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Probability theory

Lévy and Poisson approximations of switched stochastic systems by a semimartingale approach

Approximations de Lévy et de Poisson des systèmes stochastiques modulés, via une approche basée sur les semi-martingales

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A B S T R A C T

In this Note, we present the weak convergence of additive functionals of processes with locally independent increments and Markov switching in Lévy and Poisson approximation schemes. The singular perturbation problem for the generators of switched processes is used to prove the semimartingales' predictable characteristics convergence.

R É S U M É

Nous étudions dans cette Note la convergence faible des fonctionnelles additives des processus à accroissements localement indépendants, avec modulation markovienne, vers des processus de Lévy et de Poisson, sous différentes hypothèses et rééchelonnements de temps. Nous utilisons des techniques de perturbation singulière des opérateurs pour établir des résultats de convergence faible concernant les caractéristiques prédictibles des semi-martingales.

Version française abrégée

Soit la fonctionnelle additive $\int_0^t \eta(ds; x(s))$ rééchelonnée en temps de deux différentes façons – voir les relations (1) et (2) ci-dessous –, dépendant du paramètre $\varepsilon \downarrow 0$, arbitrairement petit.

Les processus $\eta_\varepsilon(ds; x(t/\varepsilon))$, $t \geq 0$, $\varepsilon > 0$, sont des processus *à accroissements localement indépendants*, définis par les générateurs Γ^ε – voir la relation (3). Le processus $x(t)$, $t \geq 0$, est un processus de Markov de purs sauts défini par le générateur $\mathbf{Q}$ – voir la relation (4).

Nous avons les résultats de convergence suivants.

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Théorème 0.1. *Sous les conditions* **C1–C4***, la convergence faible suivante a lieu*

$$\zeta^{\varepsilon}(t) \Rightarrow \zeta^{0}(t), \quad \varepsilon \downarrow 0.$$

Le processus limite $\zeta^0(t)$, $t \geq 0$, est défini par le générateur $\widehat{\Gamma}$ – voir la relation (5) – dans lequel le drift détérministe est défini par $\widehat{b}(u) = \int_E \pi(\mathrm{d}x) b(u; x)$ et le noyau d'intensité moyenne de sauts est défini par $\widehat{\Gamma}(u, \mathrm{d}v) = \int_E \pi(\mathrm{d}x)\Gamma(u, \mathrm{d}v; x)$.

Théorème 0.2. *Sous les conditions* **C1–C3** *et* **C4**′*, la convergence faible suivante a lieu*

$$\xi^{\varepsilon}(t) \Rightarrow \xi^{0}(t), \quad \varepsilon \downarrow 0.$$

Le processus limite $\xi^0(t)$, $t \geq 0$, est un processus de Lévy défini par le générateur $\widehat{\mathbf{L}}$ – voir la relation (6). L'opérateur R_0 est l'opérateur potentiel du processus de Markov, défini par $\mathbf{Q}R_0 = R_0\mathbf{Q} = \Pi - I$.

1. Introduction

The Lévy and Poisson approximations comprise a widely studied research field, while several theoretical results along with applications exist in the literature. Since Lévy processes are now standard, the Lévy approximation is quite useful for analyzing complex systems (see, e.g., [1,12]). Moreover, they are involved in many applications, e.g., risk theory, finance, queueing, physics, etc. For the Lévy process framework, see, e.g., [1,12,5].

Processes with locally independent increments (PLII) in the Poisson and Lévy approximation schemes were studied in [7,9,10]. The main idea of the Lévy approximation scheme is that the jumps of the stochastic system are split into two parts: small jumps with probabilities close to one and large jumps with probabilities tending to zero along with the series parameter $\varepsilon \downarrow 0$. So, in the Lévy approximation principle, the probabilities (or intensities) of jumps are normalized by the series parameter $\varepsilon > 0$.

