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Functional analysis

KK-theory of A-valued semi-circular systems

KK-théorie des systèmes semi-circulaires A-valués

Emmanuel Germain [a], Pierre Umber [b]

[a] *LMNO UMR 6139, Université de Caen et CNRS, France*
[b] *ENS Lyon, France*

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ABSTRACT

We compute in this article the KK-theory of A-valued semi-circular systems thanks to tools developed by Pimsner (see [1]) to study generalized Toeplitz algebras.

RÉSUMÉ

On calcule dans cet article la KK-théorie de systèmes semi-circulaires A-valués à l'aide d'outils développés par Pimsner (voir [1]) pour étudier les algèbres de Toeplitz généralisées.

To begin with, we will need a result in Hilbert module theory.

Proposition 0.1. *Let A, C be C^*-algebras, B a sub-C^*-algebra of C, E a Hilbert module over A, and $\phi : A \to B$ a *-morphism. Let $j : B \to C$ be the inclusion, $\psi \stackrel{\text{def}}{=} j \circ \phi$ and $E_0 \stackrel{\text{def}}{=} \overline{\{\sum x_i \otimes b_i, x_i \in E, b_i \in B\}} \subset E \otimes_\psi C$. Then E_0 is naturally endowed with a structure of Hilbert module over B and $E_0 \simeq E \otimes_\phi B$.*

Indeed, let $x_i, x'_k \in E$, $b_i, b'_k \in B$. We have:

$$< \sum x_i \otimes b_i, \sum x'_k \otimes b'_k > = \sum_{i,k} b_i^* \phi(< x_i, x'_k >) b'_k \in B.$$

As B is closed in C, we have: $\forall x, y \in E_0, < x, y > \in B$. As a result, E_0 is naturally endowed with a structure of pre-Hilbert module over B, which is complete because E_0 is a closed subspace of the Hilbert module $E \otimes_\psi C$.

For the second part of the proposition, let $\pi : (x, b) \in E \times B \mapsto x \otimes b \in E_0$. If $a \in E$, then $\pi(x \cdot a, b) = x \cdot a \otimes b = x \otimes j \circ \phi(a) b = x \otimes \phi(a) b = \pi(x, a \cdot b)$. Then π induces $\tilde{\pi} : E \otimes_{alg} B \to E_0$. We clearly have:

$$< \tilde{\pi}(\sum x_i \otimes b_i), \tilde{\pi}(\sum x_i \otimes b_i) > = < \sum x_i \otimes b_i, \sum x_i \otimes b_i >,$$

E-mail addresses: emmanuel.germain@unicaen.fr (E. Germain), pierre.umber@ens-lyon.fr (P. Umber).

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so $\tilde{\pi}$ is an isometry. As E_0 is complete, $\tilde{\pi}$ extends to an isometry $\widehat{\pi}$ on $E \otimes_\phi B$. As $\widehat{\pi}$ is an isometry, $Im(\widehat{\pi})$ is closed in E_0, but $Im(\widehat{\pi})$ contains a dense subspace of E_0, so $\widehat{\pi}$ is an isomorphism and $E_0 \simeq E \otimes_\phi B$.

Let's turn now to our main result. Let A be a C^*-algebra with unit, and E be a Hilbert module over A with an isometric *-morphism $\phi : A \to L_A(E)$ which endowed E with a left action. The algebra A is supposed to be separable and E countably generated. We will denote by $\mathcal{F}(E)$ the Fock space associated to E, which is $\mathcal{F}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$ (where $E^{\otimes 0} = A$). Each $E^{\otimes n}$ is a left A-module, thus $\mathcal{F}(E)$ is endowed with a diagonal left action over A.

Let $\xi \in E$ and T_ξ be the left creation operator $\eta \mapsto \xi \otimes \eta$. Then $T_\xi \in L_A(\mathcal{F}(E))$ and the annihilation operator is given by $T_\xi^*(\eta_1 \otimes \ldots \otimes \eta_n) = <\xi, \eta_1> \eta_2 \otimes \ldots \otimes \eta_n$.

We denote by $\mathcal{T}_E$ the associated Toeplitz algebra, which is the C^*-algebra generated by A and the operators T_ξ.

If E is also endowed with an anti-linear involution $\xi \mapsto \xi^*$ then there is a natural *-subalgebra of $\mathcal{T}_E$, that we denote by $\mathcal{S}_E$, and is generated by A and elements $T_\xi + T_{\xi^*}^*$. This algebra is mainly studied in a Von Neumann algebra context (see for example [2] and [3]). We will here compute its KK-theory as a particular case of the following theorem.

