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Complex analysis

Growth of proper holomorphic maps and tropical power series

Croissance d'applications holomorphes propres et séries entières tropicales

Evgeny Abakumov [a], Evgueni Doubtsov [b,c]

[a] *Université Paris-Est, LAMA (UMR 8050), 77454 Marne-la-Vallée, France*
[b] *St. Petersburg Department of V.A. Steklov Mathematical Institute, Fontanka 27, St. Petersburg 191023, Russia*
[c] *Department of Mathematics and Mechanics, St. Petersburg State University, Universitetski pr. 28, St. Petersburg 198504, Russia*

A R T I C L E   I N F O

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A B S T R A C T

Motivated by a question of J. Globevnik, we show that a proper holomorphic immersion of the unit disk $\mathbb{D}$ into $\mathbb{C}^2$ or a proper holomorphic embedding $f : \mathbb{D} \to \mathbb{C}^3$ may have arbitrary growth. Also, using tropical power series, we characterize those radial weights w on the complex plane for which there exist $n \in \mathbb{N}$ and a proper holomorphic map $f : \mathbb{C} \to \mathbb{C}^n$ such that $|f(z)|$ is equivalent to $w(z)$.

R É S U M É

Motivés par une question de J. Globevnik, nous montrons qu'une immersion holomorphe propre du disque unité $\mathbb{D}$ dans $\mathbb{C}^2$ ou un plongement holomorphe propre $f : \mathbb{D} \to \mathbb{C}^3$ peut avoir une croissance arbitraire. En outre, en utilisant les séries entières tropicales, nous caractérisons les poids radiaux w sur le plan complexe pour lesquels il existe $n \in \mathbb{N}$ et une application holomorphe propre $f : \mathbb{C} \to \mathbb{C}^n$ tels que $|f(z)|$ soit équivalente à $w(z)$.

1. Introduction

Let $\mathcal{D}$ denote $\mathbb{C}^d$ or the unit ball B_d of $\mathbb{C}^d$, $d \geq 1$. For the unit disk B_1, we also use the symbol $\mathbb{D}$.

1.1. Proper holomorphic maps

A holomorphic map $f : \mathcal{D} \to \mathbb{C}^n$, $n \geq 2$, is called *proper* if the preimage of every compact set is compact. A proper holomorphic map is *an immersion* if its Jacobian is non-degenerate everywhere. By definition, *a proper holomorphic embedding* is a proper holomorphic immersion which is one-to-one. It is not easy to construct a proper holomorphic embedding

E-mail addresses: evgueni.abakoumov@u-pem.fr (E. Abakumov), dubtsov@pdmi.ras.ru (E. Doubtsov).

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$f:\mathbb{D}\to\mathbb{C}^2$; the first example was obtained by K. Kasahara and T. Nishino (see [13,21]). Later the unit disk was replaced by an annulus (see [17]), the punctured disk (see [3]) and more sophisticated planar domains. However, the following problem remains open (see [9,11]): Can any planar domain be properly holomorphically embedded into $\mathbb{C}^2$?

J. Globevnik [10] proves that a proper holomorphic embedding $f:\mathbb{D}\to\mathbb{C}^2$ may grow arbitrarily rapidly; also, he asks whether such an embedding may grow arbitrarily slowly. Motivated by this question, we will show that results from [1] provide quantitative assertions related to the growth of proper holomorphic immersions and embeddings of the unit disk $\mathbb{D}$. In fact, a proper holomorphic immersion $f:\mathbb{D}\to\mathbb{C}^2$ or a proper holomorphic embedding $f:\mathbb{D}\to\mathbb{C}^3$ may have arbitrary growth.

For $d\ge 2$, proper holomorphic embeddings $f:B_d\to\mathbb{C}^n$ have been investigated by many authors in a more general setting of Stein manifolds M_d of dimension d. The following essentially sharp result was obtained by Y. Eliashberg and M. Gromov [7]: every Stein manifold M_d of dimension d can be properly holomorphically embedded into $\mathbb{C}^{n(d)}$ for the minimal integer $n(d)>(3d+1)/2$.

