C. R. Acad. Sci. Paris, Ser. I 354 (2016) 907–911

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

Quantum ergodicity for Eisenstein functions

Ergodicité quantique des fonctions d'Eisenstein

Yannick Bonthonneau [a], Steve Zelditch [b,1]

[a] *CIRGET, UQÀM, 201, av. Président-Kennedy, Montréal, Québec, H2X 3Y7, Canada*
[b] *Department of Mathematics, Northwestern University, Evanston, IL 60208, USA*

A R T I C L E I N F O

Article history:
Received 21 December 2015
Accepted after revision 28 June 2016
Available online 5 August 2016

Presented by the Editorial Board

A B S T R A C T

A new proof is given of Quantum Ergodicity for Eisenstein Series for cusped hyperbolic surfaces. This result is also extended to higher dimensional examples, with variable curvature.

R É S U M É

On donne une nouvelle preuve de l'ergodicité quantique des séries d'Eisenstein pour les surfaces de Riemann à pointes. On étend aussi ce résultat en plus grande dimension, en autorisant la courbure variable.

1. Introduction

The purpose of this note is to prove a quantum ergodicity theorem for Eisenstein series on a rather general cusp manifold with ergodic geodesic flow. In the special case of finite area hyperbolic surfaces with cusps, a similar quantum ergodicity theorem was proved in [10]. In this note, we give a new, simpler and more general proof of that result, which generalizes to any cusped manifold with ergodic geodesic flow. In particular, the manifolds may have variable negative curvature.

To state the result, we introduce some notation and terminology. A $(d+1)$-dimensional *cusp manifold* is a Riemannian manifold M that decomposes as a compact manifold M_0 whose boundaries are torii, and a finite number κ of topological half-cylinders $Z_1, \dots, Z_\kappa$, such that the metric on these Z_i's is

$$ds^2 = \frac{dy^2 + d\theta^2}{y^2}, \tag{1}$$

E-mail addresses: yannick.bonthonneau@gmail.com (Y. Bonthonneau), zelditch@math.northwestern.edu (S. Zelditch).
[1] Research partially supported by NSF grant DMS-1541126.

http://dx.doi.org/10.1016/j.crma.2016.06.006

where y is the vertical coordinate, starting at $a_i > 0$, and θ is the horizontal coordinate, living in $\mathbb{R}^d/\Lambda_i$, where Λ_i is a unimodular lattice. On such a manifold, we can pick a global coordinate y_M, that coincides with the vertical coordinate sufficiently high in the cusps, and is some positive constant in M_0.

For such a manifold, the spectrum of the Laplace–Beltrami operator $-\Delta$ acting on $L^2(M, g)$ decomposes into the pure spectrum part ($\mu_0 = 0 < \mu_1 \leq \dots$) possibly finite, possibly infinite, and the absolutely continuous spectrum $[d^2/4, +\infty)$.

With each eigenvalue μ, we can associate an eigenfunction u_μ (repeating eventual multiple eigenvalues). For the continuous spectrum, we recall the existence of meromorphic families [3,5] of smooth, non-L^2 eigenfunctions $E_i(s)$ that satisfy the following. For each i, E_i decomposes as

$$E_i(s) = I_i y^s + G_i(s) \tag{2}$$

where I_i is the characteristic function of Z_i, and $G_i(s)$ is a function in $L^2(M)$ whenever $\Re s > d/2$. We also require that for all s,

$$(-\Delta - s(d-s))E_i(s) = 0. \tag{3}$$

Such a collection is unique, and they are called the Eisenstein functions. The E_i's do not have poles on $\{\Re s = d/2\}$. We denote $E(s) = (E_1(s), \dots, E_\kappa(s))$. The scattering matrix is defined in the following way. Let $f(s, y)$ be the square matrix whose i line is the zero Fourier coefficient in cusp Z_i, at a fixed height $y > a_i$, of E; let I be the identity matrix. Then

$$f(s, y) = y^s I + \phi(s) y^{1-s}. \tag{4}$$

The matrix $\phi(s)$ is called the *scattering matrix*, and $\varphi(s)$, its determinant, is the *scattering determinant*. When $\Re s = d/2$, ϕ is unitary, and $|\varphi| = 1$.

