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Partial differential equations

Multiplicity for a 4-sublinear Schrödinger–Poisson system with sign-changing potential via Morse theory ☆

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A B S T R A C T

In this paper, we study a 4-sublinear Schrödinger–Poisson system with sign-changing potential. Under some suitable assumptions, the existence of two nontrivial solutions are obtained by using the Morse theory. Our result improves the recent ones of Chen and Zhang (2014) [6].

1. Introduction and main results

In this paper, we mainly consider the multiplicity of the nontrivial solutions for the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the potential $V(x)$ satisfies the following condition:

(V) $V(x) \in C(\mathbb{R}^3)$ is a bounded function.

Under condition (V), we consider the following increasing sequence $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ of minimax values defined by

$$\lambda_n := \inf_{V \in \mathcal{V}_n} \sup_{u \in V, u \neq 0} \frac{\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V u^2\right) \mathrm{d}x}{\int_{\mathbb{R}^3} u^2 \, \mathrm{d}x},$$

where $\mathcal{V}_n$ denotes the family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^3)$. Denote $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Following [15], λ_∞ is the bottom of the essential spectrum of $-\Delta + V$ if it is finite and for every $n \in \mathbb{N}$ the inequality $\lambda_n < \lambda_\infty$ implies that λ_n is an eigenvalue of $-\Delta + V$ of finite multiplicity. Throughout this paper, we assume that there exists $k \ge 1$ such that

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$$\lambda_k < 0 < \lambda_{k+1}. \tag{1.2}$$

Problem (1.1) and a similar problem have been widely studied by many researchers in recent years. There are many results on the existence, nonexistence or multiplicity of solutions for problem (1.1) with positive potential $V(x)$. For example, when $V(x) \equiv 1$, $f(x,u) = a(x)|u|^{p-1}u$, Cerami and Vaira [2] employed the Nehari manifold and the linking theorem to prove that the problem had a positive solution. Huang et al. [9] considered the problem (1.1) with $V(x) \equiv 1$ and $f(x,u) = k(x)|u|^{p-2}u + \mu h(x)u$. By the mountain-pass theorem, they verified that the problem has two positive solutions, which improves the results of [1]. When $f(x,u) = u^p$, Ruiz [16] get both existence and nonexistence of a solution to problem (1.1). When $V(x) = (1+\mu g(x))$, $f(x,u) = |u|^{p-1}u$, Jiang and Zhou [10] proved that the problem has a ground sated solution by combining a priori estimates and approximation methods. Chen [4] get the multiple positive solutions to problem (1.1) by the variational method. When $V(x) \not\equiv 1$, but $\inf_{\mathbb{R}^3} V(x) > 0$, there are also many results (see, e.g., [18,17,5,21]). For other interesting results on the Schrödinger–Poisson system, we refer readers to [12,19,20,11,7,8,13] and references therein.

In a very recent paper [6], Chen and Zhang studied the existence of nontrivial solutions to problem (1.1) under condition (V) and the following assumptions imposed on the nonlinearity f.

$(f1)$ $f \in C^1(\mathbb{R}^3 \times \mathbb{R})$ and there exist $p \in (2,6)$ and $c_1 > 0$ such that

$$|f(x,t)| \le c_1(1+|t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{1.3}$$

$(f2)$ $f(x,t) = o(t)$ as $t \to 0$ uniformly in $x \in \mathbb{R}^3$.
$(f3)$ $\limsup_{|t|\to\infty} \frac{F(x,t)}{t^4} \le 0$ uniformly in $x \in \mathbb{R}^3$.
$(f4)$ There exists $0 < h < \lambda_\infty$ such that

$$4F(x,t) \le tf(x,t) + ht^2, \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.4}$$

where and in the sequel $F(x,u) = \int_{\mathbb{R}^3} f(x,s)\,ds$.

By using a finite-dimensional approximation method, the authors obtained the following result.

