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Genus one enumerative invariants in del-Pezzo surfaces with a fixed complex structure

Invariants énumératifs de genre un avec une structure complexe fixée pour des surfaces de del Pezzo

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ABSTRACT

We obtain a formula for the number of genus one curves with a fixed complex structure of a given degree on a del-Pezzo surface that pass through an appropriate number of generic points of the surface. This enumerative problem is expressed as the difference between the symplectic invariant and an intersection number on the moduli space of rational curves.

RÉSUMÉ

Nous obtenons une formule pour le nombre de courbes de genre un avec une structure complexe fixée, de degré donné, et passant par un nombre approprié de points génériques de la surface. La solution est exprimée comme la différence entre l'invariant symplectique et un nombre d'intersection sur l'espace de modules de courbes rationnelles.

1. Introduction

Enumerative Geometry of rational curves in $\mathbb{P}^2_{\mathbb{C}}$ is a classical question in algebraic geometry. A natural generalization is to ask how many elliptic curves, with a fixed j-invariant, are there of a given degree that pass through the right number of generic points. In [8] and [4], using methods of algebraic and symplectic geometry respectively, Pandharipande and Ionel obtain a formula for the number of degree d genus one curves with a fixed complex structure in $\mathbb{P}^2_{\mathbb{C}}$ that pass through $3d-1$ generic points. In this paper, we extend their result to del-Pezzo surfaces.

Let X be a complex del-Pezzo surface and $\beta \in H_2(X, \mathbb{Z})$ a given homology class. Let $n_{0,\beta}$ denote the number of rational curves of degree β in X that pass through δ_β generic points, where $\delta_\beta := \langle c_1(TX), \beta\rangle - 1$. We prove the following.

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Theorem 1.1. *Let X and β be as above. Let $n^j_{1,\beta}$ denote the number of elliptic curves with fixed j invariant of degree β in X that pass through δ_β generic points. Then*

$$n^j_{1,\beta} = \frac{2g_\beta}{|\mathrm{Aut}(\Sigma_1, j)|} n_{0,\beta} \qquad \textit{where} \qquad g_\beta := \frac{\beta\cdot\beta - c_1(TX)\cdot\beta + 2}{2},$$

$|\mathrm{Aut}(\Sigma_1, j)|$ denotes the number of automorphisms of a genus one Riemann surface with fixed j invariant that fixes a point and "$\cdot$" denotes topological intersection.

Note that g_β in Theorem 1.1 coincides with the genus of a smooth degree β curve on X. The numbers $n_{0,\beta}$ are computed in [6, p. 29] and [2, Theorem 3.6] using a recursive formula. When $X := \mathbb{P}^2$, our formula for $n^j_{1,\beta}$ is consistent with the formula of Pandharipande and Ionel in [8] and [4]. In [6], the authors actually give a formula to compute the genus 0 Gromov–Witten invariants of the del-Pezzo surfaces, which a priori need not be enumerative. It is shown in [2, page 63, last paragraph] that the numbers obtained in [6] are actually equal to $n_{0,\beta}$.

The results of Pandharipande and Ionel generalize the result of P. Aluffi; in [1], he computes the number of genus one cubics with a fixed complex structure in $\mathbb{P}^2$ through 8 generic points.

The problem of enumerating elliptic curves with a fixed j-invariant has also been studied by tropical geometers. In [5], Kerber and Markwig compute the number of tropical elliptic curves in $\mathbb{P}^2$ with a fixed j-invariant. Combined with the correspondence theorem [7, Theorem A], one can conclude that the number computed is indeed the same as the number of plane elliptic curves with a fixed j-invariant. Currently, this question is also being studied for other surfaces. In [7], Len and Ranganathan obtain a formula for the number of elliptic curves with a fixed j-invariant of a given degree for Hirzebruch surfaces, using methods from tropical geometry.