Theorem 0.2. *Let S be any sub-C^*-algebra of $\mathcal{T}_E$ which contains A and is generated by linear combinations of creation and annihilation operators. Then S is KK-equivalent to A*

According to Pimsner (see Proposition 3.3 in [1]), Toeplitz algebras satisfy the following universal property:

Proposition 0.3. *Let B be a C^*-algebra and $\sigma : A \to B$ a *-morphism. We suppose that there is a family $(t_\xi)_{\xi \in E}$ in B such that:*

1) $\xi \mapsto t_\xi$ *is* $\mathbb{C}$*-linear*
2) $t_\xi \sigma(a) = t_{\xi a}$ *and* $\sigma(a) t_\xi = t_{\phi(a)\xi}$
3) $t_\xi^* t_\zeta = \sigma(<\xi, \zeta>)$

Then σ extends to a unique morphism on $\mathcal{T}_E$ such that $\sigma(T_\xi) = t_\xi$.

We denote by i_A the inclusion of A in S, i_S the inclusion of S in $\mathcal{T}_E$ and $j \overset{\text{def}}{=} i_S \circ i_A$. Let P be the projection in $\mathcal{F}(E)$ onto $E^{\otimes 0} = A$ and $Q \overset{\text{def}}{=} 1 - P$. Let $\pi_0 : A \to L_A(\mathcal{F}(E))$ given by the diagonal left action of $\mathcal{F}(E)$, and $\tilde{\pi}_1 \overset{\text{def}}{=} Q\pi_0 = \pi_0 Q$. We also define $\tilde{T}_\xi \overset{\text{def}}{=} Q T_\xi Q$. Then $(\tilde{\pi}_1, \tilde{T}_\xi)$ satisfies conditions in Proposition 0.3, so $\tilde{\pi}_1$ extends to a representation π_1 of $\mathcal{T}_E$.

Let $\beta \overset{\text{def}}{=} (\mathcal{F}(E) \oplus \mathcal{F}(E), (\pi_0, \pi_1), F)$ where $F : \mathcal{F}(E) \oplus \mathcal{F}(E) \to \mathcal{F}(E) \oplus \mathcal{F}(E)$ is defined by $F(\xi \oplus \zeta) = \zeta \oplus \xi$. Then β is an element of $KK(\mathcal{T}_E, A)$ (see Lemma 4.2 and Definition 4.3 in [1]).

We have the relations $j \otimes_{\mathcal{T}_E} \beta = 1_A$ and $\beta \otimes_A j = 1_{\mathcal{T}_E}$, where 1_C, for every C^*-algebra C, is the multiplicative unit in the ring $KK(C, C)$ (see Theorem 4.4 in [1]). We consider $\alpha \overset{\text{def}}{=} i_S \otimes_{\mathcal{T}_E} \beta \in KK(S, A)$.

Proposition 0.4. *We have the relations $i_A \otimes_S \alpha = 1_A$ and $\alpha \otimes_A i_A = 1_S$.*

Indeed, for the first one we have $i_A \otimes_S \alpha = i_A \otimes_S i_S \otimes_{\mathcal{T}_E} \beta = j \otimes_{\mathcal{T}_E} \beta = 1_A$.

For the second one, we first recall all the tools which are introduced in Pimsner's article in the proof of Theorem 4.4. Let $\tau_1 : \mathcal{T}_E \to L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$ be the operator such that, for $T \in \mathcal{T}_E$, $\tau_1(T)$ acts on $A \otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$ by $\tau_1(T)(S) \overset{\text{def}}{=} TS$ and is equal to zero on $\bigoplus_{n \geq 1} E^{\otimes n} \otimes_j \mathcal{T}_E$ (note that τ_1 is a *-morphism). Let $\tau_0 : \mathcal{T}_E \to L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$ be the operator such that, for $T_\xi \in \mathcal{T}_E$, $\tau_0(T_\xi)$ acts on $A \otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$ by $\tau_1(T_\xi)(S) \overset{\text{def}}{=} \xi \otimes S$ and is equal to zero on $\bigoplus_{n \geq 1} E^{\otimes n} \otimes_j \mathcal{T}_E$. Note that $(\tau_0(T_\xi))^*(\eta \otimes S) = <\xi, \eta> S$ on $E \otimes_j \mathcal{T}_E$ and is equal to zero on $A \otimes_j \mathcal{T}_E$ and $\bigoplus_{n \geq 2} E^{\otimes n} \otimes_j \mathcal{T}_E$.

Lemma 0.5. *Consider $T_\xi \in \mathcal{T}_E$ and $t \in [0, 1]$. We define*

$$\tilde{T}_{\xi,t} \overset{\text{def}}{=} \cos(\frac{\pi}{2}t)\tau_0(T_\xi) + \sin(\frac{\pi}{2}t)\tau_1(T_\xi) + \pi_1(T_\xi) \otimes 1_{\mathcal{T}_E}.$$

The couple $(\pi_0 \otimes 1_{\mathcal{T}_E}, \tilde{T}_{\xi,t})$ satisfies the conditions in Proposition 0.3, and thus $\pi_0 \otimes 1_{\mathcal{T}_E}$ extends to a representation $\pi_t : \mathcal{T}_E \to L_{\mathcal{T}_E}(\mathcal{F}(E) \otimes_j \mathcal{T}_E)$.