The Weyl law of W. Mueller for a cusp manifold states [5],

$$\frac{1}{4\pi} \int_{-h^{-1}}^{h^{-1}} -\frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}r\right) \mathrm{d}r + \sum_{h\sqrt{\mu} \leq 1} 1 = \frac{\mathrm{vol}(B^*M)}{(2\pi)^{d+1}} h^{-d-1} + o(h^{-d-1}). \tag{5}$$

Mueller obtained it using the small-time asymptotics of the heat kernel. In [1], a semi-classical machinery is used to generalize the Weyl law to a local Weyl law, which gives an approximation of the trace of pseudo-differential operators (see Proposition 2.3 in [1]). For semi-classical pseudo-differential operators with compactly supported symbols σ: As $h \to 0$,

$$\frac{1}{4\pi} \int_{\mathbb{R}} \left\langle \mathrm{Op}_h(\sigma) E\left(\frac{d}{2} + \mathrm{i}r\right), E\left(\frac{d}{2} + \mathrm{i}r\right) \right\rangle \mathrm{d}r + \sum_{\mu} \langle \mathrm{Op}_h(\sigma) u_\mu, u_\mu \rangle \sim \frac{1}{(2\pi h)^{d+1}} \int_{T^*M} \sigma. \tag{6}$$

Combining (5) and (6), we get

$$\begin{aligned} &\frac{h^{d+1}}{4\pi} \int_{\mathbb{R}} \left(\left\langle \mathrm{Op}_h(\sigma) E\left(\frac{d}{2} + \mathrm{i}r\right), E\left(\frac{d}{2} + \mathrm{i}r\right) \right\rangle + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}r\right) \int_{\lambda S^*M} \sigma \right) \mathrm{d}r \\ &\qquad + h^{d+1} \sum_{\mu} \langle \mathrm{Op}_h(\sigma) u_\mu, u_\mu \rangle - \int_{h\sqrt{\mu} S^*M} \sigma \to 0. \end{aligned} \tag{7}$$

The local Weyl law gives the average value of the matrix elements of $\mathrm{Op}_h(\sigma)$ over the spectrum. The main result of this note is "the mean absolute deviation" goes to zero:

Theorem 1.1. *Let M be a cusp manifold whose geodesic flow is* ergodic. *Then, for all compactly supported symbol σ, we have the following convergence, as $h \to 0$.*

$$\begin{aligned} &\frac{h^{d+1}}{4\pi} \int_{\mathbb{R}} \left| \left\langle \mathrm{Op}_h(\sigma) E\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right), E\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \right\rangle + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \int_{\lambda S^*M} \sigma \right| \frac{\mathrm{d}\lambda}{h} \\ &\qquad + h^{d+1} \sum_{\mu} \left| \langle \mathrm{Op}_h(\sigma) u_\mu, u_\mu \rangle - \int_{h\sqrt{\mu} S^*M} \sigma \right| \to 0. \end{aligned} \tag{8}$$

(the measure on S^*M is the usual normalized Liouville measure).

Remark 1. Henceforth we let $\varepsilon(\sigma, h)$ denote the quantity in the LHS of (8).

Note that we only sum absolute values and not squares as in the now-standard quantum variances (see [11] for background). This is because Eisenstein functions are not L^2, so that absolute values are easier to control than squares. The theorem thus states that "the mean absolute deviation" goes to zero.

