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Unique continuation estimates for the Kolmogorov equation in the whole space

Inégalités de continuation unique pour l'équation de Kolmogorov dans l'espace tout entier

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A B S T R A C T

We prove in this Note an observation estimate at one point in time for the Kolmogorov equation in the whole space. Such estimate implies the observability and the null controllability for the Kolmogorov equation with a control region which is sufficiently spread out throughout the whole space.

R É S U M É

Nous démontrons dans cette Note des inégalités d'observation traduisant la continuation unique pour l'équation de Kolmogorov définie sur l'espace tout entier.

1. Introduction and the main result

Consider the following Kolmogorov equation in the whole space ($d \in \mathbb{N}^+$)

$$\begin{cases} (\partial_t + v \cdot \nabla_x - \Delta_v) g(t,x,v) = 0, & (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ g(0,x,v) = g_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{1}$$

The well-posedness of the solution to (1) was proved in Propositions 2.1 and 2.2 in [3]. In [3], the authors considered the following definition.

Definition 1.1. (See Definition 1.1 in [3].) An open set O of $\mathbb{R}^n$ ($n \in \mathbb{N}^+$) is said to be an observability open set on the whole space $\mathbb{R}^n$ if there exist $\delta > 0$ and $r > 0$ such that

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$$\forall\, y \in \mathbb{R}^n,\ \exists\, y' \in O \text{ such that } B_{\mathbb{R}^n}(y', r) \subset O \text{ and } |y - y'| \le \delta. \tag{2}$$

Here $B_{\mathbb{R}^n}(x, r)$ denotes an open ball in $\mathbb{R}^n$ of radius r centered at x.

From this definition, the authors in [3] proved the following estimate: assume that $\omega_x \subset \mathbb{R}^d$ and $\omega_v \subset \mathbb{R}^d$ both verify the property (2) with $n = d$. Then for all $T > 0$, there exists $C > 0$ so that for each $g_0 \in L^2(\mathbb{R}^{2d})$, the solution to (1) satisfies that

$$\|g|_{t=T}\|_{L^2(\mathbb{R}^{2d})} \le C\|g\|_{L^2((0,T)\times\omega_x\times\omega_v)}. \tag{3}$$

In [3], the proof of (3) is based on a spectral inequality, a Carleman inequality with respect to the variable v and a decay inequality for the Fourier transform of the solution to (1) with respect to the variable x. The geometric condition (2) plays an important role in proving (3). The authors in [3] pointed out the following fact: there exists an open set $\mathcal{O}$ of $\mathbb{R}^{2d}$, which is an observability open set in the whole $\mathbb{R}^{2d}$, and does not contain any Cartesian product $\mathcal{O}_1 \times \mathcal{O}_2$, where each $\mathcal{O}_1$ and $\mathcal{O}_2$ are both observability open sets in the whole space $\mathbb{R}^d$.

In this note, assuming that $\omega \subset \mathbb{R}^{2d}$ verifies (2) with $n = 2d$, we get a unique continuation estimate for the Kolmogorov equation. Such kind of estimate has been studied in [1] and [6]. Our proof combines the spectral inequality given in [3] and a decay inequality on the Fourier transform of the solution to (1) with respect to the variables x and v. The main result is as follows.

Theorem 1.2. *Let $\omega \subset \mathbb{R}^{2d}$ be an observability open set on the whole space $\mathbb{R}^{2d}$. Then there exists $C = C(\omega, d) > 0$ so that for all $T > 0$, $\alpha \in (0, 1)$ and $g_0 \in L^2(\mathbb{R}^{2d})$, the solution to (1) satisfies that*

$$\big\|g|_{t=T}\big\|_{L^2(\mathbb{R}^{2d})} \le e^{\frac{C}{\alpha}(1+\frac{1}{T^3})}\big\|g|_{t=T}\big\|^{1-\alpha}_{L^2(\omega)}\|g_0\|^{\alpha}_{L^2(\mathbb{R}^{2d})}. \tag{4}$$

By a telescoping series method (see [6, Theorem 1.1]), a direct consequence of (4) is the following observability estimate.