However, the method used here to prove the weak convergence is quite different from the one proposed by other authors: the aim is to prove the convergence of predictable characteristics of semimartingales that are integral functionals of some switching Markov processes. However, the drawback is that the predictable characteristics of semimartingales themselves depend upon the process we study. Thus, in order to prove the weak convergence of the processes, we should also prove the convergence of predictable characteristics that depend on the initial process. Classical methods cannot cope with this situation.

We propose functionals of PLII [7] (also known as Piecewise deterministic Markov processes – PDMP, [3]) using a combination of two methods. The first one of these methods is based on weak convergence theory for semimartingales, and the second one relies on a solution to the singular perturbation problem instead of the ergodic theorem. Therefore, the proofs include two steps.

In the first one, we prove the relative compactness of the semimartingales $\xi^{\varepsilon}(t)$, $\varepsilon > 0$, and in the second one we prove convergence of the processes $\xi^{\varepsilon}(t)$ by using a singular perturbation technique as presented in [7].

2. Basic definitions and assumptions

Let us consider the space $\mathbb{R}^d$ endowed with a norm $|\cdot|$ ($d \geq 1$), and $(E, \mathcal{E})$, a *standard state space* (i.e., E is a Polish space and $\mathcal{E}$ its Borel σ-algebra, [7]). For a vector $v \in \mathbb{R}^d$ and a matrix $c \in \mathbb{R}^{d\times d}$, denote, respectively, by v^* and c^* the corresponding transposed matrices. Let $C_3(\mathbb{R}^d)$ be a measure-determining class of real-valued bounded functions, such that $g(u)/|u|^2 \to 0$, as $|u| \to 0$ for $g \in C_3(\mathbb{R}^d)$ (see [6,7]). Denote by $\Rightarrow$ the weak convergence in the Skorohod space $D[0, \infty)$ [4].

In order to prove weak convergence for different approximation schemes we should use adequate time-rescaling. Namely, in the case of Poisson approximation, the additive functional $\zeta^{\varepsilon}(t)$, $t \geq 0$, $\varepsilon > 0$ on $\mathbb{R}^d$ in the series scheme, with series parameter $\varepsilon \downarrow 0$, is defined by the stochastic additive functional

$$\zeta^{\varepsilon}(t) = \zeta_0^{\varepsilon} + \int_0^t \eta_{\varepsilon}(\mathrm{d}s; x(s/\varepsilon)), \tag{1}$$

while in the case of the Lévy approximation, we normalize the additive functional $\xi^{\varepsilon}(t)$, $t \geq 0$, $\varepsilon > 0$ on $\mathbb{R}^d$ in another way

$$\xi^{\varepsilon}(t) = \xi_0^{\varepsilon} + \int_0^t \eta_{\varepsilon}(\mathrm{d}s; x(s/\varepsilon^2)). \tag{2}$$

The family of Markov jump processes with *locally independent increments* $\eta_{\varepsilon}(t; x)$, $t \geq 0$, $x \in E$, on $\mathbb{R}^d$, is defined by the generators [7, Section 3.3.1] (see also [8])

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(u) = \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u)]\Gamma^{\varepsilon}(u, \mathrm{d}v; x), \quad x \in E, \tag{3}$$

or, equivalently

$$\mathbf{\Gamma}^{\varepsilon}(x)\varphi(u) = b_{\varepsilon}(u;x)\varphi'(u) + \frac{1}{2}c_{\varepsilon}(u;x)\varphi''(u) + \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u) - v\varphi'(u) - \frac{vv^*}{2}\varphi''(u)]\Gamma^{\varepsilon}(u,\mathrm{d}v;x),$$

where $b_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} v\Gamma^{\varepsilon}(u,\mathrm{d}v;x)$, $c_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^{\varepsilon}(u,\mathrm{d}v;x)$, and $\Gamma^{\varepsilon}(u,\mathrm{d}v;x)$ is the intensity kernel.