Conditions 1) and 2) are easy to check.

As regards condition 3), we have: $\tilde{T}_{\xi,t}^* \tilde{T}_{\zeta,t} = I + J + K$ where

$$I = \cos^2(\frac{\pi}{2}t)(\tau_0(T_\xi))^*\tau_0(T_\zeta) + \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)(\tau_0(T_\xi))^*\tau_1(T_\zeta) + \cos(\frac{\pi}{2}t)(\tau_0(T_\xi))^*\pi_1(T_\zeta)\otimes 1_{\mathcal{T}_E},$$

$$J = \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t)(\tau_1(T_\xi))^*\tau_0(T_\zeta) + \sin^2(\frac{\pi}{2}t)(\tau_1(T_\xi))^*\tau_1(T_\zeta) + \sin(\frac{\pi}{2}t)(\tau_1(T_\xi))^*\pi_1(T_\zeta)\otimes 1_{\mathcal{T}_E},$$

$$K = \cos(\frac{\pi}{2}t)((\pi_1(T_\xi))^*\otimes 1_{\mathcal{T}_E})\tau_0(T_\zeta) + \sin(\frac{\pi}{2}t)((\pi_1(T_\xi))^*\otimes 1_{\mathcal{T}_E})\tau_1(T_\zeta) + ((\pi_1(T_\xi))^*\pi_1(T_\zeta))\otimes 1_{\mathcal{T}_E}).$$

Then we compute each term on the subspace where it doesn't vanish. Remark that the subspaces $E^{\otimes n}\otimes_j \mathcal{T}_E$ of $\mathcal{F}(E)\otimes_j \mathcal{T}_E$ are stable for $\pi_1 \otimes 1_{\mathcal{T}_E}$. Let $T \in A\otimes_j \mathcal{T}_E \simeq \mathcal{T}_E$, $\eta \in E$. We have:

$(\tau_0(T_\xi))^*\tau_0(T_\zeta)(T) = (\tau_0(T_\xi))^*(\zeta\otimes T) = <\xi,\zeta> T;$
$(\tau_0(T_\xi))^*\tau_1(T_\zeta)(T) = 0;$
$(\tau_0(T_\xi))^*(\pi_1(T_\zeta)\otimes 1_{\mathcal{T}_E})(\eta\otimes T) = (\tau_0(T_\xi))^*(QT_\zeta Q\eta\otimes T) = 0;$
$(\tau_1(T_\xi))^*\tau_0(T_\zeta)(T) = 0;$
$(\tau_1(T_\xi))^*\tau_1(T_\zeta)(T) = T_\xi^* T_\zeta T = <\xi,\zeta> T;$
$(\tau_1(T_\xi))^*(\pi_1(T_\zeta)\otimes 1_{\mathcal{T}_E}) = 0;$
$((\pi_1(T_\xi))^*\otimes 1_{\mathcal{T}_E})\tau_0(T_\zeta)(T) = (\pi_1(T_\xi^*)\zeta)\otimes T = 0;$
$((\pi_1(T_\xi))^*\otimes 1_{\mathcal{T}_E})\tau_1(T_\zeta)(T) = (\pi_1(T_\xi^*)1)\otimes T_\zeta T = 0.$

For the last two statements, we use the fact that $\pi_1(T_\xi^*)$ vanishes on the subspaces $A = E^{\otimes 0}$ and $E = E^{\otimes 1}$ of $\mathcal{F}(E)$. As regards the last term, let $\eta \in \mathcal{F}(E)$. We have: $(\pi_1(T_\xi^*)\pi_1(T_\zeta))\eta\otimes T = <\xi,\eta>(Q\eta)\otimes T$. Finally:

$$\tilde{T}_{\xi,t}^*\tilde{T}_{\zeta,t}(\eta\otimes T) = (\cos^2(\frac{\pi}{2}t) + \sin^2(\frac{\pi}{2}t)) <\xi,\eta> (P\eta)\otimes T + <\xi,\eta>(Q\eta)\otimes T$$
$$= <\xi,\zeta> \eta\otimes T$$

so $\tilde{T}_{\xi,t}^*\tilde{T}_{\zeta,t} = (\pi_0\otimes 1_{\mathcal{T}_E})(<\xi,\zeta>)$.

We now focus on $\alpha\otimes_A i_A$. Likewise, we can define $\tau_1^S : S \to L_S(\mathcal{F}(E)\otimes_{i_A} S)$ and $\tau_0^S : S \to L_S(\mathcal{F}(E)\otimes_{i_A} S)$.

