



Partial differential equations

Identifiability for a severely ill-posed oxygen balance model

*Identifiabilité pour un système de désoxygénation–réoxygénation sévèrement mal posé*Naïma Débit^a, Souad Khiari^{b,c}^a Université Claude-Bernard, Lyon-1, UMR CNRS 5208, ICJ, 69100 Villeurbanne, France^b Sorbonne Universités, UTC, EA 2222, LMAC, 60205 Compiègne, France^c Université de Tunis El Manar, École nationale d'ingénieurs de Tunis, LAMSIN, 1002, Tunis, Tunisie

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ABSTRACT

We are interested in recovering boundary data in a dispersive oxygen-balance model. The missing boundary condition is the flux of the biochemical oxygen demand (the amount of oxygen necessary for the oxidation of organic matter) at one extreme point. The observations are collected on the dissolved oxygen at the other extremity. This problem turns out to be severely ill-posed. We perform the mathematical analysis of it. We prove a uniqueness result owing to Pazy's theorem for parabolic boundary value problems and we prove that the compatible data set is dense.

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R É S U M É

Nous nous intéressons au problème inverse de complétion de données pour un modèle parabolique de biodégradation, basé sur deux traceurs : la demande biochimique en oxygène (DBO) et l'oxygène dissous (OD). La donnée manquante est le flux de la DBO à l'extrémité amont du cours d'eau. La contrepartie est que l'on dispose de deux conditions à l'extrémité aval sur l'OD. Le problème résultant est mal posé. Nous vérifions qu'il souffre d'une forte instabilité ; il est donc sévèrement mal posé. Ensuite, nous réalisons l'analyse mathématique du problème pour prouver un résultat d'unicité de la solution, et nous montrons que l'ensemble des données compatibles est dense.

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La modélisation de la pollution organique des eaux s'appuie essentiellement sur deux traceurs (cf. [10]). L'un est l'oxygène dissous, et l'autre est la demande biochimique en oxygène, c'est-à-dire la quantité d'oxygène nécessaire à la

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biodégradation de la matière organique. Dans les procédés d'autoépuration, certaines bactéries aérobies jouent un rôle principal; elles décomposent la matière organique polluante en absorbant une fraction de l'oxygène dissous dans le milieu. La DBO est donc un indicateur pertinent de la présence de matières organiques biodégradables dans les cours d'eau. Cela permet aussi d'évaluer les qualités organoleptiques de l'eau.

L'objectif de cette note est l'analyse de la reconstitution du flux de pollution (de DBO) à partir d'une observation sur l'oxygène dissous (OD). La particularité de ce problème réside dans le fait que la station d'observation est loin de l'endroit où l'on souhaite retrouver la donnée manquante. Ce problème est gouverné mathématiquement par un modèle de complétion de données au bord pour un système de deux équations paraboliques où seuls les phénomènes de réaction et de dispersion sont pris en compte. Nonobstant son importance physique, l'advection est mise de côté, car elle ne cause pas de difficultés mathématiques majeures. Tout ce qui sera démontré ici peut être étendu au système de transport complet. Le problème inverse qui nous préoccupe est donné par les équations (1)–(5). Les conditions aux limites sont abondantes sur la concentration d'OD (c), alors que certaines données sont manquantes sur la densité de DBO (b). Nous débutons par l'étude du caractère mal posé de ce problème de complétion de données, après l'avoir exprimé sous sa forme réduite. Dans le problème réduit, le flux de pollution est l'inconnue principale, ce qui nous amène à mettre en évidence un opérateur sur l'ensemble des flux admissibles. La difficulté d'analyser, et éventuellement de résoudre le problème, est intrinsèquement liée aux propriétés de cet opérateur. Nous établissons, dans un cas particulier, que cet opérateur est finalement un opérateur de convolution de noyau régulier et très plat à l'origine. Nous examinons par la suite l'identifiabilité. Un résultat d'unicité est démontré grâce à la théorie de point-selle développée dans [4,3] et à un résultat d'unicité dû à A. Pazy [9].