Theorem 1.1. *(See [6].) Suppose that (V) and $(f1)$–$(f4)$ are satisfied, then the problem (1.1) has a nontrivial solution.*

Inspired by [6], in the present paper, we revisit the existence of nontrivial solutions to problem (1.1) under conditions (V), $(f1)$, $(f2)$ and the following condition:

$(f5)$ $\limsup_{|t|\to\infty} \frac{F(x,t)}{t^4} < 0$ uniformly in $x \in \mathbb{R}^3$.

By using the Morse theory, we get two nontrivial solutions. More precisely, the following theorem is our main result in this paper.

Theorem 1.2. *Suppose that (V), $(f1)$, $(f2)$ and $(f5)$ are satisfied, then the problem (1.1) has at least two nontrivial solutions.*

Remark 1.1. From (V), one can easily see that the potential $V(x)$ is allowed to be sign-changing. This makes this problem more difficult than the positive ones [2,4,18,17,9,5,21], because under our assumption on $V(x)$, the variational functional related to problem (1.1) does not satisfy the (PS)-condition in general. We delicately analyze the norm of the working space E and use the uniqueness of the limit to overcome this difficulty (see, Lemmas 3.1 and 3.2).

Remark 1.2. From [6], we know that the nonlinearity $f(u) = F'(u) = \beta \ln(1+|u|^3)$ satisfies the conditions $(f1)$, $(f2)$, and $(f5)$ if β is a sufficiently small positive number.

Remark 1.3. The condition $(f4)$ is a variant version Ambrosetti–Rabinowitz (AR for short) condition that is always assumed to prove the boundedness of the Palais–Smale ((PS) for short) sequences of the energy functional. However, in the present paper, we can verify the boundedness of (PS) sequences without the condition $(f4)$ if the limit in $(f3)$ is strictly negative, that is, condition $(f5)$ holds. Moreover, by using the Morse theory, we obtain two different nontrivial solutions.

The remainder of this paper is organized as follows. In Section 2, some important preliminaries and variational setting are presented, while the proofs of the main result are given in Section 3.

2. Preliminaries and variational setting

Throughout this paper, we denote by C, C_i positive constants that may vary from line to line and denote by $\to$ ($\rightharpoonup$) the strong (weak) convergence. E^* denotes the dual space of E.

Let

$$E = \left\{ u | u \in H^1(\mathbb{R}^N), Vu^2 \in L^1(\mathbb{R}^3) \right\}.$$

Corresponding to the eigenvalue λ_k, we let W^- and W^+ be the negative space and positive space of the quadratic form

$$\int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2]\,\mathrm{d}x.$$

From (1.2), we deduct that $E = W^- \bigoplus W^+$. For any $u, v \in E$, we define

$$(u, v) = \int_{\mathbb{R}^3} (\nabla \widehat{u}^+ \nabla \widehat{v}^+ + V(x)\widehat{u}^+ \widehat{v}^+)\,\mathrm{d}x - \int_{\mathbb{R}^3} (\nabla \widehat{u}^- \nabla \widehat{v}^- + V(x)\widehat{u}^- \widehat{v}^-)\,\mathrm{d}x,$$

where $u = \widehat{u}^+ + \widehat{u}^-$, $v = \widehat{v}^+ + \widehat{v}^-$, $\widehat{u}^+, \widehat{v}^+ \in W^+$ and $\widehat{u}^-, \widehat{v}^- \in W^-$. Then $(\cdot,\cdot)$ is an inner product in E. Therefore, E is a Hilbert space with the norm

$$\|u\| = (u, u)^{\frac{1}{2}} = \left(\|u^+\|^2 - \|u^-\|^2 \right)^{\frac{1}{2}}.$$

For $u \in H^1(\mathbb{R}^3)$, it is well known that the Poisson equations $-\Delta\phi = u^2$ has a unique solution

$$\phi(x) = \phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\mathrm{d}y, \quad \text{in } D^{1,2}(\mathbb{R}^3).$$

Now we define a functional J on E by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(x, u)\mathrm{d}x, \tag{2.1}$$

for all $u \in E$. Under the assumptions (V) and $(f1)$, J is a C^1 functional in E and for any $u, v \in E$, the derivative of it is given by

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)uv]\,\mathrm{d}x + \int_{\mathbb{R}^3} \phi_u uv\,\mathrm{d}x - \int_{\mathbb{R}^3} f(x, u)v\,\mathrm{d}x. \tag{2.2}$$

Clearly, if u is a critical point of J, then (u, ϕ_u) is the solution to problem (1.1).