The switching Markov process $x(t)$, $t \geq 0$, on the standard state space $(E,\mathcal{E})$, is defined by the generator

$$\mathbf{Q}\varphi(x) = q(x)\int_E P(x,\mathrm{d}y)[\varphi(y) - \varphi(x)], \tag{4}$$

where $q(x)$, $x \in E$, is the intensity of the jumps of $x(t)$, $t \geq 0$, and $P(x,\mathrm{d}y)$ is the transition kernel of the embedded Markov chain x_n, $n \geq 0$, defined as $x_n = x(\tau_n)$, $n \geq 0$, with $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n \leq \ldots$ the jump times of $x(t)$, $t \geq 0$.

There are three conditions common to both approximation schemes:

C1: the Markov process $x(t)$, $t \geq 0$, is uniformly ergodic with stationary distribution $\pi(B)$, $B \in \mathcal{E}$;

C2: *uniform square-integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} vv^*\Gamma(u,\mathrm{d}v;x) = 0,$$

where the kernel $\Gamma(u,\mathrm{d}v;x)$ is defined in **PA2** below;

C3: *growth condition*: there exists a positive constant L such that

$$|b(u;x)| \leq L(1+|u|), \quad \text{and} \quad |c(u;x)| \leq L(1+|u|^2),$$

and for any real-valued nonnegative function $f(x)$, $x \in \mathbb{R}^d$, such that $\int_{\mathbb{R}^d\setminus\{0\}}(1+f(x))|x|^2\mathrm{d}x < \infty$, we have

$$|\Lambda(u,v;x)| \leq Lf(v)(1+|u|),$$

where $\Lambda(u,v;x)$ is the Radon–Nikodym derivative of $\Gamma(u,B;x)$ with respect to the Lebesgue measure $\mathrm{d}v$ in $\mathbb{R}^d$, that is,

$$\Gamma(u,\mathrm{d}v;x) = \Lambda(u,v;x)\mathrm{d}v.$$

2.1. Poisson approximation conditions

C4: The family of processes with locally independent increments $\eta_{\varepsilon}(t;x)$, $t \geq 0$, $x \in E$ satisfies the Poisson approximation conditions [7, Section 7.2.3].

PA1: Approximation of the mean values and second moments:

$$b_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} v\Gamma^{\varepsilon}(u,\mathrm{d}v;x) = \varepsilon[b(u;x) + \theta_b^{\varepsilon}(u;x)],$$

and

$$c_{\varepsilon}(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^{\varepsilon}(u,\mathrm{d}v;x) = \varepsilon[c(u;x) + \theta_c^{\varepsilon}(u;x)].$$

PA2: Poisson approximation condition for intensity kernel

$$\Gamma_g^{\varepsilon}(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma^{\varepsilon}(u,\mathrm{d}v;x) = \varepsilon[\Gamma_g(u;x) + \theta_g^{\varepsilon}(u;x)]$$

for all $g \in C_3(\mathbb{R}^d)$, and the kernel $\Gamma_g(u;x)$ is bounded for all $g \in C_3(\mathbb{R}^d)$, that is,

$$|\Gamma_g(u;x)| \leq \Gamma_g \quad \text{(a constant depending on } g\text{)},$$

where the kernel $\Gamma(u,\mathrm{d}v;x)$ is defined on the class $C_3(\mathbb{R}^d)$ by the relation

$$\Gamma_g(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma(u,\mathrm{d}v;x), \quad g \in C_3(\mathbb{R}^d).$$

The above negligible terms θ_a^{ε}, θ_b^{ε}, θ_c^{ε} satisfy the following condition

$$\sup_{x\in E} |\theta_{\cdot}^{\varepsilon}(u;x)| \to 0, \quad \varepsilon \downarrow 0.$$

PA3: Initial value condition:

$$\sup_{\varepsilon>0} E|\zeta_0^\varepsilon| \le C < \infty$$

and

$$\zeta_0^\varepsilon \Rightarrow \zeta_0.$$

2.2. Lévy approximation conditions

C4′: The family of processes with locally independent increments $\eta_\varepsilon(t;x)$, $t \ge 0$, $x \in E$, satisfies the Lévy approximation conditions [7, Section 9.2].