2. Enumerative versus symplectic invariant

We now explain the basic idea to compute $n^j_{1,\beta}$. Let (X, J, ω) be a compact semi-positive symplectic manifold, with a compatible almost complex structure J of dimension $2m$ and $\beta \in H_2(X,\mathbb{Z})$ be a homology class. Let k be a nonnegative integer such that $k + 2g \geq 3$. Let $\alpha_1, \cdots, \alpha_k$ and $\gamma_1, \cdots, \gamma_l$ be integral homology classes in $H_*(X,\mathbb{Z})$ such that

$$\sum_{i=1}^{k} 2m - \deg(\alpha_i) + \sum_{j=1}^{l}(2m - 2 - \deg(\gamma_j)) = 2m(1-g) + 2\langle c_1(TX),\ \beta\rangle\,. \tag{2.1}$$

Fix pseudocycles A_i, $1 \leq i \leq k$, and B_j, $1 \leq j \leq l$, on X representing the homology classes α_i and γ_j. Fix a compact Riemann surface Σ_g of genus g; its complex structure will be denoted by j. Define

$$\begin{aligned}\mathcal{M}^{\nu,j}_{g,k}(X,\beta;\alpha_1,\cdots,\alpha_k;\gamma_1,\cdots,\gamma_k) := \{&(u, y_1, \cdots, y_k) \in \mathcal{C}^\infty(\Sigma_g, X)\times X^k \mid u_*[\Sigma_g] = \beta,\\ &\bar\partial_{j,J} u = \nu,\ u(y_i)\in A_i\ \forall\, i = 1,\cdots,k,\quad \mathrm{Im}(u)\cap \mathrm{B}_j \neq \emptyset\ \forall j = 1,\cdots,l\},\end{aligned}$$

where $\nu : \Sigma_g \times X \longrightarrow T^*\Sigma_g \otimes TX$ is a generic smooth perturbation and

$$\bar\partial_{j,J} u := \frac{1}{2}\Big(du + J\circ du\circ j\Big).$$

The symplectic invariant (or the Ruan–Tian invariant) is defined to be the signed cardinality of the above set, i.e.,

$$\mathrm{RT}_{g,\beta}(\alpha_1,\cdots,\alpha_k;\gamma_1,\cdots,\gamma_l) := \pm|\mathcal{M}^{\nu,j}_{g,k}(X,\beta;\alpha_1,\cdots,\alpha_k;\gamma_1,\cdots,\gamma_k)|\,.$$

When $k = 0$, we denote the invariant as

$$\mathrm{RT}_{g,\beta}(\emptyset;\gamma_1,\cdots,\gamma_l).$$

Furthermore, when $\gamma_1,\ldots,\gamma_l$ all denote the class of a point, then we abbreviate the invariant as $\mathrm{RT}_{g,\beta}$. Similarly, when $l = 0$ we denote the invariant as

$$\mathrm{RT}_{g,\beta}(\alpha_1,\cdots,\alpha_k;\emptyset).$$

If (2.1) is not satisfied, then we formally define the invariant to be zero.

A natural question is to ask whether the symplectic invariant $\mathrm{RT}_{g,\beta}$ is equal to the enumerative invariant $n^j_{g,\beta}$. For $\mathbb{P}^2$, and more generally for del-Pezzo surfaces, the genus zero symplectic invariant is equal to the enumerative invariant [9, page 267]. However, even for $\mathbb{P}^2$, the genus one symplectic invariant is not enumerative. In general, the following fact is true ([11, Theorem 1.1]),

$$\mathrm{RT}_{1,\beta} = |\mathrm{Aut}(\Sigma_1, j)| n^j_{1,\beta} + \mathrm{CR}\,, \tag{2.2}$$