1.2. Organization of the paper

We apply results from [1] in Section 2 to study the possible growth of proper holomorphic immersions and embeddings of certain bounded planar domains and of the unit ball B_d, $d\ge 2$. In Section 3, the corresponding condition of approximation by finite sums of moduli of holomorphic functions on $\mathcal{D}$ is shown to be equivalent to several natural or well-known approximation properties. The arguments are essentially known for $\mathcal{D}=B_d$, $d\ge 1$; in the interesting case $\mathcal{D}=\mathbb{C}^d$, $d\ge 1$, we use tropical power series.

2. Growth of proper holomorphic maps of the unit ball

For $R=1$ or $R=+\infty$, let $w:[0,R)\to(0,+\infty)$ be *a weight function*, that is, let w be non-decreasing, continuous and unbounded. Setting $w(z)=w(|z|)$ for $z\in\mathcal{D}$, we extend w to *a radial weight* on $\mathcal{D}$. In what follows, we freely exchange a weight function and its extension to a radial weight.

Given a set X and functions $u,v:X\to(0,+\infty)$, we write $u\asymp v$ and we say that u and v are *equivalent* if $C_1u(x)\le v(x)\le C_2u(x)$, $x\in X$, for some constants $C_1,C_2>0$.

Let $\mathcal{H}ol(\mathcal{D})$ denote the space of holomorphic functions on $\mathcal{D}$.

Definition 2.1. A radial weight w on $\mathcal{D}$ is called *approximable by a finite sum of moduli* if there exist $f_1,f_2,\dots,f_n\in\mathcal{H}ol(\mathcal{D})$, $n\in\mathbb{N}$, such that

$$|f_1(z)|+|f_2(z)|+\dots+|f_n(z)|\asymp w(z),\quad z\in\mathcal{D}.$$

Clearly, a radial weight w has the above property if and only if there exists a proper holomorphic map $f:\mathcal{D}\to\mathbb{C}^n$ for which $|f(z)|\asymp w(z)$.

A function $v:[0,R)\to(0,+\infty)$ is called *log-convex* if $\log v(t)$ is a convex function of $\log t$, $0<t<R$. We have the following theorem for $\mathcal{D}=B_d$.

Theorem 2.2. *(See [2, Theorems 1.2 and 1.3].) Let w be a radial weight on B_d, $d\ge 1$. Then the following properties are equivalent:*

- *w is approximable by a finite sum of moduli on B_d;*
- *w is equivalent to a log-convex weight function.*

Moreover, if $d=1$, then any log-convex radial weight is approximable by the sum of moduli of two holomorphic functions.

Let $f:\mathcal{D}\to\mathbb{C}^n$ be a proper holomorphic immersion such that $|f|\asymp w$ for a radial weight w. For $\mathcal{D}=B_d$, Theorem 2.2 guarantees that w is equivalent to a log-convex weight function. Applying Hadamard's three-circle theorem, we obtain the same conclusion for $\mathcal{D}=\mathbb{C}^d$. The results of the present section show that this is the only restriction on w for $\mathcal{D}=B_d$ and sufficiently large $n=n(B_d)$. However, the restrictions on w are more stringent for $\mathcal{D}=\mathbb{C}^d$; see Section 3.3.

Corollary 2.3. *Let w be a log-convex radial weight on $\mathbb{D}$. Then there exists a proper holomorphic immersion $f:\mathbb{D}\to\mathbb{C}^2$ such that $|f|\asymp w$.*