Corollary 1.3. *Let $\omega \subset \mathbb{R}^{2d}$ be an observability open set on the whole space $\mathbb{R}^{2d}$. Let $T > 0$ and $E \subset (0, T)$ be a measurable set of positive measure. Then there exists $C_{\text{obs}} = C(\omega, d, T, E) > 0$ so that for each $g_0 \in L^2(\mathbb{R}^{2d})$, the solution to (1) verifies that*

$$\big\|g|_{t=T}\big\|_{L^2(\mathbb{R}^{2d})} \le C_{\text{obs}} \int_E \|g(t,\cdot,\cdot)\|_{L^2(\omega)}\, dt. \tag{5}$$

When $E = (0, T)$, $C_{\text{obs}} = e^{C(1+\frac{1}{T^3})}$ where C only depends on ω and d.

Such observability estimate implies by duality the null controllability for the Kolmogorov equation.

2. A spectral inequality

The following spectral inequality plays a key role to deduce the estimate (4). Here $\hat{f}$ denotes the Fourier transform of f.

Theorem 2.1. *(See Theorem 3.1 in [3].) Let $\omega \subset \mathbb{R}^{2d}$ be an observability open set on the whole space $\mathbb{R}^{2d}$. Then there exists $C = C(\omega, d) > 0$ such that for all $N > 0$, every $f \in L^2(\mathbb{R}^{2d})$ verifies that*

$$\int_{|\zeta|\le N} |\hat{f}(\zeta)|^2\, d\zeta \le e^{C(1+N)} \int_{\omega} \Big| \int_{|\zeta|\le N} \hat{f}(\zeta) e^{iz\cdot\zeta}\, d\zeta \Big|^2 dz. \tag{6}$$

We mention that, for a smooth compact and connected Riemannian manifold M with metric g and boundary ∂M, the following inequality was obtained in [4]: let $\omega \subset M$ be an open nonempty subset. There exists $C > 0$ such that the Laplace–Beltrami operator $-\Delta_g$ on M satisfies that

$$\|u\|_{L^2(M)} \le C\, e^{C\sqrt{\lambda}}\|u\|_{L^2(\omega)} \text{ for all } \lambda > 0 \text{ and } u \in \text{span}\{e_j;\ \lambda_j \le \lambda\}, \tag{7}$$

where $\{\lambda_j\}$ and $\{e_j\}$ are the eigenvalues and the corresponding eigenvectors of $-\Delta_g$ with the zero Dirichlet boundary condition. Based on this type of inequality (7), a similar estimate to (4) was obtained for the heat equation in a bounded domain (see [1, Theorem 6]). The strategy in this note also works for the heat equation in the whole space. This can be compared with [5], where M is non-compact with a Ricci curvature bounded below. The author in [5] proves that, under an interpolation inequality in [5, (6) on p. 40], (2) implies the spectral inequality (6), which yields the observability for the heat equation in M.

3. A decay inequality

We apply the Fourier transform, with respect to the variables x and v, to Eq. (1). Then we get the following equation in the corresponding frequency space

$$\begin{cases} (\partial_t - \xi \cdot \nabla_\eta + |\eta|^2)\hat{g}(t,\xi,\eta) = 0, & (t,\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d, \\ \hat{g}(0,\xi,\eta) = \hat{g}_0(\xi,\eta), & (\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d. \end{cases} \tag{8}$$

The solution to (8) has an explicit representation, which has been obtained in [2, Section 7.6, pp. 210–211]. Based on this, we get a decay estimate for the Kolmogorov equation as follows.

Proposition 3.1. *There exist $C > 0$ and $C' = C'(d) > 0$ such that for all $N, T > 0$ and each $g_0 \in L^2(\mathbb{R}^{2d})$, the solution to (8) verifies that*

$$\int_{|(\xi,\eta)|>N} |\hat{g}(T,\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \le \mathrm{e}^{C' - CN^2 \min\{T,T^3\}} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} |g_0(x,v)|^2 \,\mathrm{d}x\,\mathrm{d}v. \tag{9}$$

Proof. Let g be a solution to (8). One can directly compute that

$$\hat{g}(t,\xi,\eta) = \hat{g}_0(\xi,\eta+\xi t)\exp\big(-|\eta|^2 t - \eta\cdot\xi t^2 - |\xi|^2 t^3/3\big),\ \forall (t,\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d.$$

This yields that for all $(t,\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|\hat{g}(t,\xi,\eta)| \le |\hat{g}_0(\xi,\eta+\xi t)| \exp\big[-(|\eta|^2 + |\xi|^2)\min\{t,t^3\}/30\big].$$

From this, we see that for all $N, T > 0$,

$$\int_{|(\xi,\eta)|>N} |\hat{g}(T,\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \le \exp\big(-N^2\min\{T,T^3\}/15\big) \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} |\hat{g}_0(\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta,$$

which leads to (9). This ends the proof. □

4. Proofs of Theorem 1.2 and Corollary 1.3

In this section, we first prove Theorem 1.2 by combining Theorem 2.1 and Proposition 3.1 as follows.