By comparison, Theorem 5.1 of [10] states that, for a finite area hyperbolic surface $X_\Gamma = \mathbb{H}^2/\Gamma$ (with $r = \frac{\lambda}{h}$), whenever $\mathrm{Op}(\sigma)$ is a compactly supported homogeneous order 0 pseudo-differential operator,

$$\begin{aligned}\frac{h^2}{4\pi} \int\limits_{|r|\le h^{-1}} &\left| \left\langle \mathrm{Op}_h(\sigma) E\left(\frac{1}{2}+\mathrm{i}r\right), E\left(\frac{1}{2}+\mathrm{i}r\right)\right\rangle + \frac{\varphi'}{\varphi}\left(\frac{1}{2}+\mathrm{i}r\right) \int\limits_{rhS^*M} \sigma \right| \mathrm{d}r \\ &+ h^2 \sum_{\mu:\sqrt{\mu}\le h^{-1}} \left| \langle \mathrm{Op}_h(\sigma) u_\mu, u_\mu\rangle - \int\limits_{h\sqrt{\mu}S^*M} \sigma \right| \to 0. \end{aligned} \tag{9}$$

Here, $\mathbb{H}^2$ is the upper half-plane, $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete co-finite subgroup and $\sigma \in C_0^\infty(S^*X_\Gamma)$. Theorem 6.1 of [10] extends the result to more general homogeneous symbols, including Eisenstein series.

In this article, we use semi-classical operators. The asymptotic (8) is a priori stronger than (9) since it involves integrals/sums of positive quantities over the full spectrum with no change in the power of h. However, if $\sigma \in C_0^\infty(T^*M)$ is supported for $2|\xi| < \lambda_0$, then the contribution from the λ's and $h\sqrt{\mu}$'s larger than λ_0 is $\mathcal{O}(h^\infty)$. Indeed, by results of [1], $\mathrm{Op}(\sigma)\mathbb{1}(-h^2\Delta > \lambda_0^2)$ is a *negligible* operator — i.e. smoothing *and* $\mathcal{O}(h^\infty)$ — thus its trace norm (after removing the zero-Fourier mode coefficient in the cusps) is $\mathcal{O}(h^\infty)$ (again from [1]). Hence the result is essentially the same if one integrates over $\mathbb{R}$ or over $[-Ch^{-1}, Ch^{-1}]$. In the following, we will mostly consider the integral over $[-\lambda_0 h^{-1}, \lambda_0 h^{-1}]$.

For compact manifolds, variance (or absolute sum) decay estimates such as Theorem 1.1 imply that there is a subsequence of density one of the eigenvalues, so that the individual terms tend to zero for every observable $\mathrm{Op}_h(\sigma)$. We refer to [4], [9] and [11] for background. The presence of continuous spectrum complicates this conclusion because the relative densities of discrete and continuous spectrum is itself not known. As discussed in the remarks on page 38 of [10], one may define the density of a family of Eisenstein series and cusp forms by using the measure $M_\Gamma(T) := -\frac{1}{4\pi}\int_{-T}^{T} \frac{\varphi'}{\varphi}(\frac{1}{2}+\mathrm{i}r)\mathrm{d}r$ on intervals of the continuous spectrum and the counting measure $N_\Gamma(T)$ for discrete eigenvalues. One could then define a normalized density on intervals of the union of the discrete and continuous spectrum by dividing the sum of the two measures by $M_\Gamma(T) + N_\Gamma(T)$. In this sense, there should exist a density-one subset of the spectrum so that individual integrands or summands should tend to zero for every σ. Since we work in a very general context, we omit further discussion of sifting out density one subsets.

Since the symbol σ is assumed to be compactly supported, the result is independent of the choice of quantization Op. Especially convenient quantizations are the hyperbolic pseudo-differential ones of [8] that were used in [10] and the related quantization defined in [1]. To keep this article brief, we will use the latter; we do not review the many constructions and definitions in [10] and [1], but refer the reader to those articles for further background.