1. Introduction and setting of the problem

The model we deal with is centered on the indicator b , for the Biochemical Oxygen Demand and the Dissolved Oxygen concentration c ; their respective acronyms are BOD and DO. The BOD is the amount of oxygen necessary for the biodegradation of the organic matter, while the DO is the oxygen housed in the water. In the sequel, the symbol x is used for the curvilinear abscissa, while t stands for the time variable. The stream water is thus represented by $I = (0, L)$, while $T > 0$ is the final instant. The couple of concentrations (b, c) is the solution to the following boundary value system:

$$\partial_t b - (db')' + rb = f \quad \text{in } I \times (0, T), \tag{1}$$

$$\partial_t c - (dc')' + r_*c + rb = g \quad \text{in } I \times (0, T), \tag{2}$$

$$db'(L, t) = dc'(L, t) = 0 \quad \text{in } (0, T), \tag{3}$$

$$db'(0, t) = \gamma(t), \quad dc'(0, t) = 0 \quad \text{in } (0, T), \tag{4}$$

$$b(x, 0) = 0, \quad c(x, 0) = c_S \quad \text{in } I. \tag{5}$$

The longitudinal dispersion coefficient d and the parameters of reaction rates (r, r_*) lie in $L^\infty(I)$ and are positive. The dispersion and reaction parameters d and r are bounded away from zero. The term rb appearing in the transport equation on c is the depletion of oxygen. It is of course possible to add advection to the transport equations. In spite of its important physical role, we choose not to do so for simplicity and because they have no real influence on the mathematics we expose here. Hence, the results we provide are still valid for the advective system. Neumann conditions (4) tell that no oxygen supply (on c) occurs at point $x = 0$ and says also that a polluting flux (on b) is taking place.

The direct system (1)–(5) is triangular and can be studied using well-known tools from the theory of parabolic differential equations (see [9,8]). The specific fact here is that, in real-life situations, measurements on the polluting flux $\gamma(\cdot)$ are too hard to obtain. On the contrary, recording the values of c at the border is easy and may be realized instantaneously. Condition (4) on b at $x = 0$ is therefore replaced by a Dirichlet condition on c at $x = L$, so that

$$c(L, t) = \alpha(t). \tag{6}$$

The effect of the double boundary conditions on c at $x = L$ is dramatic. The nature of the problem is entirely altered. With (4), it was well posed, with (6) it becomes ill posed. It was a direct problem, it becomes an inverse problem. A diagram for the boundary conditions to deal with is provided in Fig. 1.

Ill-posedness degree—To briefly address this important issue, we consider the flux $\gamma(\cdot)$ as a full unknown. Let then $\gamma \in L^2(0, T)$ be given and denote by b_γ the unique solution to the direct problem obtained by assembling equation (1), (the first) boundary condition (3), (the first) condition (4) and (the first) initial condition (5). We turn afterward to c_γ that we construct as the solution to the parabolic problem formed by equation (2) (with $-rb_\gamma$ as a source term), (the second) boundary condition (3), (the second) condition (4) and (the second) initial condition (5). The polluting flux γ to look for is then the one that satisfies the following observation:

$$S\gamma(t) = c_\gamma(L, t) = \alpha(t), \quad \text{in } (0, T). \tag{7}$$

With this formulation, one may carry out Fourier computations to derive an explicit expression of S , in the case of constant parameters. If $d = 1$ and $r = r_* = 1$ then Fourier computations can be conducted and we obtain:

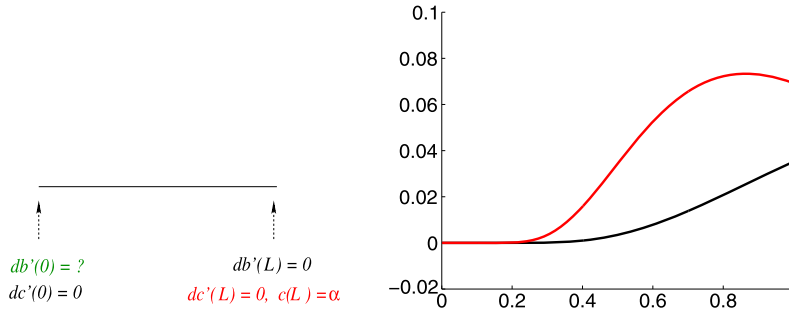


Fig. 1. The representative curves of the kernels K and K' (left).
Fig. 1. Les courbes représentatives des noyaux K et K' (gauche).