where CR denotes a correction term. Let us explain what this term means. First we note that the factor of $|\mathrm{Aut}(\Sigma_1, j)|$ is there because we do not mod out by automorphisms in the definition of $\mathcal{M}^{\nu,j}_{g,k}$. Hence, if $u : (\Sigma_1, j) \longrightarrow X$ is a solution to the $\overline{\partial}$-equation and the complex structure on X is genus one regular, then there will be $|\mathrm{Aut}(\Sigma_1, j)|$ new solutions close to u to the perturbed $\overline{\partial}$-equation. Next, we note that as $\nu \to 0$, a sequence of (J, ν)-holomorphic maps can also converge to a bubble tree whose base (the torus) is a constant (ghost) map [4, page 2]. These maps will also contribute to the computation of $\mathrm{RT}_{1,\beta}$ invariant. This extra contribution is defined to be the correction term CR.

We now explain how to compute the correction term. Let $\mathcal{M}_{0,n}(X, \beta)$ denote the moduli space of rational degree β curves on X that represent the class $\beta \in H_2(X, \mathbb{Z})$ and are equipped with n ordered marked points, modulo equivalence. In other words,

$$\mathcal{M}_{0,n}(X,\beta) := \{(u, y_1, \cdots, y_n) \in \mathcal{C}^\infty(\mathbb{P}^1, X) \times (\mathbb{P}^1)^n \mid \overline{\partial} u = 0,\ u_*[\mathbb{P}^1] = \beta\}/\mathrm{PSL}(2, \mathbb{C}),$$

with $\mathrm{PSL}(2,\mathbb{C})$ acting diagonally on $\mathbb{P}^1 \times (\mathbb{P}^1)^n$. Let $\overline{\mathcal{M}}_{0,n}(X,\beta)$ denote the stable map compactification of $\mathcal{M}_{0,n}(X,\beta)$.

Let us now focus on $\overline{\mathcal{M}}_{0,1}(X,\beta)$, the moduli space of curves with one marked point. Let $\mathcal{H}$ be the divisor in $\overline{\mathcal{M}}_{0,1}(X,\beta)$ corresponding to the extra condition that the curve passes through a given point. Let $\mathcal{L} \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$ and $\mathrm{ev} : \overline{\mathcal{M}}_{0,1}(X,\beta) \longrightarrow X$ be the universal tangent bundle and the evaluation map at the marked point. Following the same argument as in [4, Lemma 1.23], we conclude that the bundle $\mathrm{ev}^*TX \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta) \cap \mathcal{H}^{\delta_\beta}$ admits a nowhere vanishing section ν. This is because the rank of ev^*TX is two, while the dimension of the variety $\overline{\mathcal{M}}_{0,1}(X,\beta) \cap \mathcal{H}^{\delta_\beta}$ is one. Hence $\mathrm{ev}^*TX \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta) \cap \mathcal{H}^{\delta_\beta}$ admits a trivial sub bundle spanned by ν, which we denote as $\mathbb{C}\langle\nu\rangle$. When $X := \mathbb{P}^2$, it is shown in [4, Lemma 1.25], that the correction term is given by

$$\mathrm{CR} = \langle c_1(\mathcal{L}^* \otimes \mathrm{ev}^*TX/\mathbb{C}\langle\nu\rangle),\ [\overline{\mathcal{M}}_{0,1}(X,\beta)] \cap \mathcal{H}^{\delta_\beta}\rangle. \tag{2.3}$$

A more detailed justification of (2.3) is given in [11], by using the results of [12]. Furthermore, the gluing construction in [12] is valid in general for Kähler manifolds [12, page 8]. Hence, we conclude that (2.3) holds for del-Pezzo surfaces as well. Zinger also pointed out this fact to the second author of this paper in a personal communication ([13]).

In the next section we will obtain a formula for $c_1(\mathcal{L}^*)$ and compute the right-hand side of (2.3). The left-hand side of (2.2) is computed using the formula given in [9]. Hence, we obtain $n^j_{1,\beta}$.

