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Number theory

On the distribution modulo 1 of the sum of powers of a Salem number

Sur la répartition modulo 1 de la somme des puissances d'un nombre de Salem

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ABSTRACT

It is well known that the sequence of powers of a Salem number θ, modulo 1, is dense in the unit interval, but is not uniformly distributed. Generalizing a result of Dupain, we determine, explicitly, the repartition function of the sequence $(P(\theta^n) \bmod 1)_{n\geq 1}$, where P is a polynomial with integer coefficients and θ is quartic. Also, we consider some examples to illustrate the method of determination.

RÉSUMÉ

Il est bien connu que la suite des puissances d'un nombre de Salem θ, modulo 1, est dense dans l'intervalle unité, sans être uniformément distribuée. Généralisant un résultat de Dupain, on détermine explicitement la fonction de répartition de la suite $(P(\theta^n) \bmod 1)_{n\geq 1}$, où P est un polynôme à coefficients entiers et θ est quartique. On illustre également la méthode de détermination par quelques exemples.

1. Introduction

Studying the distribution modulo 1 of the powers of a fixed real number θ greater than 1 has been of interest for some time. In his monograph [7], Salem considered certain special algebraic integers. For instance, he showed that the sequence $(\theta^n)_{n\geq 1}$ tends to zero in $\mathbb{R}/\mathbb{Z}$ when θ is a Pisot number. If θ is a Salem number then $(\theta^n)_{n\geq 1}$ is dense in $\mathbb{R}/\mathbb{Z}$, i.e. the fractional parts of θ^n are dense in the unit interval [0, 1], but are not uniformly distributed. (See [2], pp. 87–89.) Moreover, Salem numbers are the only known numbers whose powers are dense in $\mathbb{R}/\mathbb{Z}$. A Pisot number is a real algebraic integer greater than 1 whose other conjugates have modulus less than 1. A Salem number is a real algebraic integer greater than 1 whose other conjugates have modulus less than or equal to 1 and a conjugate with modulus one. It is easy to see that a

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Salem number θ has one conjugate, namely θ^{-1}, inside the unit disc, while the others are on the boundary. The degree, say $2t$, of θ is even and is at least 4. Throughout, we will use the following notation where x and x' designate real numbers:

(i) the integer part function: $[x] = \max\{n \in Z : n \le x\}$,
(ii) the fractional part function: $\{x\} = x - [x]$,
(iii) congruence modulo 1: $x \equiv x' \bmod 1 \Leftrightarrow x - x' \in \mathbb{Z}$,
(iv) distance from x to the nearest integer: $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$.

Definition 1.1. Let $(u_n)_{n\ge1}$ be a sequence of real numbers and let $x \in [0,1]$. Then, the quantity $f(x) = \lim_{N\to\infty} \frac{\operatorname{card}\{n<N|\{u_n\}<x\}}{N}$, when it exists, is called the **repartition function** (also called the asymptotic distribution function [3]) of the sequence $(u_n)_{n\ge1}$ evaluated at x.

Here, we consider only those x for which $f(x)$ and its derivative $f'(x)$, called the **density function**, exist, i.e. almost everywhere.

From now on, suppose that θ is a Salem number, $u_n = P(\theta^n)$, where $P(x)$ is a polynomial with integer coefficients. Denote conjugates of θ by θ^{-1}, $\exp(\pm 2i\pi\omega_1), \dots, \exp(\pm 2i\pi\omega_{t-1})$. Since the sum of an algebraic integer and its conjugates is an integer, for all $n \in \mathbb{N}$, $\theta^n + \theta^{-n} + 2\sum_{j=1}^{t-1}\cos 2\pi n\omega_j \equiv 0 \pmod 1$ so that the distribution of $\theta^n \pmod 1$ is essentially that of $-2\sum_{j=1}^{t-1}\cos 2\pi n\omega_j$.