$$(S\gamma)(t) = \int_0^t K(t-s)\gamma(s) ds, \quad \forall t \in (0, T).$$

The convolution Kernel $K(\cdot)$ is given by (after setting $\lambda_k = (\frac{k\pi}{L})^2 + 1$):

$$K(s) = \frac{1}{L} s e^{-s} + \frac{2}{L} \sum_{k \geq 1} (-1)^k s e^{-\lambda_k s} \quad \forall s \in (0, T).$$

Equation (7) turns out to be of Volterra type and solving it may be seen as a deconvolution process. Various tools have been developed for studying such a problem [7]. In particular, on account of the easily checked fact that $K \in L^2(0, T)$, the operator S is compact. As a result, problem (7) is ill posed. If we are involved in its degree of ill-posedness, we need to check out the decaying rate of its singular values. This point is known to be related to the regularity of the kernel K at the vicinity of the origin. It is precisely dependent on the flatness of K at $s = 0$. To have a better insight, we plot in Fig. 1 the kernel K so as its derivative K' . The kernel K being strongly flat, the sequence of the singular values of the operator S decays fast toward zero. This makes problem (7) be severely ill posed.

2. Identifiability

The identifiability question can be handled following the methodology developed in [1,2]. The main difference with those works is that, here, we intend to recover the polluting flux at one extremity of the river from observations made on the other extremity. The mathematical process consists first in putting the problem under an abstract form. After setting $Y = (b, c)^T$ and $F = (f, g)^T$, the inverse problem can be written under

$$\partial_t Y + AY = F. \tag{8}$$

The operator A is defined by: for all $\Upsilon = (\varphi, \psi)^T$,

$$A\Upsilon = \begin{pmatrix} -(\text{d}\varphi')' + r\varphi \\ -(\text{d}\psi')' + r_*\psi + r\varphi \end{pmatrix}.$$

It is an unbounded linear operator in $L^2(I)$. The domain $D(A)$ is given by

$$D(A) = \left\{ \Upsilon \in H^1(I), \quad ((\text{d}\varphi')', (\text{d}\psi')') \in L^2(I) \quad \text{d}\psi'(0) = 0, (\psi, \text{d}\psi', \text{d}\varphi')(L) = (0, 0, 0) \right\}.$$

We start by the analysis of the quasi-steady problem, where the time derivatives $(\partial_t b, \partial_t c)$ are replaced by $(\lambda b, \lambda c)$, with λ a positive real number. This is connected to the resolvent $R(\lambda) = (\lambda + A)^{-1}$ (see [5]).

2.1. The resolvent $R(\lambda)$

We need first to describe the spaces where (b, c) are sought for and where the test functions (in the variational formulation) are selected. We introduce therefore three Hilbert spaces

$$V = H^1(I), \quad Q = \left\{ \psi \in H^1(I) \quad \psi(0) = 0 \right\}, \quad Q_* = \left\{ \psi \in H^1(I) \quad \psi(L) = 0 \right\}.$$

We also define three bilinear forms

$$a(\chi, \varphi) = \int_I r\chi\varphi \, dx, \quad \forall (\chi, \varphi) \in V \times V,$$

$$m(\psi, \varphi) = \int_I (\lambda + r)\psi\varphi \, dx + \int_I d\psi'\varphi' \, dx, \quad \forall (\psi, \varphi) \in Q \times V.$$

λ is a fixed real-number. The notation m_* is used when r is replaced by r_* and Q by Q_* in the expression of m . All these forms are obviously continuous.