Proof of Theorem 1.2. Let g be the solution to Eq. (1) with the initial data $g_0 \in L^2(\mathbb{R}^{2d})$. For each $N > 0$, write

$$\hat{g}(t,\xi,\eta) = \chi_{B_N}(\xi,\eta)\hat{g}(t,\xi,\eta) + \chi_{B_N^c}(\xi,\eta)\hat{g}(t,\xi,\eta),\ \forall (t,\xi,\eta) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d,$$

where χ_{B_N} and $\chi_{B_N^c}$ denote the characteristic functions of the set $B_N \triangleq \{(\xi,\eta) \in \mathbb{R}^{2d};\ |(\xi,\eta)| \le N\}$ and its complement, respectively. Let $T > 0$. We observe that for all $N > 0$,

$$(2\pi)^d \|g|_{t=T}\|_{L^2(\mathbb{R}^{2d})} = \|\hat{g}|_{t=T}\|_{L^2(\mathbb{R}^{2d})} \le \|\chi_{B_N}\hat{g}|_{t=T}\|_{L^2(\mathbb{R}^{2d})} + \|\chi_{B_N^c}\hat{g}|_{t=T}\|_{L^2(\mathbb{R}^{2d})}. \tag{10}$$

On one hand, we apply (6) to g to get the existence of a positive constant $C_1 = C_1(\omega,d)$ so that for all $N > 0$,

$$\begin{aligned} \int_{B_N} |\hat{g}(T,\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta \le \mathrm{e}^{2C_1(N+1)}\Big[&\int_\omega \Big| \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} \hat{g}(T,\xi,\eta)\mathrm{e}^{\mathrm{i}(x\cdot\xi + v\cdot\eta)} \,\mathrm{d}\xi\,\mathrm{d}\eta\Big|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &+ \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \Big| \int_{B_N^c} \hat{g}(T,\xi,\eta)\mathrm{e}^{\mathrm{i}(x\cdot\xi + v\cdot\eta)} \,\mathrm{d}\xi\,\mathrm{d}\eta\Big|^2 \,\mathrm{d}x\,\mathrm{d}v\Big]. \end{aligned} \tag{11}$$

On the other hand, let $f(\xi,\eta) \triangleq \chi_{B_N^c}(\xi,\eta)\hat{g}(T,\xi,\eta)$, $(\xi,\eta) \in \mathbb{R}^d_\xi \times \mathbb{R}^d_\eta$. It follows from the inverse Fourier transform formula that $\int f(\xi,\eta)\mathrm{e}^{\mathrm{i}(x\cdot\xi+v\cdot\eta)}\,\mathrm{d}\xi\,\mathrm{d}\eta$ is the inverse Fourier transform of f. Then

$$\begin{aligned} \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \Big| \int_{B_N^c} \hat{g}(T,\xi,\eta)\mathrm{e}^{\mathrm{i}(x\cdot\xi + v\cdot\eta)} \,\mathrm{d}\xi\,\mathrm{d}\eta\Big|^2 \,\mathrm{d}x\,\mathrm{d}v &= \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_v} \Big| \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} f(\xi,\eta)\mathrm{e}^{\mathrm{i}(x\cdot\xi + v\cdot\eta)} \,\mathrm{d}\xi\,\mathrm{d}\eta\Big|^2 \,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d_\xi \times \mathbb{R}^d_\eta} |f(\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{B_N^c} |\hat{g}(T,\xi,\eta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{12}$$

Meanwhile, we apply (9) to g to obtain that there exist $C_2 > 0$ and $C_3 = C_3(d) > 0$ so that for all $N > 0$,

$$\int_{B_N^c} |\hat{g}(T,\xi,\eta)|^2 \,d\xi\, d\eta \le e^{2[C_3 - C_2 N^2 \min\{T,T^3\}]} \int_{\mathbb{R}_x^d \times \mathbb{R}_v^d} |g_0(x,v)|^2 \,dx\,dv. \tag{13}$$