If θ is a quartic Salem number, Dupain [5] explicitly determined the repartition function for $(\theta^n)_{n\ge1}$, modulo 1. Namely,

$$f(x) = \frac{5}{2} - \frac{1}{\pi}\left(\arccos\frac{x-2}{2} + \arccos\frac{x-1}{2} + \arccos\frac{x}{2} + \arccos\frac{x+1}{2}\right).$$

It follows that

$$f'(x) = \frac{1}{2\pi}\left(\frac{1}{\sqrt{1-\left(\frac{x-2}{2}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x-1}{2}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}}\right).$$

If θ is a Salem number of degree $2t$, $t \ge 2$, Doche, Mendès France and Ruch [4] determined the density function for $(\theta^n)_{n\ge1}$, modulo 1:

$$f'(x) = 1 + 2\sum_{k=1}^{\infty} J_0(4k\pi)^{t-1}\cos 2\pi kx \tag{1}$$

on $(0,1)$. Here $J_0(\cdot)$ is the Bessel function of the first kind of index 0.

2. The main theorem

The aim of this paper is to obtain explicit forms of the repartition and of the density functions of the sequence $(P(\theta^n) \bmod 1)_{n\ge1}$, where θ is a quartic Salem number.

For a non-constant polynomial $P(x) = \sum_{j=0}^{m} a_j x^j$ with integer coefficients, let $Q_1^{-1}, Q_2^{-1}, \dots, Q_K^{-1}$ be all branches of the inverse function of $Q(x) = -2\sum_{j=0}^{m} a_j T_j(x)$, restricted to $[-1,1]$, where T_j is Chebyshev polynomial of the first kind with degree j. Since $Q(x)$ is a polynomial, we can introduce a partition of $[-1,1]$, namely $-1 = x_0 < x_1 < \cdots < x_K = 1$ such that $Q'(x_1) = Q'(x_2) = \cdots = Q'(x_{K-1}) = 0$ and $Q'(x)$ is positive or negative on each sub-interval (x_{k-1}, x_k), $k = 1, 2, \dots, K$. Let us introduce α_k, β_k: if $Q'(x)$ is positive on (x_{k-1}, x_k), then $Q(x_{k-1}) = \alpha_k$, $Q(x_k) = \beta_k$; if $Q'(x)$ is negative on (x_{k-1}, x_k), then $Q(x_{k-1}) = \beta_k$, $Q(x_k) = \alpha_k$. Now we define $Q_k^{-1}(x)$ as the inverse function of $Q(x)$ on $[x_{k-1}, x_k]$. Then $[\alpha_k, \beta_k]$ will be the domain of Q_k^{-1}, where $\alpha_k < \beta_k$ and $k \in \{1, 2, \dots, K\}$. Let S_k be the extension of Q_k^{-1} to $\mathbb{R}$, defined as follows:

$$S_k(x) = \begin{cases} Q_k^{-1}(x) \text{ for } x \in [\alpha_k, \beta_k] \\ Q_k^{-1}(\alpha_k) \text{ for } x \in (-\infty, \alpha_k) \\ Q_k^{-1}(\beta_k) \text{ for } x \in (\beta_k, \infty) \end{cases}$$

in the case $Q_k^{-1}(x)$ is decreasing on $[\alpha_k, \beta_k]$,

$$S_k(x) = \begin{cases} -Q_k^{-1}(x) \text{ for } x \in [\alpha_k, \beta_k] \\ -Q_k^{-1}(\alpha_k) \text{ for } x \in (-\infty, \alpha_k) \\ -Q_k^{-1}(\beta_k) \text{ for } x \in (\beta_k, \infty) \end{cases}$$

in the case $Q_k^{-1}(x)$ is increasing on $[\alpha_k, \beta_k]$. Setting $g(x) = \pi^{-1}\sum_{k=1}^{K}(\arccos(S_k(x)))$ and $M = [\max_{1\le k\le K}\max\{|\alpha_k|, |\beta_k|\}]+1$, then we have the following result.

Theorem 2.1. *The repartition and the density functions of the sequence* $(P(\theta^n) \bmod 1)_{n\ge1}$, *where* θ *is a quartic Salem number are defined by the equations:*

$$f(x) = \sum_{i=-M}^{M}(g(x+i) - g(i))$$

and

$$f'(x) = \pi^{-1}\sum_{i=-M}^{M}\sum_{k=1}^{K}(\arccos(S_k(x+i)))'. \tag{2}$$

Proof. Let conjugates of θ be θ^{-1}, $\exp(2i\pi\omega)$, $\exp(-2i\pi\omega)$. Since for any natural n

$$a_j(\theta^{nj} + \theta^{-nj} + 2\cos 2\pi nj\omega) \equiv 0 \pmod 1 \quad j = 0, 1, \ldots, m$$
$$P(\theta^n) = \sum_{j=0}^{m} a_j\theta^{nj} \equiv -\sum_{j=0}^{m} a_j(\theta^{-nj} + 2\cos 2\pi nj\omega) \pmod 1.$$