Proof. Given a log-convex radial weight w, the proof of Theorem 1.2 in [2] provides a holomorphic map $(g_1,g_2):\mathbb{D}\to\mathbb{C}^2$ such that $g_j(0)\ne 0$, $j=1,2$, and $|g_1(z)|+|g_2(ze^{i\theta})|\asymp w(z)$, $z\in\mathbb{D}$, $\theta\in[0,2\pi)\setminus Q$, where Q is a finite set. Put $h_1(z)=g_1(z)$ and $h_2(z)=zg_2(z)$. Then $h_1(0)\ne 0$ and $h_2'(0)\ne 0$. Observe that two strictly positive continuous functions are equivalent in $\mathbb{D}$ if they are equivalent in a neighborhood of the unit circle. Also, the zero sets of h_j and h_j', $j=1,2$, are countable ones. Therefore, there exists $\theta\in[0,2\pi)$ such that, for $f_1(z)=h_1(z)$ and $f_2(z)=h_2(ze^{i\theta})$, we have $(f_1(z),f_2(z))\ne(0,0)$ for all $z\in\mathbb{D}$, $|f_1|+|f_2|\asymp w$ and $(f_1'(z),f_2'(z))\ne(0,0)$ for all $z\in\mathbb{D}$. $\square$

Corollary 2.4. *Let w be a log-convex radial weight on $\mathbb{D}$. Then there exists a proper holomorphic embedding $f:\mathbb{D}\to\mathbb{C}^3$ such that $|f|\asymp w$.*

Proof. Corollary 2.3 provides a proper holomorphic immersion $(f_1, f_2):\mathbb{D}\to\mathbb{C}^2$ such that $|f_1|+|f_2|\asymp w$. It suffices to set $f=(f_1, f_2, f_3)$, where $f_3(z)=z$, $z\in\mathbb{D}$. □

Remark 1. For the radial weights on $\mathbb{D}$, the log-convexity is a regularity condition, not a growth one. In particular, a log-convex weight function may grow arbitrarily slowly or arbitrarily rapidly. So, a proper holomorphic immersion $f:\mathbb{D}\to\mathbb{C}^2$ may grow arbitrarily slowly. It would be interesting to know whether Corollary 2.3 extends to appropriate *embeddings*.

Remark 2. There are certain analogs of Corollaries 2.3 and 2.4 for multiply connected planar domains. For example, let $D_j\Subset\mathbb{D}$, $j=1,\dots,J$, be open disks such that $\overline{D}_j\cap\overline{D}_k=\varnothing$, $j\neq k$. Put $\Omega=\mathbb{D}\setminus\bigcup_{j=1}^J D_j$. Assume that a continuous function $w:\Omega\to(0,+\infty)$ has appropriate radial log-convex growth in certain neighborhoods of $\partial\mathbb{D}$ and ∂D_j, $j=1,\dots,J$, with respect to the corresponding centers. Then, using Theorem 2.2 with $d=1$, it is possible to construct a proper holomorphic embedding $f:\Omega\to\mathbb{C}^{2J+3}$ such that $|f|\asymp w$.

For arbitrary $d\geq 1$, Theorem 2.2 implies the following assertion.

Corollary 2.5. *Let w be a log-convex radial weight on B_d. Then there exists a number $n=n(d)$ and a proper holomorphic embedding $f:B_d\to\mathbb{C}^{n(d)}$ such that $|f|\asymp w$.*

It would be interesting to learn the optimal value of $n(d)$ in the above corollary.

3. Holomorphic approximation of radial weights

In this section, we show that the property of being approximable by a finite sum of moduli is equivalent to several natural or well-known conditions formulated below. The proofs will be given elsewhere.

3.1. Related approximation problems and properties

Recall that $\mathcal{D}$ denotes $\mathbb{C}^d$ or B_d, $d\geq 1$. Given a radial weight w on $\mathcal{D}$, *the associated weight* $\widetilde{w}$ is defined as $\widetilde{w}(z)=\sup\{|f(z)|: f\in\mathcal{H}ol(\mathcal{D}),\ |f|\leq w \text{ on } \mathcal{D}\}$, $z\in\mathcal{D}$. The notion of associated weight naturally arises in the study of the growth space $\mathcal{A}^w(\mathcal{D})$, which consists of those $f\in\mathcal{H}ol(\mathcal{D})$ for which $|f|\leq Cw$ on $\mathcal{D}$ with some constant $C>0$. The definition of $\widetilde{w}$ was formally introduced in [4] in a more general setting; see [4] for basic properties of $\widetilde{w}$. In particular, $\widetilde{w}$ is a radial weight, so the associated weight function $\widetilde{w}:[0,R)\to(0,+\infty)$ is correctly defined ($R=1$ for $\mathcal{D}=B_d$ and $R=+\infty$ for $\mathcal{D}=\mathbb{C}^d$). Also, $\widetilde{w}$ is known to be log-convex.