3. Computation of the correction term

We will now give a self-contained proof of obtaining a formula for $c_1(\mathcal{L}^*)$ and hence computing the correction term. Alternatively, one can also compute the Chern classes by using the dilaton equation and the divisor equation as given in [3, Section 26.3].

Lemma 3.1. *On $\overline{\mathcal{M}}_{0,1}(X,\beta)$, the following equality of divisors holds:*

$$c_1(\mathcal{L}^*) = \frac{1}{(\beta \cdot x_1)^2}\Big((x_1 \cdot x_1)\mathcal{H} - 2(\beta \cdot x_1)ev^*(x_1) + \sum_{\substack{\beta_1+\beta_2=\beta,\\ \beta_1,\beta_2\neq 0}} \mathcal{B}_{\beta_1,\beta_2}(\beta_2 \cdot x_1)^2\Big), \tag{3.1}$$

where $\mathcal{H}$ is the locus satisfying the extra condition that the curve passes through a given point, $\mathcal{B}_{\beta_1,\beta_2}$ denotes the boundary stratum corresponding to the splitting into a degree β_1 curve and degree β_2 curve with the last marked point lying on the degree β_1 component and $x_i := c_i(TX)$.

Proof. The proof is similar to the one given in [4]. Let $\mu_1, \mu_2 \in X$ be two generic pseudocycles in X that represent the class Poincaré dual to x_1. Let $\widetilde{\mathcal{M}}$ be a cover of $\overline{\mathcal{M}}_{0,1}(X,\beta)$ with two additional marked points with the last two marked points lying on μ_1 and μ_2 respectively. More precisely,

$$\widetilde{\mathcal{M}} := \mathrm{ev}_2^{-1}(\mu_1) \cap \mathrm{ev}_3^{-1}(\mu_2) \subset \overline{\mathcal{M}}_{0,3}(X,\beta),$$

where ev_2 and ev_3 denote the evaluation maps at the second and third marked points respectively. Note that the projection $\pi : \widetilde{\mathcal{M}} \longrightarrow \overline{\mathcal{M}}_{0,1}(X,\beta)$ that forgets the last two marked points is a $(\beta \cdot x_1)^2$-to-one map. We now construct a meromorphic section

$$\phi : \widetilde{\mathcal{M}} \longrightarrow \pi^*\mathcal{L}^* \qquad \text{given by} \qquad \phi([u, y_1; y_2, y_3]) := \frac{(y_2 - y_3)dy_1}{(y_1 - y_2)(y_1 - y_3)}. \tag{3.2}$$

The right-hand side of (3.2) involves an abuse of notation: it is to be interpreted in an affine coordinate chart and then extended as a meromorphic section on the whole of $\mathbb{P}^1$. Note that on $(\mathbb{P}^1)^3$, the holomorphic line bundle

$$\eta := q_1^* K_{\mathbb{P}^1} \otimes \mathcal{O}_{(\mathbb{P}^1)^3}(\Delta_{12} + \Delta_{13} - \Delta_{23})$$

is trivial, where $q_1 : (\mathbb{P}^1)^3 \longrightarrow \mathbb{P}^1$ is the projection to the first factor and $\Delta_{jk} \subset (\mathbb{P}^1)^3$ is the divisor consisting of all points (z_i, z_2, z_3) such that $z_j = z_k$. The diagonal action of $\mathrm{PSL}(2, \mathbb{C})$ on $(\mathbb{P}^1)^3$ lifts to η preserving its trivialization. The section ϕ in (3.2) is given by this trivialization of η.