Since $\sum_{j=0}^{m} a_j\theta^{-nj}$ tends to the integer a_0 as n tends to infinity the distribution of $P(\theta^n) \pmod 1$ is essentially that of

$$-2\sum_{j=0}^{m} a_j\cos 2\pi nj\omega = -2\sum_{j=0}^{m} a_jT_j(\cos 2\pi n\omega) = Q(\cos 2\pi n\omega),$$

where $T_j(x) = \sum_{k=0}^{j} b_k^{\langle j\rangle}x^k$ is Chebyshev polynomial of the first kind. Hence we have

$$Q(w) = -2\sum_{j=0}^{m} a_j\sum_{k=0}^{j} b_k^{\langle j\rangle}w^k = -2(c_mw^m + c_{m-1}w^{m-1} + \cdots + c_0).$$

where we denoted $w = \cos 2\pi n\omega$ and

$$\begin{aligned}
c_m &= a_mb_m^{\langle m\rangle},\\
c_{m-1} &= a_{m-1}b_{m-1}^{\langle m-1\rangle} + a_mb_{m-1}^{\langle m\rangle},\\
&\ldots\\
c_j &= a_jb_j^{\langle j\rangle} + a_{j+1}b_j^{\langle j+1\rangle} + \cdots + a_mb_j^{\langle m\rangle},\\
&\ldots\\
c_0 &= a_0b_0^{\langle 0\rangle} + a_1b_0^{\langle 1\rangle} + a_2b_0^{\langle 2\rangle} + \ldots + a_mb_0^{\langle m\rangle}.
\end{aligned} \tag{3}$$

It is obvious that $Q(w) \in [-M, M]$ thus $\{Q(w)\} < x \Leftrightarrow$ there is an $i \in \{-M, -M+1, \ldots, M-1\}$ such that $i \le Q(w) \le i + x$. Now we conclude that there is a $k \in \{1, 2, \ldots, K\}$ such that: if Q^{-1} is increasing on $[\alpha_k, \beta_k]$ then we have $Q_k^{-1}(\max(\alpha_k, i)) \le w \le Q_k^{-1}(\min(\beta_k, i+x))$ and $S_k(i) \ge -w \ge S_k(i+x)$; if Q^{-1} is decreasing on $[\alpha_k, \beta_k]$ then $Q_k^{-1}(\max(\alpha_k, i)) \ge w \ge Q_k^{-1}(\min(\beta_k, i+x))$ and $S_k(i) \ge w \ge S_k(i+x)$. It is easy to verify that $(2\pi)^{-1}\arccos(\cos 2\pi t) = \|t\|$ and, as a consequence of this, that $(2\pi)^{-1}\arccos(-\cos 2\pi t) = (2\pi)^{-1}\arccos\cos(2\pi t + \pi) = \|t + 1/2\|$, for that reason we have

$$(2\pi)^{-1}\arccos(Q_k^{-1}(\max(\alpha_k, i))) \le \|n\omega\| \le (2\pi)^{-1}\arccos(Q_k^{-1}(\min(\beta_k, i+x)))$$

in the case Q_k^{-1} is decreasing, or

$$(2\pi)^{-1}\arccos(-Q_k^{-1}(\max(\alpha_k, i))) \le \|n\omega + 1/2\| \le (2\pi)^{-1}\arccos(-Q_k^{-1}(\min(\beta_k, i+x)))$$

in the case Q_k^{-1} is increasing.

It is fulfilled that $\|n\omega\|$ and $\|n\omega + 1/2\|$ are uniformly distributed on $[0, 1/2]$ because 1, ω are $\mathbf{Q}$-linearly independent [2], Theorem 5.3.2 so we can use [2], Theorem 4.6.3. Consequently, for all L, R such that $0 \le L < R \le 1/2$,

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{card}\{n<N|L\le\|n\omega\|\le R\}=2(R-L),$$
$$\lim_{N\to\infty}\frac{1}{N}\mathrm{card}\{n<N|L\le\|n\omega+\frac{1}{2}\|\le R\}=2(R-L).$$