The quantization procedure Op that was given in [1] enables one to quantize a class of symbols including the compactly supported functions of $(x,\xi) \in T^*M$, the constants, and the functions of the energy $f(|\xi|^2)$, with $f \in C_c^\infty(\mathbb{R})$.

2. Overview of the proof

We follow the organization of the proof of Theorem 5.1 in [10]. There are two parts of the proof. The first step is to prove the following lemma.

Lemma 2.1. *The result of Theorem 1.1 holds if the symbol σ has mean value zero in every energy level $\{|\xi| = constant\}$.*

As the symbol has mean value zero, and has compact support, the proof of this lemma is very similar to the proof of the compact case [4,9]. The proof is given in detail in [10] for hyperbolic cusp surfaces, and the proof is essentially the same for all cusp manifolds. Hence we omit the proof. The main contribution of this note is to prove the second part of the proof where the symbol does not have mean zero. In [10], this part of the proof is given in pp. 39–43 and is quite complicated. The proof we now give is much simpler and more general.

Let us explain now why the lemma is useful. Let $\sigma \in C_c^\infty(T^*M)$. Then, one can define a $C^\infty(\mathbb{R}^+)$ function by

$$\overline{\sigma}(\lambda) = \int\limits_{|\xi|=\lambda} \sigma.$$

Also, as above, let $\varepsilon(\sigma, h)$ be the quantity in the LHS of (8).

If the average of $\chi \in C_c^\infty(M)$ is 1 over M, then σ and $\tilde{\sigma} : (x,\xi) \mapsto \chi(x)\overline{\sigma}(|\xi|)$ have the same average in each energy layer. Hence, to prove the theorem, separate σ into $\tilde{\sigma}$ and $\sigma - \tilde{\sigma}$. The latter can be treated using the lemma. Then we find, using the triangular inequality

$$\limsup_{h\to 0} \varepsilon(\sigma, h) = \limsup_{h\to 0} \varepsilon(\tilde{\sigma}, h). \tag{10}$$

Now, $\tilde{\sigma}$ has a very convenient structure, even more so if we pick carefully the base profile χ, as we will see in the next section.

However, before we get to the next step, we recall some facts that will be very useful. Let Π_y^* be the projector on functions whose zero-Fourier mode vanishes for $y_M > y$. We start with the famous Maass–Selberg relation ([7, section 2], or [2]), which is a clever application of the Stokes formula

$$\int_{y_M \le y} \left| E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 = 2\kappa \log y - \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right) + \operatorname{Tr} \frac{y^{2i\lambda/h}\phi^* - y^{-2i\lambda/h}\phi}{2i\lambda/h} + \int_{y_M > y} |\Pi_y^* E|^2. \tag{11}$$

We use the following Poisson formula [6]:

$$\frac{\varphi'}{\varphi}\left(\frac{d}{2} + ir\right) = Q(r) + \sum_{\rho \text{ pole of } \varphi} \frac{2\Re\rho - d}{(\Re\rho - d/2)^2 + (r - \Im\rho)^2} \tag{12}$$

for some polynomial Q of power less or equal to $2\lfloor d/2 \rfloor$. Notice all but a finite number of poles of φ are on the right of $\{\Re s < d/2\}$.

Lastly, in the sum/integral defining the quantity $\epsilon(\sigma, h)$, consider the contribution from the λ's and $h\sqrt{\mu}$'s larger than λ_0 where σ is supported for $2|\xi| < \lambda_0$. As mentionned above $\mathcal{O}(h^\infty)$ because $\operatorname{Op}(\sigma)\mathbb{1}(-h^2\Delta > \lambda_0^2)$ is a *negligible* operator – i.e. smoothing *and* $\mathcal{O}(h^\infty)$ – hence its trace norm (after removing the zero-Fourier mode coefficient) is $\mathcal{O}(h^\infty)$ (again from [1]).