Write $T_3^1 \triangleq \min\{T, T^3\}$. By the inverse Fourier transform formula, we see from (10)–(13) that for all $N > 0$,

$$\|g|_{t=T}\|_{L^2(\mathbb{R}^{2d})} \le e^{C_1(N+1)} \|g|_{t=T}\|_{L^2(\omega)} + 2\,e^{C_1(N+1)+C_3-C_2N^2T_3^1} \|g_0\|_{L^2(\mathbb{R}^{2d})}. \tag{14}$$

Let $\alpha \in (0,1)$. We set $k(\alpha) \triangleq \alpha/(1-\alpha)$. Then we have that for all $N > 0$,

$$C_1 N \le \frac{C_1^2}{2k(\alpha)C_2T_3^1} + k(\alpha)\frac{C_2N^2T_3^1}{2} \text{ and } C_1N - C_2N^2T_3^1 \le \frac{C_1^2}{2C_2T_3^1} - \frac{C_2N^2T_3^1}{2}.$$

These, together with (14), yield that for all $\varepsilon \in (0,1)$,

$$\|g|_{t=T}\|_{L^2(\mathbb{R}^{2d})} \le \widetilde{C}_1 \Big[\varepsilon^{-k(\alpha)} \|g|_{t=T}\|_{L^2(\omega)} + \varepsilon \|g_0\|_{L^2(\mathbb{R}^{2d})}\Big], \tag{15}$$

where

$$\widetilde{C}_1 \triangleq \max\left\{ e^{C_1 + \frac{C_1^2}{2k(\alpha)C_2T_3^1}}, 2\,e^{C_1+C_3+\frac{C_1^2}{2C_2T_3^1}} \right\} \le 2\,e^{\frac{(C_1+C_2+C_3)^2}{\alpha C_2}(1+\frac{1}{T^3})}.$$

Since $\|g|_{t=T}\|_{L^2(\mathbb{R}^{2d})} \le \|g_0\|_{L^2(\mathbb{R}^{2d})}$, the minimization of the right side of (15), with respect to the variable ε over $\mathbb{R}^+$, leads to (4). This completes the proof. □

We next use the telescoping series method to deduce Corollary 1.3 from Theorem 1.2.

Proof of Corollary 1.3. Let g be the solution to Equation (1) with the initial data $g_0 \in L^2(\mathbb{R}^{2d})$. We take $\alpha = 1/2$ in (4) and then see from the Young inequality that there exists $C_1 = C_1(\omega, d) > 0$ so that

$$\|g|_{t=T}\| \le \frac{1}{\varepsilon} e^{C_1(1+\frac{1}{T^3})} \|g(T,\cdot,\cdot)\|_{L^2(\omega)} + \varepsilon \|g|_{t=0}\|, \ \forall \varepsilon > 0.$$

Generally, for each $0 < t_1 < t_2$, we have that

$$\|g|_{t=t_2}\| \le \frac{1}{\varepsilon} e^{C_1[1+\frac{1}{(t_2-t_1)^3}]} \|g(t_2,\cdot,\cdot)\|_{L^2(\omega)} + \varepsilon \|g|_{t=t_1}\|, \ \forall \varepsilon > 0. \tag{16}$$

Let l be a Lebesgue density point of E. Then by [6, Proposition 2.1], we know that for each $\lambda \in (1/\sqrt[6]{2}, 1)$, there exists a sequence $\{l_m\} \subset (l, T)$ so that for each $m \in \mathbb{N}^+$,

$$l_m - l = \lambda^{m-1}(l_1 - l) \text{ and } 3|E \cap (l_{m+1}, l_m)| \ge |l_{m+1} - l_m|. \tag{17}$$

Take a $m \in \mathbb{N}^+$ and let $0 < l_{m+2} < l_{m+1} \le s < l_m < T$. Since $\|g|_{t=l_m}\| \le \|g|_{t=s}\|$ and $l_{m+1} - l_{m+2} \le s - l_{m+2}$, we apply (16), where $t_1 = l_{m+2}$ and $t_2 = s$, to get that

$$\|g|_{t=l_m}\| \le \frac{1}{\varepsilon} e^{C_1[1+\frac{1}{(l_{m+1}-l_{m+2})^3}]} \|g(s,\cdot,\cdot)\|_{L^2(\omega)} + \varepsilon \|g|_{t=l_{m+2}}\|, \ \forall \varepsilon > 0.$$