Let $K_1\le K$ be natural number such that $Q_1^{-1}, Q_2^{-1},\dots,Q_{K_1}^{-1}$ are decreasing and $Q_{K_1+1}^{-1}, Q_{K_1+2}^{-1},\dots,Q_K^{-1}$ are increasing. Now we can determine the repartition function

$$f(x)=\lim_{N\to\infty}\frac{1}{N}\mathrm{card}\{n<N|\{Q(\cos(2\pi n\omega))\}<x\}=$$
$$\lim_{N\to\infty}\frac{1}{N}\mathrm{card}\bigcup_{i=-M}^{M}\Bigg(\bigcup_{k=1}^{K_1}\left\{n<N|\frac{\arccos(Q_k^{-1}(\max(\alpha_k,i)))}{2\pi}\le\|n\omega\|\le\frac{\arccos(Q_k^{-1}(\min(\beta_k,i+x)))}{2\pi}\right\}\bigcup$$
$$\bigcup_{k=K_1+1}^{K}\left\{n<N|\frac{\arccos(-Q_k^{-1}(\max(\alpha_k,i)))}{2\pi}\le\|n\omega+\frac{1}{2}\|\le\frac{\arccos(-Q_k^{-1}(\min(\beta_k,i+x)))}{2\pi}\right\}\Bigg)=$$
$$\sum_{i=-M}^{M}\sum_{k=1}^{K}\left(\frac{1}{\pi}\arccos(S_k(i+x))-\frac{1}{\pi}\arccos(S_k(i))\right)=$$
$$\sum_{i=-M}^{M}(g(x+i)-g(i))$$

because all sets in the double union are disjoint. Now it is obvious that the density function is

$$f'(x)=\sum_{i=-M}^{M}g'(x+i).\qquad\square\tag{4}$$

Remark 1. For practical implementation of the algorithm presented in Theorem 2.1, it is more convenient to determine M using (3) with $M=2\sum_{j=0}^{m}|c_j|$.

Remark 2. Since $\{P(\theta^n)\}=\{P(\theta^n)+l\}$, $l\in\mathbb{Z}$ we can take, without loss of generality, that $a_0=0$.

Corollary 2.2. *Let the line $x=v$, $v\in[0,1]$ be a vertical asymptote of the graph of the density function $y=f(x)$. Then $\lim_{x\to v-0}f'(x)=\infty$ if and only if $v=1$ or $v=\{\beta_k\}$ and $\lim_{x\to v+0}f'(x)=\infty$ if and only if $v=0$ or $v=\{\alpha_k\}$, $k=1,2,\dots,K$.*

Proof. We proved in the Theorem 2.1 that

$$f'(x)=\sum_{i=-M}^{M}g'(x+i)=\sum_{i=-M}^{M}\sum_{k=1}^{K}\pi^{-1}(\arccos(S_k(i+x)))'=$$
$$=-\pi^{-1}\sum_{i=-M}^{M}\sum_{k=1}^{K}(1-S_k^2(i+x))^{-1/2}S_k'(i+x).$$

Now we conclude that $\lim_{x\to v-0}f'(x)=\infty$ if and only if there are i_0, k_0 such that

$$\lim_{[\alpha_{k_0},\beta_{k_0}]\ni x\to v-0}(1-(Q_{k_0}^{-1})^2(i_0+x))^{-1/2}S_{k_0}'(i_0+x)=\infty.$$

There are two cases: either $Q_{k_0}^{-1}(i_0+v)=\pm1$ or $\lim_{x\to v-0}S_{k_0}'(i_0+x)=\infty$. If $Q_{k_0}^{-1}(i_0+v)=\pm1$ then $i_0+v=Q_{k_0}(\pm1)$. Since $Q_{k_0}(\pm1)$ is an integer and $v\in[0,1]$, we conclude that either $v=0$ or $v=1$. But $v=0$ is impossible because $v-0$ will be out of the domain of $Q_{k_0}^{-1}(x)$. If $\lim_{x\to v-0}S_{k_0}'(i_0+x)=\infty$ then, using the last remark, either $i_0+v=\alpha_{k_0}$ or $i_0+v=\beta_{k_0}$. Again $i_0+v=\alpha_{k_0}$ is impossible because i_0+v-0 will be out of the domain of $S_{k_0}'(x)$. If $i_0+v=\beta_{k_0}$ then we conclude that $v=\{\beta_{k_0}\}$ with an exception: if β_{k_0} is an integer then its fractional part is 0 but, as we have seen, $v=0$ is impossible. Nevertheless, the claim is true because, in that case, the line $x=1$ should be a vertical asymptote of the graph.

