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Topology

Kameko's homomorphism and the algebraic transfer

Le morphisme de Kameko et le transfert algébrique

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A R T I C L E I N F O

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A B S T R A C T

Let $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ be the graded polynomial algebra over the prime field of two elements $\mathbb{F}_2$, in k generators $x_1, x_2, \dots, x_k$, each of degree 1. Being the mod-2 cohomology of the classifying space $B(\mathbb{Z}/2)^k$, the algebra P_k is a module over the mod-2 Steenrod algebra $\mathcal{A}$. In this Note, we extend a result of Hưng on Kameko's homomorphism $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \longrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. Using this result, we show that Singer's conjecture for the algebraic transfer is true in the case $k = 5$ and the degree $7.2^s - 5$ with s an arbitrary positive integer.

R É S U M É

Soit $P_k := \mathbb{F}_2[x_1, x_2, \dots, x_k]$ l'algèbre polynomiale graduée à k générateurs sur le corps à deux éléments $\mathbb{F}_2$, chaque générateur étant de degré 1. En tant que cohomologie mod-2 du classifiant $B(\mathbb{Z}/2)^k$, l'algèbre P_k est dotée d'une structure naturelle de module sur l'algèbre de Steenrod $\mathcal{A}$. Dans cette Note, nous généralisons un résultat de Hưng pour le morphisme de Kameko $\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \longrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. En appliquant ce résultat, nous montrons que la conjecture de Singer pour le transfert algébrique est vraie pour $k = 5$ et le degré $7{,}2^s - 5$ avec $s > 0$.

Let $(\mathbb{Z}/2)^k$ be the elementary Abelian 2-group of rank k. Denote by $B(\mathbb{Z}/2)^k$ the classifying space of $(\mathbb{Z}/2)^k$. Then,

$$P_k := H^*(B(\mathbb{Z}/2)^k) \cong \mathbb{F}_2[x_1, x_2, \dots, x_k],$$

a polynomial algebra in k variables $x_1, x_2, \dots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

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Being the cohomology of a topological space, P_k is a module over the mod-2 Steenrod algebra, $\mathcal{A}$. The action of $\mathcal{A}$ on P_k is explicitly given by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & \text{otherwise}, \end{cases}$$

and subject to the Cartan formula $Sq^n(fg) = \sum_{i=0}^n Sq^i(f)Sq^{n-i}(g)$, for $f, g \in P_k$ (see Steenrod and Epstein [12]).

Let GL_k be the general linear group over the field $\mathbb{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an inherited action of GL_k on $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$.

Denote by $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ the subspace of $\mathbb{F}_2 \otimes_{\mathcal{A}} P_k$ consisting of the classes represented by the homogeneous polynomials of degree n in P_k. In [10], Singer defined the algebraic transfer, which is a homomorphism

$$\varphi_k : \mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n^{GL_k}$$

from the homology of the Steenrod algebra to the subspace of $(\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n$ consisting of all the GL_k-invariant classes. It is a useful tool in describing the homology groups of the Steenrod algebra, $\mathrm{Tor}^{\mathcal{A}}_{k,k+n}(\mathbb{F}_2, \mathbb{F}_2)$. The Singer algebraic transfer was studied by many authors (see Boardman [2], Hưng [5], Chơn–Hà [4], Nam [9], Hưng–Quỳnh [6], and others).

Singer showed in [10] that φ_k is an isomorphism for $k = 1, 2$. Boardman showed in [2] that φ_3 is also an isomorphism. However, for any $k \geqslant 4$, φ_k is not a monomorphism in infinitely many degrees (see Singer [10], Hưng [5]). Singer made the following conjecture.

Conjecture 1 *(see Singer [10]). The algebraic transfer φ_k is an epimorphism for any $k \geqslant 0$.*

The conjecture is true for $k \leqslant 3$. Based on the results in [13,14], it can be verified for $k = 4$.

In this Note, we extend Hưng's result in [5] on Kameko's homomorphism

$$\widetilde{Sq}^0_* : \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \longrightarrow \mathbb{F}_2 \otimes_{\mathcal{A}} P_k.$$

This homomorphism is an GL_k-homomorphism induced by the $\mathbb{F}_2$-linear map, also denoted by $\widetilde{Sq}^0_* : P_k \to P_k$, given by

$$\widetilde{Sq}^0_*(x) = \begin{cases} y, & \text{if } x = x_1x_2 \dots x_k y^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in P_k$. Note that $\widetilde{Sq}^0_*$ is not an $\mathcal{A}$-homomorphism. However, $\widetilde{Sq}^0_* Sq^{2t} = Sq^t \widetilde{Sq}^0_*$ and $\widetilde{Sq}^0_* Sq^{2t+1} = 0$ for any non-negative integer t.

