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Remarks on the symplectic invariance of Aubry–Mather sets

Remarques sur l'invariance symplectique des ensembles d'Aubry–Mather

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ABSTRACT

In this note, we discuss and clarify some issues related to the generalization of Bernard's theorem on the symplectic invariance of Aubry, Mather and Mañé sets, to the cases of non-zero cohomology classes or non-exact symplectomorphisms, not necessarily homotopic to the identity.

RÉSUMÉ

On discute et clarifie quelques questions liées à la généralisation du théorème de Bernard sur l'invariance symplectique des ensembles d'Aubry, de Mather et de Mañé aux cas de classes de cohomologie non nulles et de symplectomorphismes non exacts et non nécessairement homotopes à l'identité.

1. Introduction

In the study of Hamiltonian dynamical systems, *Aubry–Mather theory* refers to a series of variational techniques, related to the Principle of Least Action, that singled out particular orbits and, more generally, invariant sets obtained as minimizing solutions to a variational problem. These sets are nowadays called the *Mather*, *Aubry* and *Mañé sets*, and as a result of their action-minimizing property, they enjoy many interesting dynamical properties and a rich geometric structure.

Symplectic aspects of Aubry–Mather theory and its relation to symplectic geometry have soon attracted a lot of interest, starting from the work of Paternain, Polterovich, and Siburg [7]. In his seminal paper [1], Bernard established the symplectic invariance of the Aubry–Mather sets corresponding to the zero cohomology class, under the action of exact symplectomorphisms that preserve the condition of being of Tonelli type. See also, just to mention a few papers in the literature that followed [2,3,8–12].

In this note, starting from Bernard's result, we would like to discuss and clarify some aspects related to the generalization of his theorem to other cohomology classes and to non-exact symplectomorphisms, not necessarily homotopic to the

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identity (see Theorem 9 and Corollary 10). As we shall see, in fact, one has to keep into account two distinct issues: the cohomology class of the symplectomorphism, as well as the action of the symplectomorphism on de Rham cohomology classes.

2. Notation and setting

Let M be a closed manifold, and let us denote by $\mathrm{T}M$ and T^*M, respectively, its tangent and cotangent bundles. A *Tonelli Hamiltonian* is a C^2 function $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ that is strictly convex and superlinear in each fiber. For each de Rham cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we denote by $\mathcal{M}^*_c(H)$, $\mathcal{A}^*_c(H)$ and $\mathcal{N}^*_c(H)$, respectively the *Mather*, *Aubry* and *Mañé* sets of cohomology class c, associated with H. Moreover, we denote by $\alpha_H : \mathrm{H}^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ and $\beta_H : \mathrm{H}_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ the so-called *Mather's minimal average actions*. We refer to [5,6,9] for a precise definition of these objects and for a discussion of their properties.

Let λ be the Liouville form on T^*M, which can be written in local coordinates as $\sum_j p_j \,\mathrm{d}q_j$. The projection onto the base $\pi : \mathrm{T}M \longrightarrow M$ is a homotopy equivalence whose homotopy inverse is given by the inclusion of the 0-section $\iota : M \hookrightarrow \mathrm{T}^*M$. From now on, we tacitly identify the de Rham cohomology groups $\mathrm{H}^1(M;\mathbb{R})$ and $\mathrm{H}^1(\mathrm{T}^*M;\mathbb{R})$ by means of the isomorphisms induced by π and ι. Analogously, we identify the singular homology groups $\mathrm{H}_1(M;\mathbb{R})$ and $\mathrm{H}_1(\mathrm{T}^*M;\mathbb{R})$.

Given a symplectomorphism Ψ of $(\mathrm{T}^*M, \mathrm{d}\lambda)$, we denote by $[[\Psi]]$ the cohomology class $[\Psi^*\lambda - \lambda] \in \mathrm{H}^1(M;\mathbb{R})$. Such a symplectomorphism is called *exact* when $[[\Psi]] = 0$.

Example 1.

(i) A particularly simple class of symplectomorphisms is given by translations in the fibers, that is, maps $\Theta_\alpha(q,p) = (q, p + \alpha_q)$, where α is a closed 1-form on M. These symplectomorphisms are obviously homotopic to the identity, and their cohomology class is given by $[[\Theta_\alpha]] = [\alpha]$.