Since $c_1(\pi^*\mathcal{L}^*)$ is the zero divisor minus the pole divisor of ϕ, we gather that

$$c_1(\pi^*\mathcal{L}^*) = \{y_2 = y_3\} - \{y_1 = y_2\} - \{y_1 = y_3\}.$$

When projected down to $\overline{\mathcal{M}}_{0,1}(X, \beta)$, the divisor $\{y_2 = y_3\}$ becomes

$$(x_1 \cdot x_1)\mathcal{H} + (\beta_2 \cdot x_1)^2 \mathcal{B}_{\beta_1,\beta_2},$$

while both the divisors $\{y_1 = y_2\}$ and $\{y_1 = y_3\}$ become $(\beta \cdot x_1)\mathrm{ev}^*(x_1)$. Since $\widetilde{\mathcal{M}}$ is a $(\beta \cdot x_1)^2$-to-one cover of $\overline{\mathcal{M}}_{0,1}(X, \beta)$, we obtain (3.1). □

Using Lemma 3.1, we conclude that

$$\langle c_1(\mathcal{L}^*),\ [\overline{\mathcal{M}}_{0,1}(X, \beta)] \cap \mathcal{H}^{\delta_\beta}\rangle = -2n_{0,\beta}. \tag{3.3}$$

To see why this is so, we first note that $\overline{\mathcal{M}}_{0,1}(X, \beta) \cap \mathcal{H}^{\delta_\beta+1}$ is zero. This is because the number of rational curves through $\delta_\beta + 1$ generic points is zero. Next, we note that $\overline{\mathcal{M}}_{0,1}(X, \beta) \cap \mathcal{H}^{\delta_\beta} \cap \mathcal{B}_{\beta_1,\beta_2}$ is also zero. This is because the number of β curves which pass through δ_β points cannot split into a degree β_1 curve and a degree β_2 curve. This is because such a split curve will pass through $\delta_{\beta_1} + \delta_{\beta_2}$ points, which is one less than δ_β. So a split curve cannot pass through δ_β generic points. Finally we note that for any homology class $\mu \in H_2(X, \mathbb{Z})$, the following is true

$$[\overline{\mathcal{M}}_{0,1}(X, \beta)] \cap \mathcal{H}^{\delta_\beta} \cap \mathrm{ev}^*[\mu] = n_{0,\beta}(\beta \cdot \mu). \tag{3.4}$$

To see why this is so, we note that the left-hand side of (3.4) counts the number of degree β rational curves through δ_β points and one marked point, such that the marked point lies on some cycle representing the class β. There are $\beta \cdot \mu$ choices for that marked point to lie, which gives us the right hand side of (3.4). These three facts give us (3.3). Note that, when we say $\mathrm{ev}^*[\mu]$, we mean the pullback of the cohomology class Poincaré dual to μ (inside X). Using (3.4) (with $\mu := c_1(TX)$), we conclude that

$$\langle c_1(\mathrm{ev}^*TX),\ [\overline{\mathcal{M}}_{0,1}(X, \beta)] \cap \mathcal{H}^{\delta_\beta}\rangle = \Big(\beta \cdot c_1(TX)\Big) n_{0,\beta}. \tag{3.5}$$

From (3.3), (3.5) and (2.3) it follows that

$$\mathrm{CR} = \Big(\beta \cdot c_1(TX) - 2\Big) n_{0,\beta}. \tag{3.6}$$

4. Computation of the symplectic invariant

We now compute the symplectic invariant $\mathrm{RT}_{1,\beta} := \mathrm{RT}_{1,\beta}(\emptyset; p_1, \ldots, p_{\delta_\beta})$ using the formula [9, page 263, (1.2)]. Let $e_1, e_2, \cdots, e_k$ be a basis for $H_*(X, \mathbb{Z})$. Let

$$g_{ij} := e_i \cdot e_j \qquad \text{and} \qquad g^{ij} := \Big(g^{-1}\Big)_{ij}.$$