It is obvious that $\lim_{x\to v+0}f'(x)=\infty$ if and only if $v=0$ or $v=\{\alpha_k\}$ can be proved completely analogously. $\square$

We present the next procedure for sketching the graph of the density function, resulting from the previous corollary:

(i) find a set $\overline{A}$ of local minimum points, a set $\overline{B}$ of local maximum points, and a set $\overline{S}$ of (horizontal inflection) stationary points on $[-1, 1]$ of $Q(x)$;
(ii) find a set A of fractional parts of values at local minimum points, a set B of fractional parts of values at local maximum points, and a set S of fractional parts of values at stationary points of $Q(x)$;
(iii) let $x_0, x_1, \dots, x_r$, $0 = x_0 < x_1 < \cdots < x_r = 1$ be sorted $r+1$ elements of $A \cup B \cup S$;
(iv) if $x_i \in A \cup S$, $x_{i+1} \in B \cup S$ then $f'(x)$ has vertical asymptotes $x = x_i$, $x = x_{i+1}$ on interval (x_i, x_{i+1}) so $f'(x)$ has the shape of $\cup$;
(v) If $x_i \in A \cup S$, $x_{i+1} \in A \setminus (B \cup S)$ then $f'(x)$ has vertical asymptote $x = x_i$ on interval (x_i, x_{i+1}) so $f'(x)$ has the shape of left half of $\cup$, we will denote it by $\lfloor$;
(vi) if $x_i \in B \setminus (A \cup S)$, $x_{i+1} \in B \cup S$ then $f'(x)$ has vertical asymptote $x = x_{i+1}$ on interval (x_i, x_{i+1}), so $f'(x)$ has the shape of the right half of $\cup$; we will denote it by $\rfloor$;
(vii) If $x_i \in B \setminus (A \cup S)$, $x_{i+1} \in A \setminus (B \cup S)$, then $f'(x)$ has no vertical asymptote on interval (x_i, x_{i+1}), so $f'(x)$ has the shape of $\smile$.

Remark 3. In the first item of the previous procedure, we have to solve the equation $-Q'(\cos t)\sin t = 0 \Leftrightarrow \sin t = 0 \vee Q'(\cos t) = 0$. Solutions to $\sin t = 0$ are $t = k\pi$, $k \in \mathbb{Z}$. If x is a solution to $Q'(x) = 0$ and $-1 \le x \le 1$ then $t = \arccos(x) + 2k\pi$, $k \in \mathbb{Z}$ is a stationary point of $Q(\cos(t))$. Thus, if a solution to $Q'(x) = 0$ is out of $\mathbb{R}$ or greater than 1 in modulus, we should ignore it.

In the second item, we have to find $\{Q(\cos k\pi)\} = \{Q(\pm 1)\} = 0$, $k \in \mathbb{Z}$ so that $0 \in A \cup B \cup S$. Since the fractional part of a real number is in $[0, 1)$ we should take that 0 and 1 must be both in or both out of set A, as well as B and S. We conclude that $1 \in A \cup B \cup S$.

3. Linear, quadratic and cubic polynomial

If $P(x) = a_1 x$ then, using the notation of the Theorem 2.1, we have only one branch of the inverse function of $Q(x) = -2a_1 x$ i.e. $Q^{-1}(x) = \frac{-x}{2a_1}$, so that $g(x) = \pi^{-1}\arccos(\frac{-x}{2a_1})$. The repartition function is $f(x) = \pi^{-1}\sum_{i=-2a_1}^{2a_1-1}(\arccos(-\frac{x+i}{2a_1}) - \arccos(-\frac{i}{2a_1}))$. The density function is $f'(x) =$

$$= \frac{1}{2a_1\pi}\left(\frac{1}{\sqrt{1-\frac{(x-2a_1)^2}{4a_1^2}}} + \frac{1}{\sqrt{1-\frac{(x-2a_1+1)^2}{4a_1^2}}} + \cdots + \frac{1}{\sqrt{1-\frac{(x+2a_1-1)^2}{4a_1^2}}}\right).$$

If we take $a_1 = 1$ we get the Dupain's formula, which is presented above.