(ii) Any diffeomorphism $\psi : M \longrightarrow M$ can be lifted to a diffeomorphism $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ defined by

$$\Psi(q,p) := (\psi(q), (\psi^{-1})^* p) = \Big(\psi(q), p \circ \mathrm{d}\psi^{-1}(\psi(q))\Big),$$

that preserves the Liouville form λ, *i.e.*, $\Psi^*\lambda = \lambda$. In particular, Ψ is an exact symplectomorphism. As a special instance, let $A \in \mathrm{GL}_n(\mathbb{Z})$ and consider the linear map on $\mathbb{T}^n$ given by $\psi(q) = (A^{\mathrm{T}})^{-1} q$. The associated symplectomorphism of $\mathrm{T}^*\mathbb{T}^n$ is given by $\Psi(q,p) = ((A^{\mathrm{T}})^{-1} q, Ap)$.

3. Symplectic aspects of Aubry–Mather theory

Let us start by recalling Bernard's result.

Theorem 2. *(Bernard [1].) Let $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian and $\Phi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ an exact symplectomorphism such that $H \circ \Phi$ is still of Tonelli type. Then:*

$$\begin{aligned}
\mathcal{M}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{M}^*_0(H)\big)\\
\mathcal{A}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{A}^*_0(H)\big)\\
\mathcal{N}^*_0(H \circ \Phi) &= \Phi^{-1}\big(\mathcal{N}^*_0(H)\big).
\end{aligned}$$

Remark 3. Obviously the condition that the Hamiltonian $H \circ \Phi$ be still of Tonelli type is very restrictive. For instance, if $M = S^1$ and $H(q,p) = p^2$, consider any Hamiltonian diffeomorphism $\Phi : \mathrm{T}^*S^1 \longrightarrow \mathrm{T}^*S^1$ mapping a fiber $\mathrm{T}^*_q S^1$ to a curve $(q(t), p(t))$ such that $t \mapsto p(t)$ is not monotone; then the composition $H \circ \Phi$ is not Tonelli, as its restriction to the fiber $\mathrm{T}^*_q S^1$ is not convex.

Remark 4.

(i) Bernard's theorem does not hold anymore if Φ is not exact. For example, if we consider the symplectomorphism $\Theta_\alpha(q,p) = (q, p + \alpha_q)$, where α is a closed 1-form on M, then one can easily check that

$$\mathcal{M}^*_0(H \circ \Theta_\alpha) = \Theta_\alpha^{-1}\big(\mathcal{M}^*_{[\alpha]}(H)\big),$$

which, in general, could be different from $\Theta_\alpha^{-1}\big(\mathcal{M}^*_0(H)\big)$. Similarly, for the Aubry and Mañé sets (see Proposition 7).

(ii) Even in the case of exact symplectomorphisms, Bernard's theorem may fail for non-zero cohomology classes. For example, let us consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ and consider the exact symplectomorphism

$$\Psi(q,p) = ((A^{\mathrm{T}})^{-1} q, Ap),$$

as explained in Example 1(ii). If $H(q,p)=\mathfrak{h}(p)$ is an integrable Tonelli Hamiltonian on $\mathrm{T}^*\mathbb{T}^n$, one can easily check that, after identifying the de Rham cohomology group $\mathrm{H}^1(\mathbb{T}^n;\mathbb{R})$ with $\mathbb{R}^n$, for all $c\in\mathbb{R}^n$ one has

$$\mathcal{M}^*_c(\mathfrak{h})=\mathcal{A}^*_c(\mathfrak{h})=\mathcal{N}^*_c(\mathfrak{h})=\{(q,p)\in\mathrm{T}^*\mathbb{T}^n:\ p=c\},\tag{3.1}$$

see e.g. [9] for the details. The Hamiltonian $\mathfrak{h}\circ\Psi$ is still of Tonelli type and integrable, hence (3.1) continues to hold if we replace $\mathfrak{h}$ with $\mathfrak{h}\circ\Psi$. In particular, for each $c\in\mathbb{R}^n$ we obtain:

$$\mathcal{M}^*_c(\mathfrak{h}\circ\Psi)=\Psi^{-1}\left(\mathcal{M}^*_{Ac}(\mathfrak{h})\right)\neq\Psi^{-1}\left(\mathcal{M}^*_c(\mathfrak{h})\right).$$

The above remark shows that two distinct features must be kept into account in order to generalize Bernard's theorem to general symplectomorphisms: the cohomology class of a symplectomorphism, and the action of the symplectomorphism on de Rham cohomology classes.