L1: Approximation of the mean values and second moments:

$$b_\varepsilon(u;x) = \int_{\mathbb{R}^d} v\Gamma^\varepsilon(u,\mathrm{d}v;x) = \varepsilon b_1(u;x) + \varepsilon^2[b(u;x) + \theta_b^\varepsilon(u;x)],$$

and

$$c_\varepsilon(u;x) = \int_{\mathbb{R}^d} vv^*\Gamma^\varepsilon(u,\mathrm{d}v;x) = \varepsilon^2[c(u;x) + \theta_c^\varepsilon(u;x)].$$

L2: Lévy approximation condition for the intensity kernel

$$\Gamma_g^\varepsilon(u;x) = \int_{\mathbb{R}^d} g(v)\Gamma^\varepsilon(u,\mathrm{d}v;x) = \varepsilon^2[\Gamma_g(u;x) + \theta_g^\varepsilon(u;x)]$$

for all $g \in C_3(\mathbb{R}^d)$, and all the conditions on the kernel $\Gamma_g(u;x)$ are equivalent to **PA2**.

L3: *Balance condition*: $\int_E \pi(\mathrm{d}x)b_1(u;x) = 0$.

L4: The same condition as **PA3** with ξ_0^ε instead of ζ_0^ε.

3. Main results

The following two theorems, Theorem 3.1 and Theorem 3.2, provide the Poisson approximation and Lévy approximation respectively.

Theorem 3.1. *Under conditions* **C1**–**C4** *the following weak convergence*

$$\zeta^\varepsilon(t) \Rightarrow \zeta^0(t), \quad \varepsilon \downarrow 0$$

holds. The limit process $\zeta^0(t)$, $t \ge 0$ is defined by the generator

$$\widehat{\mathbf{\Gamma}}\varphi(u) = \widehat{b}(u)\varphi'(u) + \int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u) - v\varphi'(u)]\widehat{\Gamma}(u,\mathrm{d}v), \tag{5}$$

where the average deterministic drift is defined by $\widehat{b}(u) = \int_E \pi(\mathrm{d}x)b(u;x)$, and the average intensity kernel is defined by $\widehat{\Gamma}(u,\mathrm{d}v) = \int_E \pi(\mathrm{d}x)\Gamma(u,\mathrm{d}v;x)$.

Theorem 3.2. *Under conditions* **C1**–**C3** *and* **C4′**, *the following weak convergence*

$$\xi^\varepsilon(t) \Rightarrow \xi^0(t), \quad \varepsilon \downarrow 0$$

holds. The limit process $\xi^0(t)$, $t \ge 0$ is a Lévy process defined by the generator

$$\widehat{\mathbf{L}}\varphi(u) = (\widehat{b}(u) - \widehat{b}_0(u))\varphi'(u) + \frac{1}{2}\sigma^2(u)\varphi''(u) + \lambda(u)\int_{\mathbb{R}^d} [\varphi(u+v) - \varphi(u)]\Gamma^0(u,\mathrm{d}v), \tag{6}$$

where:

$$\widehat{b}(u)=\int_E \pi(\mathrm{d}x)b(u;x),\quad \widehat{b}_0(u)=\int_E v\Gamma(u,\mathrm{d}v),\quad \Gamma(u,\mathrm{d}v)=\int_E \pi(\mathrm{d}x)\Gamma(u,\mathrm{d}v;x),$$

$$\widetilde{b}_1(u;x):=q(x)\int_E P(x,\mathrm{d}y)b_1(u;x),\quad c_0(u;x)=\int_E vv^*\Gamma(u,\mathrm{d}v;x)$$

$$\sigma^2(u)=2\int_E \pi(\mathrm{d}x)\{\widetilde{b}_1(u;x)R_0\widetilde{b}_1^*(u;x)+\frac{1}{2}[c(u;x)-c_0(u;x)]\},\qquad \sigma^2(u)\geq 0,$$

here R_0 is the potential operator of the Markov process, $\mathbf{Q}R_0=R_0\mathbf{Q}=\Pi-I$, and

$$\lambda(u)=\Gamma(u,\mathbb{R}^d),\qquad \Gamma^0(u,\mathrm{d}v)=\Gamma(u,\mathrm{d}v)/\lambda(u).$$