3. The case of non-zero mean value

Now, we concentrate on $\tilde{\sigma}(x, \xi) = \chi(x)\overline{\sigma}(|\xi|)$. We can stop considering altogether the L^2 eigenfunctions. Indeed, in their contribution ϵ_{pp} the LHS of (8), we can write $\overline{\sigma}(h\sqrt{\mu_j}) = \langle \overline{\sigma}(h\sqrt{\mu_j})u_j, u_j\rangle$. Up to a $\mathcal{O}(h)$ error term, this is $\langle \operatorname{Op}_h(\overline{\sigma}(|\xi|))u_j, u_j\rangle$, so that the main term in ϵ_{pp} is

$$h^{d+1} \sum_{\mu} |\langle \operatorname{Op}_h(\sigma - \overline{\sigma})u_\mu, u_\mu\rangle|. \tag{13}$$

With $\sigma' = \sigma - \overline{\sigma}$, we are back to the case $\overline{\sigma} = 0$, and the proof of Lemma 2.1 applies.

Now, by the localization of eigenfunctions – see [11] – $E(d/2 + i\lambda/h)$ is semi-classically microlocally supported on $\{|\xi| \sim \lambda\}$; as a consequence, $\operatorname{Op}_h(\tilde{\sigma})E(d/2 + i\lambda/h) \sim \chi \times E(d/2 + i\lambda/h) \times \overline{\sigma}(\lambda)$. This statement can be made precise by

$$\left\langle \operatorname{Op}_h(\tilde{\sigma})E\left(\frac{d}{2} + i\frac{\lambda}{h}\right), E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right\rangle = \int_M \chi \left|E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 \overline{\sigma}(\lambda) + \mathcal{O}(h)\|E\|^2_{\{\chi \neq 0\}}.$$

Call R_1 the contribution of the remainder in the RHS to $\varepsilon(\tilde{\sigma}, h)$. Let us first deal with the main term. Let λ_0 be such that σ is supported for $2|\xi| < \lambda_0$. The main contribution to $\varepsilon(\tilde{\sigma}, h)$ will be

$$h^d \int_{-\lambda_0}^{\lambda_0} |\overline{\sigma}(\lambda)| \times \left| \int_M \chi \left|E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right| d\lambda$$

Now, we choose the shape χ in the following way. We suppose that χ is constant in M_0, and that in the cusps, it writes as $c_\nu \chi_0(y\nu)$, where $\chi_0 \in C_c^\infty(\mathbb{R})$ equals 1 close to zero, ν is a small parameter and $c_\nu \in \mathbb{R}^+$ is a normalization constant. It is chosen so that χ has mean value 1, i.e. $\int_M \chi = \operatorname{vol}(M)$. Write

$$\int_M \chi \left|E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 = -c_\nu \int \nu \chi_0'(y\nu) \left\{ \int_{y_M \le y} |E|^2 \right\} dy.$$

For ν small enough, this is well defined, and we can use the Maass–Selberg formula (11):

$$\int_M \chi \left|E\left(\frac{d}{2} + i\frac{\lambda}{h}\right)\right|^2 = -c_\nu \int \nu \chi_0'(y\nu) \left\{ 2\kappa \log y - \frac{\varphi'}{\varphi} + \operatorname{Tr} \frac{y^{2i\lambda/h}\phi^* - y^{-2i\lambda/h}\phi}{2i\lambda/h} \right\} dy + \int_M (1 - \chi)\left|\Pi_0^* E\right|^2$$

In the RHS, the third term is highly oscillating. It will only contribute for $\mathcal{O}(h^\infty)$ to the final result, by non-stationary phase. The second will contribute by $-c_\nu \varphi'/\varphi$. The fourth is an integral supported in the cusps. We are left to prove that

$$h^d \int_{-\lambda_0}^{\lambda_0} |\overline{\sigma}(\lambda)| \times \left\{ \left| 2\kappa \int c_\nu \nu \chi_0'(\nu y) \cdot \log y dy \right| + \left| (1-c_\nu) \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \right| \right\} \mathrm{d}\lambda$$

$$+ h^{d+1} \operatorname{Tr}\left\{ \Pi_0^*(1-\chi)|\overline{\sigma}|(-h^2\Delta) \right\}, \quad \text{goes to 0 with } h.$$