Hereafter we suppose that $P(x) = a_2 x^2 + a_1 x$ and then, using the notation of the Theorem 2.1, we have two branches of the inverse function of $Q(x) = -4a_2 x^2 - 2a_1 x + 2a_2$, i.e. $Q_1^{-1}(x) = -\frac{a_1 + \sqrt{a_1^2 + 8a_2^2 - 4a_2 x}}{4a_2}$, $Q_2^{-1}(x) = -\frac{a_1 - \sqrt{a_1^2 + 8a_2^2 - 4a_2 x}}{4a_2}$.

Since $Q'(-\frac{a_1}{4a_2}) = 0$, $Q(x)$ has extremum $V = \frac{a_1^2}{4a_2} + 2a_2$ at $x = -\frac{a_1}{4a_2}$. Using the previous Corollary, we can conclude that the graph of the density function $y = f'(x)$ has an inner vertical asymptote $x = v$, $v = \{V\} \in (0, 1)$ if and only if

$$-1 < -\frac{a_1}{4a_2} < 1, \ a_1 \neq 0, \ V \notin \mathbb{Z}. \tag{5}$$

If $a_2 > 0$, then $Q(x)$ has maximum V so that $\lim_{x\to v-0} f'(x) = \infty$. In that case $\lim_{x\to 0+0} f'(x) = \infty$ so that the graph of $y = f'(x)$ has shape $\cup\smile$. Similarly, if conditions (5) are fulfilled and $a_2 < 0$, then the graph of $y = f'(x)$ has shape $\smile\cup$. If any of the conditions (5) is not fulfilled, then the graph of $y = f'(x)$ has shape $\cup$.

Finally, we suppose that $P(x) = a_3 x^3 + a_2 x^2 + a_1 x$, so that we have three branches of the inverse function of $Q(x) = -8a_3 x^3 - 4a_2 x^2 + (6a_3 - 2a_1)x + 2a_2$. Its explicit formulas are clumsy, so we will not cite them here. The roots of $Q'(x) = 0$ are

$$\bar{x}_{1,2} = \frac{-a_2 \pm \sqrt{9a_3^2 + a_2^2 - 3a_1 a_3}}{6a_3}.$$

In Table 1, using the procedure for sketching the graph of $f'(x)$, we represent different shapes of graphs.

4. Some polynomials of degree $m > 3$

It is clear, by the above-mentioned results of Dupain as well as those of Doche, Mendès France and Ruch, that the repartition and the density functions of the sequence $(P(\theta^n) \bmod 1)_{n\ge1}$ are well understood when θ is a Salem number and $P(x) = x^m$, since any power of a Salem number (with degree $2t$) is also a Salem number (of degree $2t$); see, for instance, [7] and [8].

Table 1
Procedure for sketching the graph of the density function.

a_3, a_2, a_1	$\bar{x}_1$	$\bar{x}_2$	$Q(\bar{x}_1)$	$Q(\bar{x}_2)$	A	B	S	$f'(x)$
1, 1, 1	−0.61	0.27	−0.11	2.63	0.89, 0, 1	0.63, 0, 1		$\cup\smile\cup$
3, 5, 6	−0.68	0.12	4.22	10.39	0.22, 0, 1	0.39, 0, 1		$\lfloor\cup\rfloor$
3, 3, 10	−0.17	−0.17	6.11	6.11	0, 1	0, 1	0.11	$\cup\cup$
1, −1, −2	−0.5	0.83	−5	4.48	0, 1	0.48, 0, 1		$\cup\rfloor$
1, 2, 3	−0.67	0	2.82	4	0.82, 0, 1	0, 1		$\lfloor\cup$
1, −2, −2	−0.39	1.06	−6.21		0.79	0, 1		$\smile\cup$
1, 2, −2	−1.06	0.39		6.21	0, 1	0.21		$\cup\smile$
1, 0, 0	−0.5	0.5	−2	2	0, 1	0, 1		$\cup$
1, 1, 4	$\notin\mathbb{R}$	$\notin\mathbb{R}$			0, 1	0, 1		$\cup$