Remark 1. The limit Lévy process consists of three parts: deterministic drift, diffusion part and Poisson part.
There are some cases of special interest:

1) if the limit $\varepsilon^{-2}\int_{|v|>\delta} yy^*\Gamma^\varepsilon(x,\mathrm{d}y)\to 0$, $\varepsilon\to 0$, for any $\delta>0$ holds (see Theorem 4.21 on page 558 of [6]), then the limit process $\xi^0(t)$ does not have a Poisson part;
2) if $\widehat{b}(u)-\widehat{b}_0(u)=0$, then the limit process does not have a deterministic drift;
3) if $\sigma^2(u)=0$ then the limit process does not have a diffusion part. As a variant of this case we note that if $c(u;x)=c_0(u;x)$ then also $b_1(u;x)=0$ and we obtain the conditions of Poisson approximation after renormalization $\varepsilon^2=\widetilde{\varepsilon}$ (see, for example, Chapter 7 in [7]).

Remark 2. The asymptotic behavior of the second moment in the condition **L1** contains the second modified characteristics $c(u;x)$ (see relation 4.2 on page 555 of [6]). These characteristics in the limit contain both the second moment of the Poisson part and dispersion of the diffusion part, namely $c=c_0+\sigma^2$. In the case of the Poisson approximation scheme, $c=c_0$, thus $\sigma^2=0$, and we have no diffusion part in the limit process $\zeta^0(t)$.

4. Brief proof of Theorem 3.1

The proof of Theorem 3.1 is based on the semimartingale representation of the additive functional process (1). According to Theorems 6.27 and 7.16 [2] the predictable characteristics of the semimartingale (1) have the following representations:

- $B^\varepsilon(t)=\varepsilon^{-1}\int_0^t b_\varepsilon(\zeta^\varepsilon(s);x_s^\varepsilon)\,\mathrm{d}s=\int_0^t b(\zeta^\varepsilon(s);x_s^\varepsilon)\,\mathrm{d}s+t\theta_b^\varepsilon$,
- $C^\varepsilon(t)=\varepsilon^{-1}\int_0^t c_\varepsilon(\zeta^\varepsilon(s);x_s^\varepsilon)\,\mathrm{d}s=\int_0^t c(\zeta^\varepsilon(s);x_s^\varepsilon)\,\mathrm{d}s+t\theta_c^\varepsilon$,
- $\Gamma^\varepsilon(t)=\varepsilon^{-1}\int_0^t\int_{\mathbb{R}^d} g(v)\Gamma^\varepsilon(\zeta^\varepsilon(s),\mathrm{d}v;x_s^\varepsilon)\,\mathrm{d}s=\int_0^t\int_{\mathbb{R}^d} g(v)\Gamma(\zeta^\varepsilon(s),\mathrm{d}v;x_s^\varepsilon)\,\mathrm{d}s+t\theta_g^\varepsilon$,

where $x_t^\varepsilon:=x(t/\varepsilon)$, $t\geq 0$, and $\sup_{x\in E}|\theta_\cdot^\varepsilon|\to 0$, $\varepsilon\to 0$.

The jump martingale part of the semimartingale (1) is represented as follows

$$\mu^\varepsilon(t)=\int_0^t\int_{\mathbb{R}^d} v[\mu^\varepsilon(\mathrm{d}s,\mathrm{d}v;x_s^\varepsilon)-\Gamma^\varepsilon(\zeta^\varepsilon(s),\mathrm{d}v;x_s^\varepsilon)\,\mathrm{d}s].$$

Here $\mu^\varepsilon(\mathrm{d}s,\mathrm{d}v;x)$, $x\in E$, is the family of counting measures with characteristics

$$\mathbf{E}\mu^\varepsilon(\mathrm{d}s,\mathrm{d}v;x)=\Gamma^\varepsilon(u,\mathrm{d}v;x)\,\mathrm{d}s.$$

We can see now that the predictable characteristics depend on the process $\zeta^\varepsilon(s)$. Thus, to prove the convergence of $\zeta^\varepsilon(s)$, we should prove the convergence of predictable characteristics dependent on $\zeta^\varepsilon(s)$. Instead, we combine the two methods.