For a positive integer n, by $\mu(n)$, one means the smallest number r for which it is possible to write $n = \sum_{1 \leqslant i \leqslant r}(2^{u_i} - 1)$, where $u_i > 0$.

Theorem 2 *(see Kameko [7]). Let m be a positive integer. If $\mu(2m + k) = k$, then*

$$\widetilde{Sq}^0_* : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{2m+k} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_m$$

is an isomorphism of GL_k-modules.

By a direct computation, we can see that for a non-negative integer d with either $d = 0$ or $\mu(d) \leqslant k$, there exists a non-negative integer t such that $\mu(k(2^s - 1) + 2^s d) = k$ for every $s > t$. Hence, Theorem 2 implies that

$$(\widetilde{Sq}^0_*)^{s-t} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^s d} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^t-1)+2^t d}$$

is an isomorphism of GL_k-modules for every $s \geqslant t$. However, this result does not confirm how large t should be.

Denote by $\alpha(n)$ the number of ones in dyadic expansion of n and by $\zeta(n)$ the greatest integer u such that n is divisible by 2^u. That means $n = 2^{\zeta(n)}m$, with m an odd integer. For any non-negative integer d, set

$$t(k, d) = \max\{0, k - \alpha(d + k) - \zeta(d + k)\}.$$

The following is one of our main results.

Theorem 3. *Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}^0_*)^{s-t} : (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^s d} \longrightarrow (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^t-1)+2^t d}$$

is an isomorphism of GL_k-modules for every $s \geqslant t$ if and only if $t \geqslant t(k, d)$.

For either $d=0$ or $\mu(d)\leqslant k$, we show that $t=t(k,d)$ is the minimum number such that $\mu(k(2^s-1)+2^sd)=k$ for every $s>t$. Then, the theorem follows from Theorem 2.

If $\mu(d)>k$, then $\mu(k(2^s-1)+2^sd)>k$ for every $s\geqslant 0=t(k,d)$. From a result of Wood [16], we have $(\mathbb{F}_2\otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^sd}=0$, for every $s\geqslant 0$. Therefore, Theorem 3 holds for an arbitrary non-negative integer d.

It is easy to see that $t(k,d)\leqslant k-2$ for every d and $k\geqslant 2$. So, one gets the following.

Corollary 4 *(see Hưng [5]). Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}^0_*)^{s-k+2}:(\mathbb{F}_2\otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^sd}\longrightarrow(\mathbb{F}_2\otimes_{\mathcal{A}} P_k)_{k(2^{k-2}-1)+2^{k-2}d}$$

is an isomorphism of GL_k-modules for every $s\geqslant k-2$.

Corollary 4 shows that the number $t=k-2$ commonly serves for every degree d. In [5], Hưng predicted that $t=k-2$ is the minimum number for this purpose and proved it for $k=5$. It is easy to see that for $d=2^k-k+1$, we have $t(k,d)=k-2$. So, his prediction is true for all $k\geqslant 2$.

An application of Theorem 3 is the following theorem.

Theorem 5. *Singer's conjecture is true for $k=5$ and the degree 7.2^s-5 with s an arbitrary positive integer.*

For $d=2$, we have $t(5,2)=2$ and $5(2^s-1)+2^sd=7.2^s-5$. So, by Theorem 3,

$$(\widetilde{Sq}^0_*)^{s-2}:(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{7.2^s-5}\longrightarrow(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{23}$$

is an isomorphism of GL_5-modules for every $s\geqslant 2$. Hence, by an explicit computation of $(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_{7.2^s-5}$ and $(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)^{GL_5}_{7.2^s-5}$ for $s=1,2$, one gets the following.

Theorem 6. *Let $m=7.2^s-5$ with s a positive integer. Then*

i) $\dim(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_m=191$ *for $s=1$, and* $\dim(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)_m=1245$ *for any $s\geqslant 2$.*
ii) $(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)^{GL_5}_m=0$ *for any $s\geqslant 1$.*

The second part of the theorem has been proved by Singer [10] for $s=1$. In [5], Hưng also proved this theorem for $s=2$ by using computer calculation. However, the detailed proof was unpublished at the time of the writing.