There are integer coefficients a_j of $P(x)$ such that $Q(x) = -2^m x^m$. We can use Eqs. (3) to find such a_j. The power x^n can be expressed in terms of the Chebyshev polynomials of degrees up to n (for proof see [6], chapter 2.3.1):

$$x^m = 2^{1-m} \sum_{k=0}^{[m/2]}{}' \binom{m}{k} T_{m-2k}(x), \tag{6}$$

where the dash ($\sum'$) denotes that the k-th term in the sum is to be halved if m is even and $k = m/2$. Let m be odd, we conclude that if

$$P(x) = \sum_{k=0}^{(m-1)/2} \binom{m}{k} x^{m-2k},$$

then $Q(x) = -2^m x^m$, its inverse function can be easily found: $Q^{-1}(x) = -\sqrt[m]{x}/2$. Using (2), we obtain that

$$f'(x) = \frac{1}{\pi} \sum_{i=-2^m}^{2^m-1} \frac{\sqrt[m]{x+i}}{2m(x+i)\sqrt{1-(\sqrt[m]{x+i})^2/4}}.$$

We will show that $f'(1/2+x) = f'(1/2-x)$, $|x| \le 1/2$. For that reason, $x = 1/2$ is a line of symmetry of the graph of $f'(x)$. It is convenient to introduce

$$g'(x) = \frac{1}{\pi} \frac{\sqrt[m]{x}}{2mx\sqrt{1-(\sqrt[m]{x})^2/4}},$$

then we have

$$\begin{aligned}
f'(1/2+x) &= \sum_{i=-2^m}^{2^m-1} g'(1/2+x+i) && (g'(x) \text{ is even}) \\
&= \sum_{i=-2^m}^{2^m-1} g'(-1/2-x-i) && (i = j-1) \\
&= \sum_{j=-2^m+1}^{2^m} g'(-1/2-x-j+1) && (j = -i) \\
&= \sum_{i=-2^m}^{2^m-1} g'(1/2-x+i) \\
&= f'(1/2-x).
\end{aligned}$$

Let m be even, we conclude from (6) that if

$$P(x) = \sum_{k=0}^{m/2-1} \binom{m}{k} x^{m-2k} + \frac{1}{2}\binom{m}{m/2},$$

then $Q(x) = -2^m x^m$, its inverse function can be easily found: $Q^{-1}(x) = \pm\sqrt[m]{x}/2$. Using the algorithm presented in the Theorem 2.1 we obtain that

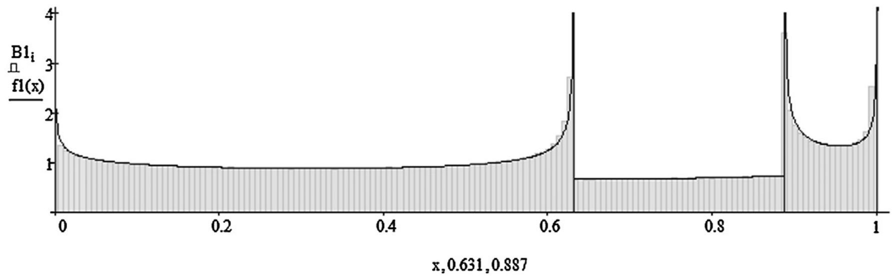

Fig. 1. $f'(x)$ (the black curve) of $(P(\theta^n) \bmod 1)_{n\ge 1}$, where $P(x) = x^3 + x^2 + x$ and the distribution histogram of the sequence. The shape of $f'(x)$ is $\bigcup\smile\bigcup$.

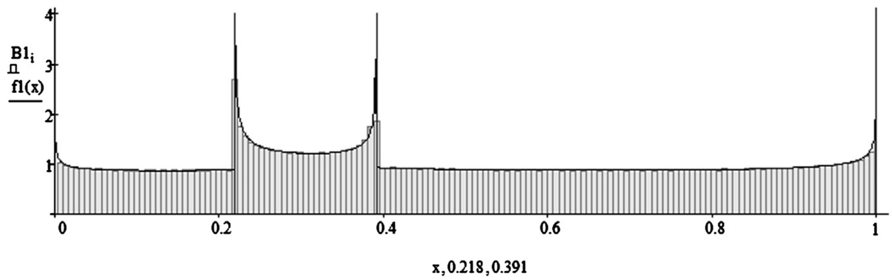

Fig. 2. $f'(x)$ (the black curve) of $(P(\theta^n) \bmod 1)_{n\ge 1}$, where $P(x) = 3x^3 + 5x^2 + 6x$ and the distribution histogram of the sequence. The shape of $f'(x)$ is $\lfloor\bigcup\rfloor$.