Now, we can express the problem under a variational form: find $Y = (b, c)^T \in V \times Q_*$ verifying

$$m(b, \psi) = (f, \psi)_{L^2} \quad \forall \psi \in Q, \tag{9}$$

$$m_*(\varphi, c) + a(b, \varphi) = (g, \varphi)_{L^2} \quad \forall \varphi \in V. \tag{10}$$

To study this non-symmetric problem, we call for the tools developed in [3]. One has first to check out that the bilinear forms $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$ fulfill inf-sup conditions on $Q \times V$ and $Q_* \times V$, respectively. Then, we need that $a(\cdot, \cdot)$ satisfies a couple of inf-sup conditions on the kernel spaces of $m(\cdot, \cdot)$ and $m_*(\cdot, \cdot)$, defined as

$$\mathcal{N} = \left\{ \varphi \in V, \quad (\lambda + r)\varphi - (d\varphi)' = 0 \quad \text{in } I, \quad d\varphi'(L) = 0 \right\},$$

$$\mathcal{N}_* = \left\{ \chi \in V, \quad (\lambda + r_*)\chi - (d\chi)' = 0 \quad \text{in } I, \quad d\chi'(0) = 0 \right\}.$$

\mathcal{N} is of dimension one and $\varphi(0)$ may be the only degree of freedom for all $\varphi \in \mathcal{N}$. Similarly, $\chi(L)$ is the degree of freedom for all $\chi \in \mathcal{N}_*$. The inf-sup conditions may be achieved through some constructive process that requires some estimates on the solutions $\varphi_1 \in \mathcal{N}_*$ and $\chi_1 \in \mathcal{N}$, where $\varphi_1(0) = 1$ and $\chi_1(L) = 1$ (φ_1 and χ_1 are respective bases of \mathcal{N} and \mathcal{N}_*).

Lemma 2.1. *There exists $\lambda_0 > 0$ and a constant, $\tau = \tau(\lambda_0) > 0$ independent of λ such that, for all $\lambda \geq \lambda_0$,*

$$a(\chi_1, \varphi_1) \geq e^{-\tau\sqrt{\lambda}}.$$

Proof. The proof for space varying d and (r, r_*) is based on comparison results concerning the solution to the Sturm-Liouville equations. It is technical and long (see [6]). To fix the ideas, we provide here a short proof for constant parameters d, r, r_* . Let us set that $d\omega^2 = \lambda + r$ and $d\omega_*^2 = \lambda + r_*$, we have:

$$\varphi_1(x) = \frac{\cosh(\omega(L-x))}{\cosh(\omega L)} \quad \text{and} \quad \chi_1(x) = \frac{\cosh(\omega_*x)}{\cosh(\omega_*L)}.$$

A straightforward calculation yields that, for large λ ,

$$a(\chi_1, \varphi_1) = \int_0^L r\chi_1(x)\varphi_1(x)dx \simeq re^{-\omega_*L}.$$

The proof is achieved after computing the norms in $H^1(I)$ of χ_1 and φ_1 , which is really easy. \square

As direct consequence and the key result of this work is the desired inf-sup conditions with accurate dependence of the inf-sup constants with respect to λ .

Proposition 2.2. *For all $\lambda \geq \lambda_0$, there exists $\tau = \tau(\lambda_0) > 0$ such that the following inf-sup conditions hold*

$$\inf_{\varphi \in \mathcal{N}} \sup_{\chi \in \mathcal{N}_*} \frac{a(\chi, \varphi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \frac{e^{-\tau\sqrt{\lambda}}}{\sqrt{\lambda}} \quad \inf_{\varphi \in \mathcal{N}_*} \sup_{\chi \in \mathcal{N}} \frac{a(\chi, \varphi)}{\|\varphi\|_{H^1} \|\chi\|_{H^1}} \geq \frac{e^{-\tau\sqrt{\lambda}}}{\sqrt{\lambda}}.$$