The proof of Theorem 6 is long and very technical. The first part is proved by determining the admissible monomials of degree m in P_5. The computations are based on some results of Kameko [7] and Singer [11] on the admissible monomials and the hit monomials (see [14]). We prove the second part by a direct computation using the admissible monomials of degree m which are determined in the first part. The computations are also based on Singer's criterion in [11] on the hit monomials.

From the results of Tangora [15], Lin [8] and Chen [3], we obtain

$$\mathrm{Tor}^{\mathcal{A}}_{5,7.2^s}(\mathbb{F}_2,\mathbb{F}_2)=\begin{cases}\langle(Ph_1)^*\rangle, & \text{if } s=1,\\ \langle(h_sg_{s-1})^*\rangle, & \text{if } s\geqslant 2,\end{cases}$$

and $h_sg_{s-1}\neq 0$, where h_s denote the Adams element in $\mathrm{Ext}^{1,2^s}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$, P is the Adams periodicity operator in [1] and $g_{s-1}\in\mathrm{Ext}^{4,2^{s+2}+2^{s+1}}_{\mathcal{A}}(\mathbb{F}_2,\mathbb{F}_2)$ for $s\geqslant 2$. Hence, by Theorem 6(ii), the homomorphism

$$\varphi_5:\mathrm{Tor}^{\mathcal{A}}_{5,7.2^s}(\mathbb{F}_2,\mathbb{F}_2)\longrightarrow(\mathbb{F}_2\otimes_{\mathcal{A}} P_5)^{GL_5}_{7.2^s-5}$$

is an epimorphism. However, it is not a monomorphism. This result confirms the one of Hưng.

Corollary 7 *(see Hưng [5]). There are infinitely many degrees in which φ_5 is not a monomorphism.*

The proofs of the results of this Note will be published in detail elsewhere.

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References

[1] J.F. Adams, A periodicity theorem in homological algebra, Math. Proc. Camb. Philos. Soc. 62 (1966) 365–377, MR0194486.

[2] J.M. Boardman, Modular representations on the homology of power of real projective space, in: M.C. Tangora (Ed.), Algebraic Topology, Oaxtepec, 1991, in: Contemp. Math., vol. 146, 1993, pp. 49–70, MR1224907.

[3] T.W. Chen, Determination of $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topol. Appl. 158 (2011) 660–689, MR2774051.

[4] P.H. Chơn, L.M. Hà, Lambda algebra and the Singer transfer, C. R. Acad. Sci. Paris, Ser. I 349 (2011) 21–23, MR2755689.

[5] N.H.V. Hưng, The cohomology of the Steenrod algebra and representations of the general linear groups, Trans. Amer. Math. Soc. 357 (2005) 4065–4089, MR2159700.

[6] N.H.V. Hưng, V.T.N. Quỳnh, The image of Singer's fourth transfer, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1415–1418, MR2588792.

[7] M. Kameko, Products of projective spaces as Steenrod modules, PhD thesis, The Johns Hopkins University, ProQuest LLC, Ann Arbor, MI, USA, 1990, 29 p., MR2638633.

[8] W.H. Lin, $\mathrm{Ext}_{\mathcal{A}}^{4,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ and $\mathrm{Ext}_{\mathcal{A}}^{5,*}(\mathbb{Z}/2,\mathbb{Z}/2)$, Topol. Appl. 155 (2008) 459–496, MR2380930.

[9] T.N. Nam, Transfert algébrique et action du groupe linéaire sur les puissances divisées modulo 2, Ann. Inst. Fourier (Grenoble) 58 (2008) 1785–1837, MR2445834.

[10] W.M. Singer, The transfer in homological algebra, Math. Z. 202 (1989) 493–523, MR1022818.

[11] W.M. Singer, On the action of the Steenrod squares on polynomial algebras, Proc. Amer. Math. Soc. 111 (1991) 577–583, MR1045150.

[12] N.E. Steenrod, D.B.A. Epstein, Cohomology Operations, Annals of Mathematics Studies, vol. 50, Princeton University Press, Princeton, NJ, USA, 1962, MR0145525.

[13] N. Sum, On the hit problem for the polynomial algebra, C. R. Acad. Sci. Paris, Ser. I 351 (2013) 565–568, MR3095107.

[14] N. Sum, On the Peterson hit problem, Adv. Math. 274 (2015) 432–489, MR3318156.

[15] M.C. Tangora, On the cohomology of the Steenrod algebra, Math. Z. 116 (1970) 18–64, MR0266205.

[16] R.M.W. Wood, Steenrod squares of polynomials and the Peterson conjecture, Math. Proc. Camb. Philos. Soc. 105 (1989) 307–309, MR0974986.