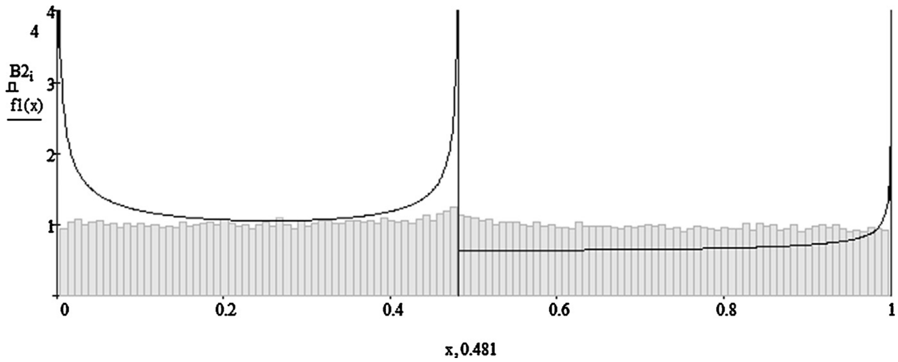

Fig. 3. $f'(x)$ (the black curve) of $(P(\theta^n) \bmod 1)_{n\ge 1}$, where $P(x) = x^3 - x^2 - 2x$ and θ is a quartic Salem number and the distribution histogram of $(P(\theta_1^n) \bmod 1)_{n\ge 1}$, where θ_1 is a Salem number of degree six. We can notice that it is close to the density function of the uniform distribution. The shape of $f'(x)$ is $\bigcup\rfloor$.

$$f'(x) = \frac{1}{\pi}\sum_{i=1}^{2^m} \frac{\sqrt[m]{-x+i}}{m(-x+i)\sqrt{1-(\sqrt[m]{-x+i})^2/4}}.$$

To illustrate the main theorem and the procedure, we give examples of distributions for some polynomials from Table 1. Using the definition of $f(x)$ and the well-known fact that the first derivative of the function $f(x)$ on a small interval $[a, b]$ could be estimated with the finite difference: $f(b) - f(a)$ divided by $b - a$, we approximate $f'(x)$. The interval $[0, 1]$ is divided into p pieces. We compute the fractional part of $P(\theta^n)$ for $1 \le n \le N$, and count the number of n so that the fractional part of $P(\theta^n)$ falls into each of subintervals. The vertical axis indicates the number of such n divided by N/p, so that this normalization results in a relative histogram that coincides with $f'(x)$, the function determined using the main theorem.

The histograms in Figs. 1, 2 are generated using the Salem number θ, which is the root of $x^4 - x^3 - x^2 - x + 1$. The histogram in Fig. 3 is generated by θ_1, the root of $x^6 - x^5 - x^4 + x^3 - x^2 - x + 1$, a Salem number of degree 6. At $x = 0.481$, where $f'(x)$ has the vertical asymptote, the histogram has only a low peak (Fig. 3). If the histogram is generated by a Salem number of degree 8, we can notice that it is almost flat. Since $J_0(x) = O(1/\sqrt{x})$, for large degrees t, the sum of the series in (1) is small, so the sequence $(\theta^n \bmod 1)_{n\ge 1}$ is close to being equidistributed, a fact that Akiyama and Tanigawa [1] make very explicit in their article. We can conclude that the same property is valid for the sequence $(P(\theta^n) \bmod 1)_{n\ge 1}$. Experimenting with different Salem numbers θ of degree six, for fixed P, we can conclude that the histogram does not depend on their size. We can explain why the formula (1) depends only on the degree $2t$ of θ: (1) numbers $1, \omega_1, \ldots, \omega_{t-1}$

are $\mathbb{Z}$-lineary independent; (2) the $(t-1)$ dimensional sequence $(\omega_1 n, \ldots, \omega_{t-1} n)$ is equidistributed in $(\mathbb{R}/\mathbb{Z})^{t-1}$; (3) the function $\cos 2\pi x$ is periodic so that Doche, Mendès France and Ruch [4] can transform the sum into the integral in the proof of their Lemma 2.1. We conclude that the repartition function of the sequence $(P(\theta^n) \bmod 1)_{n\geq 1}$ does not depend on θ because we can obtain that (3) becomes $\sum_{j=0}^{m} a_j \cos(2j\pi x)$, which is also periodic.

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