Step 1. At this step, we establish the relative compactness of the family of processes $\zeta^\varepsilon(t)$, $t\geq 0$, $\varepsilon>0$ by using the approach developed in [11]. Let us remind that the space of all probability measures defined on the standard space $(E,\mathcal{E})$ is also a Polish space; so the relative compactness and tightness are equivalent.

The following results guarantee the relative compactness of the process.

Proposition 4.1. *Under assumption* **C3** *for any* $T > 0$ *there exists a constant* $k_T > 0$*, independent of* ε *and dependent on* T*, such that*

$$\mathbf{E} \sup_{t \le T} |\zeta^\varepsilon(t)|^2 \le k_T.$$

Lemma 4.1. *Under assumption* **C3***, the following compact containment condition holds:*

$$\lim_{c \to \infty} \sup_{\varepsilon \le \varepsilon_0} \mathbf{P}\{\sup_{t \le T} |\zeta^\varepsilon(t)| > c\} = 0.$$

Lemma 4.2. *Under assumption* **C3** *there exists a constant* $k > 0$*, independent of* ε *such that*

$$\mathbf{E}|\zeta^\varepsilon(t) - \zeta^\varepsilon(s)|^2 \le k|t - s|.$$

Step 2. In this step of the proof, we solve the problem of singular perturbation for the generator of the process $\zeta^\varepsilon(t)$. We consider the three-component Markov process $A^\varepsilon(t), \zeta^\varepsilon(t), x_t^\varepsilon = x(t/\varepsilon)$, $t \ge 0$, which can be characterized by the martingale

$$\mu_t^\varepsilon = \varphi(A^\varepsilon(t), \zeta^\varepsilon(t), x_t^\varepsilon) - \int_0^t \mathbf{L}^\varepsilon \varphi(A^\varepsilon(s), \zeta^\varepsilon(s), x_t^\varepsilon)\, ds,$$

where $A^\varepsilon(s)$ is any of the three predictable characteristics and the generator $\mathbf{L}^\varepsilon$ has the following representation [7]

$$\mathbf{L}^\varepsilon = \varepsilon^{-1}\mathbf{Q} + \mathbf{\Gamma}^\varepsilon + \mathbf{A}^\varepsilon,$$

with $\mathbf{A}^\varepsilon(u; x)\varphi(u) = a_\varepsilon(u; x)\varphi'(x)$.

In order to prove the convergence of the predictable characteristics, it is sufficient to study the action of the generator $\mathbf{L}^\varepsilon$ on test functions with two variables $\varphi(v, x)$.

Thus, it has the representation

$$\mathbf{L}^\varepsilon \varphi(v, x) = [\varepsilon^{-1}\mathbf{Q} + \mathbf{A}^\varepsilon]\varphi(v, x).$$

The solution to the singular perturbation problem on the test functions $\varphi^\varepsilon(v, x) = \varphi(v) + \varepsilon\varphi_1(v, x)$ in the form $\mathbf{L}^\varepsilon\varphi^\varepsilon = \widehat{\mathbf{L}}\varphi + \theta^\varepsilon\varphi$ can be found in the same way as in Proposition 5.1 in [7]. That is

$$\widehat{\mathbf{L}} = \widehat{\mathbf{A}},$$

where $\widehat{\mathbf{A}}\varphi(v) = \widehat{a}\varphi'(v)$.

We can see now that the limit Markov process is characterized by the following predictable characteristics

$$B^0(t) = \int_0^t b(\zeta^0(s))\, ds, \quad C^0(t) = \int_0^t c(\zeta^0(s))\, ds, \quad \Gamma^0(t) = \int_0^t \Gamma_g(\zeta^0(s))\, ds.$$

So, the limit Markov process $\zeta^0(t)$ can be expressed by the generator (5).

A similar proof leads to the limit generator (6) in the case of process (2).

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