By integrating both sides over $E \cap (l_{m+1}, l_m)$ in the above inequality, we know that

$$\begin{aligned} &\Big(\varepsilon |E \cap (l_{m+1}, l_m)| e^{-\frac{C_1}{(l_{m+1}-l_{m+2})^3}}\Big) \|g|_{t=l_m}\| - \Big(\varepsilon^2 |E \cap (l_{m+1}, l_m)| e^{-\frac{C_1}{(l_{m+1}-l_{m+2})^3}}\Big) \|g|_{t=l_{m+2}}\| \\ &\le e^{C_1} \int_{E\cap(l_{m+1},l_m)} \|g(s,\cdot,\cdot)\|_{L^2(\omega)} \,ds, \ \forall \varepsilon > 0. \end{aligned} \tag{18}$$

Meanwhile, we know from (17) that

$$3|E \cap (l_{m+1}, l_m)| \ge |l_{m+1} - l_m| \ge e^{-\frac{1}{|l_{m+1}-l_m|}} \ge e^{-\frac{\lambda^3(l_1-l_2)^2}{(l_{m+1}-l_{m+2})^3}}, \ \forall m \in \mathbb{N}^+.$$

Since $l_m - l_{m+2} = (1 + \frac{1}{\lambda})(l_{m+1} - l_{m+2})$, the above, as well as (18), yields that for all $m \in \mathbb{N}^+$ and $\varepsilon > 0$,

$$\varepsilon \mathrm{e}^{-\frac{C_2}{(l_m-l_{m+2})^3}}\|g|_{t=l_m}\| - \varepsilon^2 \mathrm{e}^{-\frac{C_2}{(l_m-l_{m+2})^3}}\|g|_{t=l_{m+2}}\| \le 3\,\mathrm{e}^{C_1}\int\limits_{E\cap(l_{m+1},l_m)}\|g(s,\cdot,\cdot)\|_{L^2(\omega)}\,\mathrm{d}s, \tag{19}$$

where $C_2 = (1+\frac{1}{\lambda})^3[C_1+\lambda^3(l_1-l_2)^2]$. Let $\beta \triangleq \frac{\lambda^6}{2\lambda^6-1}$ (>0) and $\varepsilon = \mathrm{e}^{-\frac{(\beta-1)C_2}{(l_m-l_{m+2})^3}}$. Since $\lambda^2(l_m-l_{m+2}) = l_{m+2}-l_{m+4}$, $\forall m\in\mathbb{N}^+$, it follows from (19) that

$$\mathrm{e}^{-\frac{\beta C_2}{(l_m-l_{m+2})^3}}\|g|_{t=l_m}\| - \mathrm{e}^{-\frac{\beta C_2}{(l_{m+2}-l_{m+4})^3}}\|g|_{t=l_{m+2}}\| \le 3\,\mathrm{e}^{C_1}\int\limits_{E\cap(l_{m+1},l_m)}\|g(s,\cdot,\cdot)\|_{L^2(\omega)}\,\mathrm{d}s.$$

We deduce from this that

$$\begin{aligned}\mathrm{e}^{-\frac{\beta C_2}{(l_1-l_3)^3}}\|g|_{t=l_1}\| &= \sum_{k=0}^{\infty}\left[\mathrm{e}^{-\frac{\beta C_2}{(l_{2k+1}-l_{2k+3})^3}}\|g|_{t=l_{2k+1}}\| - \mathrm{e}^{-\frac{\beta C_2}{(l_{2k+3}-l_{2k+5})^3}}\|g|_{t=l_{2k+3}}\|\right]\\ &\le \sum_{k=0}^{\infty}3\,\mathrm{e}^{C_1}\int\limits_{E\cap(l_{2k+3},l_{2k+1})}\|g(s,\cdot,\cdot)\|_{L^2(\omega)}\,\mathrm{d}s \le 3\,\mathrm{e}^{C_1}\int\limits_{E\cap(l,l_1)}\|g(s,\cdot,\cdot)\|_{L^2(\omega)}\,\mathrm{d}s.\end{aligned}$$

Since $\|g|_{t=T}\| \le \|g|_{t=l_1}\|$, the above implies that

$$\|g|_{t=T}\| \le 3\,\mathrm{e}^{C_1+\frac{\beta C_2}{(l_1-l_3)^3}}\int\limits_{E}\|g(s,\cdot,\cdot)\|_{L^2(\omega)}\,\mathrm{d}s.$$

This proves (5). Especially, when $E=(0,T)$, we can take $l_1 = T$ and $l_3 = T/4$. We end the proof. □

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