A direct computations shows this for the first term. Let us call the third term R_2. The second term gives

$$h^d |1-c_\nu| \int_{-\lambda_0}^{\lambda_0} h^d \overline{\sigma}(\lambda) \left| \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \right|.$$

In the decomposition (12), the sum over the resonances is a function that is always negative for big values of λ, and the polynomial part is explicit, so we know that

$$\int_{-\lambda_0}^{\lambda_0} \left| \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \right| + \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \mathrm{d}\lambda = \mathcal{O}(h^{-d}).$$

Combining this with the Weyl law (5), we arrive at

$$|1-c_\nu| \int_{-\lambda_0}^{\lambda_0} h^d \overline{\sigma}(\lambda) \left| \frac{\varphi'}{\varphi}\left(\frac{d}{2} + \mathrm{i}\frac{\lambda}{h}\right) \right| = |1-c_\nu| \mathcal{O}(1).$$

Now, we come back to R_1 and R_2. First, R_1 can be bounded by some $\mathcal{O}(h) \operatorname{Tr} \operatorname{Op}_h(\sigma')$, with σ' compactly supported (on a set slightly bigger than $\tilde{\sigma}$). Then, we can apply the local Weyl Law (6). For R_2, we can *also* apply formula (6) (the symbol this time is not compactly supported, but it also is in the quantizable class of [1]). Hence, we find that R_1 and R_2 contribute to the lim sup by

$$\mathcal{O}(1) \operatorname{vol}\{y_M \geq 1/\nu\}.$$

At last, let us observe that when $\nu \to 0$, the assumption that χ has mean value 1 implies that $c_\nu \to 1$. Actually, this can be made more precise, as $1 - c_\nu = \mathcal{O}(\operatorname{vol}\{y > 1/\nu\})$. We deduce that, for some constant $C > 0$,

$$\limsup_{h \to 0} \varepsilon(\tilde{\sigma}, h) \leq C \operatorname{vol}\{y \geq 1/\nu\}.$$

As this does not depend on ν, we can take $\nu \to 0$, and this ends the proof.

References

[1] Y. Bonthonneau, Long time quantum evolution of observables, Commun. Math. Phys. 343 (1) (2016) 311–359.
[2] Y. Bonthonneau, Weyl laws for manifolds with hyperbolic cusps, arXiv:1512.05794, December 2015.
[3] Y. Colin de Verdière, Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein, C. R. Acad. Sci. Paris, Sér. I Math. 293 (7) (1981) 361–363.
[4] Y. Colin de Verdière, Ergodicité et fonctions propres du Laplacien, Commun. Math. Phys. 102 (3) (1985) 497–502.
[5] Werner Müller, The point spectrum and spectral geometry for Riemannian manifolds with cusps, Math. Nachr. 125 (1986) 243–257.
[6] Werner Müller, Spectral geometry and scattering theory for certain complete surfaces of finite volume, Invent. Math. 109 (2) (1992) 265–305.
[7] Peter Sarnak, Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series, Commun. Pure Appl. Math. 34 (6) (1981) 719–739.
[8] Steven Zelditch, Pseudodifferential analysis on hyperbolic surfaces, J. Funct. Anal. 68 (1) (1986) 72–105.
[9] Steven Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (4) (1987) 919–941.
[10] Steven Zelditch, Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series, J. Funct. Anal. 97 (1) (1991) 1–49.
[11] Maciej Zworski, Semiclassical Analysis, Grad. Stud. Math., vol. 138, American Mathematical Society, Providence, RI, 2012.