In applications, many results related to the growth space $\mathcal{A}^w(\mathcal{D})$ are formulated in terms of $\widetilde{w}$, so it is important to distinguish those w that are equivalent to $\widetilde{w}$.

Definition 3.1. (See [4].) A weight function $w:[0,R)\to(0,+\infty)$ is called *essential* if

$$\widetilde{w}(t)\asymp w(t),\quad 0\leq t<R.$$

Definition 3.2. A weight function $w:[0,R)\to(0,+\infty)$ is called *approximable by the maximum of a holomorphic function modulus* if there exists $f\in\mathcal{H}ol(\mathcal{D})$ such that

$$M_f(t)\asymp w(t),\quad 0\leq t<R,\quad \text{where } M_f(t)=\max\{|f(z)|:|z|=t\}.$$

Recall that Hadamard's three-circle theorem says that $M_f(t)$ is a log-convex function.

Definition 3.3. We say that a weight function $w:[0,R)\to(0,+\infty)$ is *approximable by power series with positive coefficients* if there exist $a_k\geq 0$, $k=0,1,\dots$, such that

$$\sum_{k=0}^{\infty}a_k t^k\asymp w(t),\quad 0\leq t<R.$$

Conditions related to the above property are of interest in weighted polynomial approximation problems (see, for example, [16,18,19]) and in numerical applications. To investigate this property, P. Erdös and T. Kövári [8], and U. Schmid [20] use the function

$$P_w(t) = \max\left\{ \frac{t^k}{u_k} : k = 0, 1, \dots \right\}, \quad \text{where } u_k = \sup\left\{ \frac{t^k}{w(t)} : 0 \le t < R \right\}, \quad k = 0, 1, \dots.$$

In other words, one considers the pointwise maximum of the monomials $y_k(t) = a_k t^k$ such that $y_k(t) \le w(t)$ and $y_k(t)$ reaches $w(t)$ from below. Clearly, $P_w(t) \le w(t)$. So, the reverse inequality is of interest.

Definition 3.4. We say that a weight function $w : [0, R) \to (0, +\infty)$ is *approximable from below by monomials* if

$$w(t) \le C P_w(t), \quad 0 \le t < R, \quad \text{for a constant } C > 1.$$

3.2. Holomorphic approximation on B_d

For $\mathcal{D} = B_d$, Theorem 2.2 and results from [5] and [6] provide direct relations between the properties under consideration and the log-convexity. Namely, let $w : [0, 1) \to (0, +\infty)$ be an arbitrary weight function. Then the properties in Definitions 2.1 and 3.1–3.4 with $\mathcal{D} = B_1$ (or in Definitions 2.1, 3.1 and 3.2 with $\mathcal{D} = B_d$ for all $d \ge 1$) are equivalent. In fact, w has the required properties if and only if w is equivalent to a log-convex weight function.

3.3. Tropical power series and holomorphic approximation on $\mathbb{C}^d$

Trivial examples $w_n(t) = 1 + t^{n+\frac{1}{2}}$, $t \ge 0$, show that the statement of Theorem 2.2 is no longer true for $\mathcal{D} = \mathbb{C}$. To avoid such examples, in what follows we assume that $w : [0, +\infty) \to (0, +\infty)$ is *rapid*; this means, by definition, that $\lim_{t\to\infty} t^{-n} w(t) = \infty$ for all $n \in \mathbb{N}$. However, there exists a log-convex rapid radial weight w on $\mathbb{C}$ such that w is not essential, and hence w is not approximable by finite sums of moduli; see, e.g., [4, Example 3.3]. In fact, assuming that w is rapid, we show that the equivalence to a log-convex function should be replaced by the equivalence to a *log-tropical* function.