Now, we collect some definitions and propositions about Morse theory, which are very useful for us to look for the critical points of functional J.

Let E be a real Banach space, $J \in C^1(E, \mathbb{R})$ and J satisfy (PS)-condition.

Definition 2.1. (See Chang [3].) Let u be an isolated critical point of J with $J(u) = c$, for $c \in \mathbb{R}$, and let U be a neighborhood of u, containing the unique critical point. We call

$$C_q(J, u) := H_q(J^c \cap U, J^c \cap U \setminus \{u\}), \quad q = 0, 1, 2, \cdots,$$

the qth critical group of J at u, where $J^c := \{u \in E : J(u) \le c\}$, $H_q(\cdot,\cdot)$ stands for the qth singular relative homology group with integer coefficients.

We say that u is a homological nontrivial critical point of J if at least one of its critical groups is nontrivial.

Let $K = \{u \in E | J'(u) = 0\}$ be the set of critical point of J and $a < \inf J(K)$. If $\#K < \infty$ then the Morse-type numbers of the pair (E, J^a) are defined by

$$M_q := M_q(E, J^a) = \sum_{u \in K} \dim C_q(J, u).$$

By the Morse theory [14,3] if $\beta_q := \dim C_q(J, \infty)$, then

$$\sum_{j=0}^{q} (-1)^{q-j} M_j \ge \sum_{j=0}^{q} (-1)^{q-j} \beta_j, \tag{2.3}$$

$$\sum_{j=0}^{\infty} (-1)^q M_q = \sum_{j=0}^{\infty} (-1)^q \beta_q. \tag{2.4}$$

The formal expression of the Morse inequality reads as

$$\sum_{j=0}^{\infty} M_q t^q = \sum_{j=0}^{\infty} \beta_q t^q + (1+t)\sum_{j=0}^{\infty} a_q t^q. \tag{2.5}$$

Remark 2.1. From (2.3), we easily deduce the inequalities $M_q \geq \beta_q$ for all $q \in \mathbb{Z}$. Thus, if $\beta_q \neq 0$ for some q, then J must have a critical point, say, w, with $C_q(J, w) \ncong 0$. If equality (2.4) does not hold, then J must have another critical point differing from the known ones and if u, v are two critical points of J and $C_q(J,u) \ncong C_q(J,v)$ for some q, then $u \neq v$.

3. Proofs of the main results

In this section, we are in the position to prove our main result. To complete the proof, we need the following lemmas.

Lemma 3.1. *Assume that $V(x)$ satisfies (V) and the conditions $(f1)$, $(f2)$ and $(f5)$ hold. Then any (PS) sequence $\{u_n\} \subset E$ is bounded in E.*

Proof. We first claim that J is coercive in E. In fact, if it is not true, there must exist a constant $C > 0$ and $\|u_n\| \to \infty$ such that $J(u_n) \leq C$ as $n \to \infty$. From $(f5)$, for any $\varepsilon > 0$, there exist $C(\varepsilon) > 0$ and $R_\varepsilon > 0$ such that

$$F(x,t) \leq -C(\varepsilon)t^4 + \varepsilon t^4, \quad \text{if } |t| > R_\varepsilon. \tag{3.1}$$

By $(f2)$, for ε, $C_1(\varepsilon)$ and R_ε in (3.1), we have

$$F(x,t) \leq C_1(\varepsilon)t^2, \quad \text{if } |t| \leq R_\varepsilon. \tag{3.2}$$

Then, (3.1) and (3.2) yields that

$$F(x,u) \leq -\left(C(\varepsilon) - \varepsilon\right)u^4 + C_1(\varepsilon)u^2, \quad \forall u \in E. \tag{3.3}$$