Lemma 5. *Let Ψ and Φ be symplectomorphisms of $(\mathrm{T}^*M,\mathrm{d}\lambda)$. Then*[1]

$$[[\Phi\circ\Psi]]=\Psi^*[[\Phi]]+[[\Psi]].$$

In particular, if Ψ is homotopic to the identity or if Φ is exact, then

$$[[\Phi\circ\Psi]]=[[\Phi]]+[[\Psi]].$$

Proof. It is sufficient to observe that $(\Phi\circ\Psi)^*\lambda-\lambda=\Psi^*(\Phi^*\lambda-\lambda)+\Psi^*\lambda-\lambda$. $\square$

Lemma 6. *Any symplectomorphism Ψ of $(\mathrm{T}^*M,\mathrm{d}\lambda)$ can be written as $\Psi=\Phi\circ\Theta_\eta$, where η is a closed 1-form on M and Φ is an exact symplectomorphism. In particular, $[\eta]=[[\Psi]]$.*

Proof. Let η be any closed 1-form on M, such that $[\eta]=[[\Psi]]$, and consider Θ_η, defined as in Example 1(i). Let $\Phi=\Psi\circ(\Theta_\eta)^{-1}=\Psi\circ\Theta_{-\eta}$. In order to conclude the proof, we need to check that Φ is exact. This follows from Lemma 5, since

$$[[\Phi]]=[[\Psi\circ\Theta_{-\eta}]]=\Theta^*_{-\eta}[[\Psi]]+[[\Theta_{-\eta}]]=[[\Psi]]-[\eta]=0,$$

where in the third identity we have used that $\Theta_{-\eta}$ is homotopic to the identity. $\square$

Proposition 7. *Let $H:\mathrm{T}^*M\longrightarrow\mathbb{R}$ be a Tonelli Hamiltonian and η a closed 1-form on M. Then for each $c\in H^1(M;\mathbb{R})$ we have:*

$$\begin{aligned}\mathcal{M}^*_c(H)&=\Theta_\eta\big(\mathcal{M}^*_{c-[\eta]}(H\circ\Theta_\eta)\big)\\ \mathcal{A}^*_c(H)&=\Theta_\eta\big(\mathcal{A}^*_{c-[\eta]}(H\circ\Theta_\eta)\big)\\ \mathcal{N}^*_c(H)&=\Theta_\eta\big(\mathcal{N}^*_{c-[\eta]}(H\circ\Theta_\eta)\big).\end{aligned}$$

Proof. Here, as well as in Proposition 8 and Theorem 10, we provide the proof for the Mather sets, but the same argument works for the Aubry and Mañé sets.

First of all, observe that if H is a Tonelli Hamiltonian, then $H\circ\Theta_\eta$ is a Tonelli Hamiltonian as well. The Lagrangian dual to $H\circ\Theta_\eta$ is $L-\eta$, and the associated Legendre transform $\mathcal{L}_{L-\eta}$ is $\Theta_{-\eta}\circ\mathcal{L}$. On the Lagrangian side, the identity

$$\widetilde{\mathcal{M}}_{c-[\eta]}(L-\eta)=\widetilde{\mathcal{M}}_c(L)$$

follows directly from the definition of the Mather set for a given cohomology class. Therefore

$$\mathcal{M}^*_{c-[\eta]}(H\circ\Theta_\eta):=\mathcal{L}_{L_\eta}\left(\widetilde{\mathcal{M}}_{c-[\eta]}(L-\eta)\right)=(\Theta_{-\eta}\circ\mathcal{L}_L)\big(\widetilde{\mathcal{M}}_c(L)\big)=\Theta_{-\eta}\big(\mathcal{M}^*_c(H)\big).\qquad\square$$

Proposition 8. *Let $H:\mathrm{T}^*M\longrightarrow\mathbb{R}$ be a Tonelli Hamiltonian, and $\Psi:\mathrm{T}^*M\longrightarrow\mathrm{T}^*M$ an exact symplectomorphism such that $H\circ\Psi$ is of Tonelli type. For each de Rham cohomology class $c\in\mathrm{H}^1(M;\mathbb{R})$, we have:*

$$\begin{aligned}\mathcal{M}^*_c(H)&=\Psi\big(\mathcal{M}^*_{\Psi^*c}(H\circ\Psi)\big)\\ \mathcal{A}^*_c(H)&=\Psi\big(\mathcal{A}^*_{\Psi^*c}(H\circ\Psi)\big)\\ \mathcal{N}^*_c(H)&=\Psi\big(\mathcal{N}^*_{\Psi^*c}(H\circ\Psi)\big).\end{aligned}$$