If the degrees of e_i and e_j do not add up to be the dimension of X then define g_{ij} to be zero. Using [9, page 263, (1.2)] we conclude that

$$\begin{aligned}\mathrm{RT}_{1,\beta}(\emptyset; p_1, \ldots, p_{\delta_\beta}) = \sum_{i,j} g^{ij}\mathrm{RT}_{0,\beta}(e_i, e_j; p_1, \ldots, p_{\delta_\beta}) &= \sum_{i,j} g^{ij} n_{0,\beta}(\beta \cdot e_i)(\beta \cdot e_j) \\ &= (\beta \cdot \beta) n_{0,\beta}.\end{aligned} \tag{4.1}$$

The last equality follows by writing β in the given basis e_i and using the definition of g^{ij}; the second equality follows from the same we justify (3.4). Equations (4.1), (3.6) and (2.2) give us the formula of Theorem 1.1.

5. Regularity of the complex structure for del-Pezzo surfaces

We now show that the complex structure on the del-Pezzo surfaces is genus one regular for immersion. In the statement of Theorem 1.1, the curve u passes through δ_β generic points. Hence the curve is going to be an immersion and hence it suffices to prove regularity for immersions.

Lemma 5.1. *Let X be $\mathbb{P}^2$ blown up at k points and (Σ_1, j) a compact genus 1 Riemann surface with a complex structure j. Let $u : \Sigma_1 \longrightarrow X$ be a holomorphic map representing the class $\beta := dL - m_1E_1 - \ldots - m_kE_k \in H_2(X, \mathbb{Z})$, where L and E_i denote the class of a line and the exceptional divisors respectively. Then D_u, the linearization of the $\overline{\partial}_{j,J}$ at u is surjective, provided $d > 0$ and u is an immersion. In particular, the complex structure on the del-Pezzo surface is genus 1 regular for immersions.*

Proof. We first note that if $\mathcal{L}$ is a holomorphic line bundle on (Σ_1, j) of positive degree, then the cup product

$$H^0(\Sigma_1, \mathcal{L}) \otimes H^1(\Sigma_1, \mathcal{L}^*) \longrightarrow H^1(\Sigma_1, \mathcal{L}) \otimes \mathcal{L}^*) = \mathbb{C}$$

is nondegenerate. Indeed, this coincides with the Serre duality pairing because the canonical line bundle of Σ_1 is trivial, and hence the pairing is nondegenerate. Next consider the short exact sequence of vector bundles on Σ_1 given by the differential of u

$$0 \longrightarrow T\Sigma_1 \xrightarrow{du} u^*TX \longrightarrow Q := (u^*TX)/T\Sigma_1 \longrightarrow 0\,. \tag{5.1}$$

Let

$$H^0(\Sigma_1, Q) \xrightarrow{\rho} H^1(\Sigma_1, T\Sigma_1) \longrightarrow H^1(\Sigma_1, u^*TX) \longrightarrow H^1(\Sigma_1, Q) \tag{5.2}$$

be the long exact sequence of cohomologies associated with it. We have $H^1(\Sigma_1, Q) = 0$ because $\text{degree}(Q) > 0$ (note that $\text{degree}(u^*TX) > 0$ and $\text{degree}(T\Sigma_1) = 0$). The exact sequence in (5.1) does not split; for the corresponding extension class $\psi \in H^1(\Sigma_1, (T\Sigma_1) \otimes Q^*) = H^1(\Sigma_1, Q^*)$, as observed before, there is $\psi' \in H^0(\Sigma_1, Q)$ such that $\psi \cup \psi' \neq 0$. Hence ρ in (5.2) is nonzero. This implies that ρ is surjective because $\dim H^1(\Sigma_1, u^*TX) = 1$. Hence from (5.2) we conclude that $H^1(\Sigma_1, u^*TX) = 0$, which proves that the cokernel of D_u is zero (i.e., D_u is surjective). □

When $X := \mathbb{P}^1 \times \mathbb{P}^1$, the complex structure is genus one regular; that is because for $\mathbb{P}^1$ the complex structure is genus one regular (by [10, Corollary 6.5]). Hence, the same fact holds for products of $\mathbb{P}^1$.

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