Now, we choose $\bar{h} \notin \{\lambda_i | 1 \leq i < +\infty\}$ such that $\bar{h} \geq 2C_1(\varepsilon)$. Let E^- be the space spanned by the eigenfunctions with corresponding eigenvalue less than $\bar{h}$. Then, it is easy to see that $\dim E^- < \infty$. Let E^+ be the orthogonal complement space of E^- in E. Then, for every $u \in E$, we have a unique decomposition $u = u^+ - u^-$ with $u^+ \in E^+$ and $u^- \in E^-$. Since $\bar{h} \notin \{\lambda_i | 1 \leq i < +\infty\}$, then there exists an equivalent norm of E, still denoted by $\|\cdot\|$, such that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x)u_n^2 \mathrm{d}x - \bar{h}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x = \|u_n^+\|^2 - \|u_n^-\|^2. \tag{3.4}$$

It follows from (2.1), (3.3) and (3.4) that

$$\begin{aligned} J(u_n) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2 - \bar{h}u_n^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x + \int_{\mathbb{R}^3}\left[\frac{1}{2}\bar{h}u_n^2 - F(x,u_n)\right]\mathrm{d}x \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}^3}\left[\frac{1}{2}\bar{h}u_n^2 - F(x,u_n)\right]\mathrm{d}x \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_{\mathbb{R}^3}\left[\frac{1}{2}\bar{h}u_n^2 - C_1(\varepsilon)u_n^2 + (C(\varepsilon)-\varepsilon)u_n^4\right]\mathrm{d}x \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2. \end{aligned} \tag{3.5}$$

Let $w_n = \frac{u_n}{\|u_n\|}$. Then it follows from (3.5), $\|u_n\| \to \infty$ and $J(u_n) \leq C$ that

$$\|w_n^+\|^2 \leq \|w_n^-\|^2 + o(1). \tag{3.6}$$

Since $\|w_n\| = 1$, then passing to a subsequence, we assume that $w_n \rightharpoonup w$ in E and $w_n \to w$ a.e. in $\mathbb{R}^3$. Now, we show that $w \neq 0$. Arguing indirectly, assume that $w = 0$. Then we deduce from the finite dimension of E^- that $w_n^- \to 0$ in E. This together with (3.6) yields $w_n \to 0$ in E. Obviously, this is a contradiction with $\|w_n\| = 1$ for every n. Hence, $w \neq 0$. By the Fatou lemma, we have

$$\liminf_{n\to\infty} \|u_n\|^{-4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x \geq \int_{\mathbb{R}^3} \phi_w w^2 \mathrm{d}x > 0. \tag{3.7}$$

Noting that the assumption $J(u_n) \le C$, then $\|u_n\|^{-4} J(u_n) \to 0$. Multiplying both sides of the following inequality by $\|u_n\|^{-4}$ and letting $n \to \infty$,

$$\begin{aligned} J(u_n) &\ge \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx + \int_{\mathbb{R}^3} \left[\frac{1}{2}\bar{h} u_n^2 - C_1(\varepsilon)u_n^2 + (C(\varepsilon)-\varepsilon)u_n^4\right] dx \\ &\ge \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx, \end{aligned}$$

we have $0 \ge \frac{1}{4}\int_{\mathbb{R}^3} \phi_w w^2 dx > 0$, which is a contradiction. Therefore, J is coercive in E.

Let $\{u_n\} \subset E$ be a (PS) sequence, i.e., $J(u_n) \to c$ and $J'(u_n) \to 0$ in E^*, as $n \to \infty$. Then $J(u_n) \to c$ implies that $\{u_n\}$ is bounded since J is coercive. The proof is completed. □

Lemma 3.2. *Assume that conditions (V), $(f1)$, $(f2)$ and $(f5)$ hold. Then J satisfies the (PS)-condition.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. Then Lemma 3.1 shows that $\{u_n\}$ is bounded. By (2.2) and $J'(u_n) \to 0$ we have

$$\begin{aligned} o(\|u_n\|) &= \langle J'(u_n), u_n\rangle \\ &= \int_{\mathbb{R}^3} [|\nabla u_n|^2 + V(x)u_n^2]\, dx + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx - \int_{\mathbb{R}^3} f(x,u_n)u_n\, dx \\ &= \|u_n^+\|^2 - \|u_n^-\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx - \int_{\mathbb{R}^3} f(x,u_n)u_n\, dx. \end{aligned} \tag{3.8}$$