In particular, if Ψ is homotopic to the identity, then

[1] In this equation, Ψ^* must be understood as $\iota^*\circ\Psi^*\circ\pi^*$, according to the identification of the de Rham cohomology groups $\mathrm{H}^1(M;\mathbb{R})$ and $\mathrm{H}^1(\mathrm{T}^*M;\mathbb{R})$.

$$\mathcal{M}_c^*(H) = \Psi\big(\mathcal{M}_c^*(H\circ\Psi)\big)$$
$$\mathcal{A}_c^*(H) = \Psi\big(\mathcal{A}_c^*(H\circ\Psi)\big)$$
$$\mathcal{N}_c^*(H) = \Psi\big(\mathcal{N}_c^*(H\circ\Psi)\big).$$

Proof. We choose two closed 1-forms α and β on M whose cohomology classes are equal to c and Ψ^*c respectively. Lemma 5 implies that the symplectomorphism $\Phi := \Theta_{-\alpha}\circ\Psi\circ\Theta_\beta$ is exact, since

$$[[\Phi]] = \Theta_\beta^*\Psi^*[[\Theta_{-\alpha}]] + \Theta_\beta^*[[\Psi]] + [[\Theta_\beta]] = \Psi^*[-\alpha] + [\beta] = -\Psi^*c + \Psi^*c = 0.$$

By Proposition 7 and Bernard's Theorem 2, the Mather sets transform under symplectomorphism according to

$$\begin{aligned}\mathcal{M}_{[\alpha]}^*(H) &= \Theta_\alpha(\mathcal{M}_0^*(H\circ\Theta_\alpha))\\ &= \Theta_\alpha\circ\Phi(\mathcal{M}_0^*(H\circ\Theta_\alpha\circ\Phi))\\ &= \Psi\circ\Theta_\beta(\mathcal{M}_0^*(H\circ\Psi\circ\Theta_\beta))\\ &= \Psi(\mathcal{M}_{\Psi^*[\alpha]}^*(H\circ\Psi)). \qquad \square\end{aligned}$$

Theorem 9. *Let $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian and $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ a symplectomorphism such that $H\circ\Psi$ is of Tonelli type. For each de Rham cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we have:*

$$\mathcal{M}_c^*(H) = \Psi\big(\mathcal{M}_{\Psi^*c-[[\Psi]]}^*(H\circ\Psi)\big)$$
$$\mathcal{A}_c^*(H) = \Psi\big(\mathcal{A}_{\Psi^*c-[[\Psi]]}^*(H\circ\Psi)\big)$$
$$\mathcal{N}_c^*(H) = \Psi\big(\mathcal{N}_{\Psi^*c-[[\Psi]]}^*(H\circ\Psi)\big).$$

In particular, if Ψ is homotopic to the identity, then

$$\mathcal{M}_c^*(H) = \Psi\big(\mathcal{M}_{c-[[\Psi]]}^*(H\circ\Psi)\big)$$
$$\mathcal{A}_c^*(H) = \Psi\big(\mathcal{A}_{c-[[\Psi]]}^*(H\circ\Psi)\big)$$
$$\mathcal{N}_c^*(H) = \Psi\big(\mathcal{N}_{c-[[\Psi]]}^*(H\circ\Psi)\big).$$

Proof. Let η be a closed 1-form on M whose cohomology class is equal to $[[\Psi]]$. By Lemma 6, the symplectomorphism $\Phi := \Psi\circ\Theta_{-\eta}$ is exact. This, together with Proposition 7, implies

$$\begin{aligned}\Psi\big(\mathcal{M}_{\Psi^*c-[[\Psi]]}^*(H\circ\Psi)\big) &= \Phi\circ\Theta_\eta\big(\mathcal{M}_{\Theta_\eta^*\Phi^*c-[\eta]}^*(H\circ\Phi\circ\Theta_\eta)\big)\\ &= \Phi\circ\Theta_\eta\big(\mathcal{M}_{\Phi^*c-[\eta]}^*(H\circ\Phi\circ\Theta_\eta)\big)\\ &= \Phi\big(\mathcal{M}_{\Phi^*c}^*(H\circ\Phi)\big).\end{aligned}$$

Finally, by Proposition 8, the latter term is equal to $\mathcal{M}_c^*(H)$. $\square$