Recall that *a tropical polynomial* in one variable is defined as

$$\Psi(x) = \sup_{j\in E}(a_j + jx), \quad x \in \mathbb{R},\ a_j \in \mathbb{R},\ j \in E,$$

where E is a finite subset of $\mathbb{Z}_+$. Such polynomials are natural objects of tropical geometry (see, for example, monograph [12]). Following Kiselman [15], we say that $\Psi(x)$ is *a tropical power series* if $E = \mathbb{Z}_+$ and the supremum is finite for all $x \in \mathbb{R}$.

Given a weight function $v : [0, +\infty) \to (0, +\infty)$, consider its logarithmic transformation

$$\Phi(x) = \Phi_v(x) = \log v(\mathrm{e}^x), \quad -\infty < x < +\infty.$$

Clearly, v is log-convex if and only if Φ_v is convex. We say that v is *log-tropical* if Φ_v is a tropical power series. Any tropical power series is convex, hence, any log-tropical weight is log-convex; but not every log-convex weight is equivalent to a log-tropical function, cf. Example 1 below. Also, observe that the properties of being equivalent to a log-convex function and to a log-tropical function coincide in the case of the unit disk.

Example 1. Let $\alpha > 1$, and let w_α be a weight function such that $w_\alpha(t) = \mathrm{e}^{(\log t)^\alpha}$, $t > e$. Then w_α is equivalent to a log-tropical weight function if and only if $\alpha \ge 2$.

We have $\Phi_{w_\alpha}(x) = x^\alpha$, $x > 1$, in the above example. In fact, if w is a smooth log-convex weight such that $\liminf_{x\to\infty} \Phi''_w(x) > 0$, then w is equivalent to a log-tropical function; if $\lim_{x\to\infty} \Phi''_w(x) = 0$, then w is not equivalent to a log-tropical function. Observe that log-tropical weight functions may grow arbitrarily rapidly; however, a log-tropical weight function may grow slower than any given rapid weight function.

Theorem 3.5. *Let $w : [0, +\infty) \to (0, +\infty)$ be a rapid weight function. Then the following properties are equivalent:*

- *the radial weight w on $\mathbb{C}$ is approximable by a finite sum of moduli;*
- *the radial weight w on $\mathbb{C}^d$ is approximable by a finite sum of moduli for all (some) $d \ge 1$;*
- *w is approximable by the maximum of a holomorphic function modulus on $\mathbb{C}$;*
- *w is approximable by the maximum of a holomorphic function modulus on $\mathbb{C}^d$ for all (some) $d \ge 1$;*
- *w is essential on $\mathbb{C}$;*
- *w is essential on $\mathbb{C}^d$ for all (some) $d \ge 1$;*
- *w is approximable by power series with positive coefficients;*
- *w is approximable from below by monomials;*
- *w is equivalent to a log-tropical function.*

About the proof of Theorem 3.5 We use results from [8,14,20] to prove an abridged Theorem 3.5 without two key properties related to the approximation by a finite sum of moduli. The main technical result is the following implication: if w is equivalent to a log-tropical function, then the radial weight w on $\mathbb{C}^d$ is approximable by a finite sum of moduli for all $d \geq 1$.

Remark 3. If a radial weight w on $\mathbb{C}^d$ is approximable by a finite sum of moduli for all $d \geq 1$, then standard applications are related to various concrete operators on the growth space $\mathcal{A}^w(\mathbb{C}^d)$. See [1,2] and references therein for analogous applications in the setting of the growth spaces on the unit ball of $\mathbb{C}^d$, $d \geq 1$.

Remark 4. Let $w : [0, +\infty) \to (0, +\infty)$ be a log-tropical rapid weight function. Then Theorem 3.5 implies analogs of Corollaries 2.3, 2.4 and 2.5. In fact, there exists a proper holomorphic immersion $f : \mathbb{C} \to \mathbb{C}^3$ such that $|f| \asymp w$. If w is not equivalent to a log-tropical weight function, then Theorem 3.5 guarantees that there is no proper holomorphic map $f : \mathbb{C} \to \mathbb{C}^n$, $n \in \mathbb{N}$, such that $|f(z)| \asymp w(z)$.

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