Then we deduce from (3.8) that

$$o(\|u_n\|) + \|u_n^-\|^2 = \|u_n^+\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx - \int_{\mathbb{R}^3} f(x,u_n)u_n\, dx. \tag{3.9}$$

Going if necessary to a subsequence (renamed $\{u_n\}$), we may assume $u_n \rightharpoonup u$ in E. Then u is a critical point of J. It follows that

$$0 = \langle J'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\, dx - \int_{\mathbb{R}^3} f(x,u)u\, dx,$$

which implies that

$$\|u^-\|^2 = \|u^+\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\, dx - \int_{\mathbb{R}^3} f(x,u)u\, dx. \tag{3.10}$$

Since E^- is a finite dimensional subspace of E, we get $u_n^- \to u^-$, and then $\|u_n^-\|^2 \to \|u^-\|^2$. This together with (3.9) and (3.10) implies

$$\lim_{n\to\infty} \left[\|u_n^+\|^2 + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx - \int_{\mathbb{R}^3} f(x,u_n)u_n\, dx\right] = \|u^+\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\, dx - \int_{\mathbb{R}^3} f(x,u)u\, dx. \tag{3.11}$$

Furthermore, by the Fatou lemma, we have

$$\begin{aligned} \liminf_{n\to\infty}\left[\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} f(x,u_n)u_n dx\right] &\ge \liminf_{n\to\infty}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \limsup_{n\to\infty}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \\ &\ge \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} f(x,u)u\, dx. \end{aligned} \tag{3.12}$$

Followed by (3.11) and (3.12), we have $\lim_{n\to\infty} \|u_n^+\|^2 = \|u^+\|^2$. It follows that $u_n \to u$ in E. The proof is completed. □

Proof of Theorem 1.2. By Lemma 3.2, J satisfies (PS)-condition. Now, we show that J is weakly sequentially lower semi-continuous. By Lemma 3.1, any (PS) consequence $\{u_n\} \subset E$ is bounded in E. Thus, we can assume that $u_n \rightharpoonup u$ in E. Then, we have $\liminf_{n\to\infty}\|u_n^+\| \ge \|u^+\|$. Since W^- is a finite dimensional space of E, then $u_n^- \to u^-$ in E. So, it follows from the Fatou lemma that

$$\begin{aligned}\liminf_{n\to\infty} J(u_n) &= \liminf_{n\to\infty}\left[\frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u_n)\mathrm{d}x\right]\\ &\ge \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(x,u)\mathrm{d}x\\ &= J(u),\end{aligned}$$

which shows that J is weakly sequentially lower semi-continuous in E. Therefore, we get that there exists $u^* \in E$, such that

$$J(u^*) = \inf_E J(u),$$

which means that J has global minimizer u^*. And then by Theorem 4.6 in Chapter I of [3], we have

$$C_q(J,u^*) = \delta_{q,0}\mathbb{Z}. \tag{3.13}$$

Since f is C^1 function, it follows from (1.2) and $(f2)$ that 0 is a non-degenerate critical point of J. By the implicit function theorem, we imply that 0 is an isolated critical point of J with Morse index μ_0 and nullity ν_0. So, by Theorem 4.1 in Chapter I of [3], we have

$$C_q(J,0) = \delta_{q,\mu_0}\mathbb{Z}. \tag{3.14}$$

From (3.13), (3.14) and Remark 2.1, we imply that $u^* \neq 0$. If J does not have any other critical points except for 0 and u^*, i.e., $K = \{0, u^*\}$, then equality (2.5) reads as

$$(-1)^{\mu_0+\nu_0} + (-1)^1 = (-1)^1. \tag{3.15}$$

Noticing that $\mu_0 + \nu_0 \neq 0$. So, (3.15) can not hold. Therefore, J has at least two nontrivial critical points. The proof is completed. □

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