Proposition 2.3. *Let $(f, g) \in \mathbf{L}^2(I)$. For all $\lambda \geq \lambda_0$, equations (9)–(10) have a unique solution and*

$$\|b\|_{H^1} + \|c\|_{H^1} \leq \sqrt{\lambda} e^{\tau\sqrt{\lambda}} (\|f\|_{L^2} + \|g\|_{L^2}).$$

The resolvent $R(\lambda)$ is then a bounded operator in $\mathbf{L}^2(I)$ and we have:

$$\|R(\lambda)\|_{\mathbf{L}^2(I) \rightarrow \mathbf{L}^2(I)} \leq \sqrt{\lambda} e^{\tau\sqrt{\lambda}}.$$

2.2. A uniqueness result

The overall technical conditions are assembled to guarantee the application of the uniqueness result by A. Pazy (see [9, Chapter 4, Theorem 1.2]). Recall that Y is solution to (8) if

$$Y \in \mathcal{C}([0, T]; \mathbf{L}^2(I)) \cap \mathcal{C}([0, T]; D(A)).$$

Pazy's theorem states that if $R(\lambda)$ exists for large real numbers $\lambda (> 0)$ and

$$\limsup_{\lambda \rightarrow +\infty} \frac{1}{\lambda} \log \|R(\lambda)\|_{(\mathbf{L}^2(I) \rightarrow \mathbf{L}^2(I))} = 0,$$

then the initial value problem (8) has at most one solution. It is easy to check out Pazy's assumption from Proposition 2.3. We have hence the following theorem.

Theorem 2.4. *Problem (8) has at most one solution $Y \in \mathcal{C}([0, T]; \mathbf{L}^2(I)) \cap \mathcal{C}([0, T]; \mathbf{H}^1(I))$.*

3. What about existence?

It may be important for users to obtain information about the observations $\alpha(\cdot)$ for which the inverse problem (8) has a solution. Getting partial knowledge is possible about these compatible observations. Without restricting the generality, we put to zero the data f, g so as c_S . To begin with, notice that the study presented here concludes to the fact that S is a non-closed range operator. As a consequence, the range $R(S)$ has a void interior. Otherwise, S would be bijective. Applying the open map theorem, we conclude that this fact is false. Now, the question to ask is the following: *Is the set of compatible data $\alpha = \alpha(t)$ dense in $L^2(0, T)$?* The answer is positive and is a direct consequence of the identifiability result. We have that

Lemma 3.1. *The range of the operator S is dense in $L^2(0, T)$.*

Proof. We show that the adjoint operator S^* is injective. As a result, the range of S will be dense in $L^2(0, T)$. Let η be given in $L^2(0, T)$. We consider (θ, β) to be the solution to the following backward problem:

$$\begin{aligned} -\partial_t \theta - (d\theta')' + r\theta + r\beta &= 0 && \text{in } I \times (0, T), \\ -\partial_t \beta - (d\beta')' + r_*\beta &= 0 && \text{in } I \times (0, T), \\ d\theta'(0, t) = d\beta'(0, t) &= 0 && \text{in } (0, T), \\ d\beta'(L, t) &= \eta(t) && \text{in } (0, T), \\ d\theta'(L, t) &= 0 && \text{in } (0, T), \\ \theta(x, T) = \beta(x, T) &= 0 && \text{in } I. \end{aligned}$$

Then by direct computations, it can be checked out that

$$S^*\eta(t) = \theta(0, t) \quad \text{in } (0, T).$$

Reproducing the analysis conducted for the progressive problem yields the same identifiability result. Then, the only η lying in the kernel $N(S^*)$ is the trivial function $\eta(\cdot) = 0$. The range of S is therefore dense in $L^2(0, T)$. The proof is complete. \square

4. Conclusion

The particularity here with respect to [1] lies in the fact that the reconstruction of the flux pollution at one extremity of the river is pursued from observations on the opposite extremity. The resulting inverse problem turns out to be severely ill posed and is harder to study and to solve numerically. The identifiability is based on the theory by A. Pazy. The resolvent of the problem has to be investigated. This is realized here. The study presented allows us to apply Pazy's uniqueness theorem and concludes to the identifiability result.

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