using the fact $\left|c_2+\lambda c_1^2\right| \le 2+\lambda|c_1|^2$ if $\lambda \ge -\frac{1}{2}$ (e.g., see [8, Lemma 1]), we obtain

$$\left|a_3 - a_2^2\right| \le \frac{1-\beta}{2\,(1-\alpha)\,(3-2\alpha)}\left(2 - \frac{1-\beta}{2\,(1-\alpha)}|c_1|^2\right);\ \alpha\le\beta.$$

Now Theorem 2.3.*ii* follows, upon re-applying the relation $|a_2| = \frac{1-\beta}{2(1-\alpha)}|c_1|$. □

Remark 2.4. For $\alpha = \beta$, the coefficient body (a_2, a_3) given in the above Theorem 2.3 was also obtained by Ruscheweyh [14, p. 54, equation (2.34)], but using a different technique. We note that no estimate for the coefficient body (a_2, a_3) was given by Sheil-Small, Silverman, and Silvia [17, p.188, Theorem 7] for $R(\alpha; \beta); \alpha < \beta$.

Remark 2.5. This work is dedicated to Walter Kurt Hayman (6 January 1926–...) for his pleasant sense of humor and mathematical wisdom that the first author enjoyed during the 1980s at London University and the University of York, England. Bless Walter for not missing even one Christmas greeting to date.

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