Corollary 10. *Let $H : \mathrm{T}^*M \longrightarrow \mathbb{R}$ be a Tonelli Hamiltonian and $\Psi : \mathrm{T}^*M \longrightarrow \mathrm{T}^*M$ a symplectomorphism, such that $H\circ\Psi$ is still of Tonelli type. For each de Rham cohomology class $c \in \mathrm{H}^1(M;\mathbb{R})$, we have:*

$$\alpha_H(c) = \alpha_{H\circ\Psi}(\Psi^*c - [[\Psi]]).$$

Moreover, for each homology class $h \in \mathrm{H}_1(M;\mathbb{R})$ we have:

$$\beta_H(h) = \beta_{H\circ\Psi}(\Psi_*^{-1}h) - \langle[[\Psi^{-1}]], h\rangle,$$

where[2] *$\Psi_* : \mathrm{H}_1(M;\mathbb{R}) \longrightarrow \mathrm{H}_1(M;\mathbb{R})$ is the homology homomorphism induced by Ψ.*

Proof. A classical result of Carneiro [4] implies that $\mathcal{M}_c^*(H) \subset H^{-1}(\alpha_H(c))$. From this, together with Theorem 9, we infer

$$\alpha_{H\circ\Psi}(\Psi^*c - [[\Psi]]) = H\circ\Psi(\mathcal{M}_{\Psi^*c-[[\Psi]]}^*(H\circ\Psi)) = H(\mathcal{M}_c^*(H)) = \alpha_H(c).$$

As for the Mather's β function, first notice that

$$(\Psi^{-1})^*[[\Psi]] = (\Psi^{-1})^*(\Psi^*\lambda - \lambda) = \lambda - (\Psi^{-1})^*\lambda = -[[\Psi^{-1}]].$$

[2] Here, Ψ_* must be understood as $\pi_*\circ\Psi_*\circ\iota_*$, according to the identification of the singular homology groups $\mathrm{H}_1(M;\mathbb{R})$ and $\mathrm{H}_1(\mathrm{T}^*M;\mathbb{R})$.

Since α and β are convex conjugate of each other, we have

$$\begin{aligned}
\beta_H(h) &= \sup_{c\in \mathrm{H}^1(M;\mathbb{R})} \Big(\langle c, h\rangle - \alpha_H(c)\Big) \\
&= \sup_{c\in \mathrm{H}^1(M;\mathbb{R})} \Big(\langle (\Psi^{-1})^*\Psi^* c, h\rangle - \alpha_{H\circ\Psi}(\Psi^* c - [[\Psi]])\Big) \\
&= \sup_{\tilde{c}\in \mathrm{H}^1(M;\mathbb{R})} \Big(\langle (\Psi^{-1})^*\tilde{c} + (\Psi^{-1})^*[[\Psi]], h\rangle - \alpha_{H\circ\Psi}(\tilde{c})\Big) \\
&= \sup_{\tilde{c}\in \mathrm{H}^1(M;\mathbb{R})} \Big(\langle \tilde{c}, \Psi_*^{-1}h\rangle - \alpha_{H\circ\Psi}(\tilde{c})\Big) - \langle [[\Psi^{-1}]], h\rangle \\
&= \beta_{H\circ\Psi}(\Psi_*^{-1}h) - \langle [[\Psi^{-1}]], h\rangle. \qquad \square
\end{aligned}$$

We can summarize everything in the following commutative diagram.

$$\begin{tikzcd}
\mathrm{T}^*M \arrow[loop above, "\Phi_t^{H\circ\Psi}"] \arrow[rr, "\Psi"] \arrow[dr, "H\circ\Psi"] & & \mathrm{T}^*M \arrow[loop above, "\Phi_t^{H}"'] \arrow[dl, "H"'] \\
& \mathbb{R} & \\
\mathrm{H}^1(M;\mathbb{R}) \arrow[uu, squiggly, "c\mapsto\left\{\begin{array}{l}\mathcal{M}_c^*(H\circ\Psi)\\ \mathcal{A}_c^*(H\circ\Psi)\\ \mathcal{N}_c^*(H\circ\Psi)\end{array}\right."] \arrow[ur, "\alpha_{H\circ\Psi}"'] & & \mathrm{H}^1(M;\mathbb{R}) \arrow[ll, "c\mapsto\Psi^*c-[[\Psi]]"] \arrow[ul, "\alpha_H"] \arrow[uu, squiggly, "c\mapsto\left\{\begin{array}{l}\mathcal{M}_c^*(H)\\ \mathcal{A}_c^*(H)\\ \mathcal{N}_c^*(H)\end{array}\right."']
\end{tikzcd}$$

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