The element $\alpha\otimes_A i_A$ is given by the Kasparov module

$$(\mathcal{F}(E)\otimes_{i_A} S \oplus \mathcal{F}(E)\otimes_{i_A} S, (\pi_0\otimes 1_S\circ i_S)\oplus(\pi_1\otimes 1_S\circ i_S), F\otimes 1_S).$$

Then the element $\alpha\otimes_A i_A - 1_B$ can be represented by the Kasparov module $\gamma \overset{\text{def}}{=} ((\mathcal{F}(E)\otimes_{i_A} S)\oplus(\mathcal{F}(E)\otimes_{i_A} S), \pi_0^S\oplus\pi_1^S, F\otimes 1_S)$ where $\pi_1^S = \tau_1^S\oplus(\pi_1\otimes 1_S\circ i_S)$ and $\pi_0^S = \pi_0\otimes 1_S\circ i_S$.

We also have $\pi_0\otimes 1\circ i_S = \tau_0^S\oplus(\pi_1\otimes 1\circ i_S)$.

Lemma 0.6. *Consider the $\mathbb{C}$-subspace $\mathcal{F}(E)\otimes_{i_S} S$ of $\mathcal{F}(E)\otimes_j \mathcal{T}_E$ (see Proposition 0.1) and let $t\in[0,1]$. Then the representation π_t in Lemma 0.5 induces a representation $\pi_t^S : S \to L_S(\mathcal{F}(E)\otimes_{i_A} S)$.*

Indeed, let $g \overset{\text{def}}{=} \sum_{i=1}^n \lambda_i T_{\xi_i} + \sum_{i=1}^m \mu_i T_{\zeta_i}^*$ be a generator of the C^*-algebra S. We first show that $\pi_t(g)$ stabilizes $\mathcal{F}(E)\otimes_{i_A} S$.

Let $E_0 \overset{\text{def}}{=} \{\sum \xi_i\otimes b_i, x_i\in\mathcal{F}(E), b_i\in S\}\subset \mathcal{F}(E)\otimes_{i_A} S$.

We have $\pi_t(g) = L + M + N$ where

$$L = \sum_{i=1}^n \lambda_i\tau_1(T_{\xi_i}) + \sum_{i=1}^m \mu_i(\tau_1(T_{\zeta_i}))^*$$

$$M = \sum_{i=1}^n \lambda_i\tau_0(T_{\xi_i}) + \sum_{i=1}^m \mu_i(\tau_0(T_{\zeta_i}))^*$$

$$N = \sum_{i=1}^n \lambda_i(\pi_0(T_{\xi_i}))\otimes 1_{\mathcal{T}_E} + \sum_{i=1}^m \mu_i(\pi_0(T_{\xi_i}))^*)\otimes 1_{\mathcal{T}_E}$$

As in the proof of Lemma 0.5 we only pay attention on the subspaces where terms do not vanish. Let $b\in A\otimes_{i_A} S\simeq S$, $\eta\in E$. Then we have:

$$(\sum_{i=1}^{n}\lambda_i\tau_1(T_{\xi_i})+\sum_{i=1}^{m}\mu_i(\tau_1(T_{\zeta_i}))^*)b=(\sum_{i=1}^{n}\lambda_iT_{\xi_i}+\sum_{i=1}^{m}\mu_iT^*_{\zeta_i})b\in A\otimes_{i_A}S\backsimeq S;$$
$$(\sum_{i=1}^{n}\lambda_i\tau_0(T_{\xi_i})+\sum_{i=1}^{m}\mu_i(\tau_0(T_{\zeta_i}))^*)b=(\sum_{i=1}^{n}\lambda_i\xi_i)\otimes b\in E\otimes S;$$
$$(\sum_{i=1}^{n}\lambda_i\tau_0(T_{\xi_i})+\sum_{i=1}^{m}\mu_i(\tau_0(T_{\zeta_i}))^*)\eta\otimes b=<\sum_{i=1}^{m}\mu_i\zeta_i,\eta>b\in A\otimes_{i_A}S\backsimeq S.$$

The last term clearly stabilizes E_0. By linearity, $\pi_t(g)$ stabilizes E_0. As $\pi_t(g)$ is continuous on E_0 and $(\pi_t(g))^*=\pi_t(g^*)$, π_t induces a *-morphism π_t^S on the involutive algebra generated by g valued in $L_S(\mathcal{F}(E)\otimes_{i_A}S)$. We now have to extend π_t^S to a morphism on S. We note that $\|\pi_t^S(g)\|\le\|\pi_t(g)\|\le\|g\|$ because π_t is a *-morphism between C^*-algebras. Then π_t^S is continuous and extends to a (unique) morphism on S, still denoted by π_t^S. For $t=0$ or $t=1$, we find the same π_0^S and π_1^S introduced before.

To end the proof, we will show that the family π_t^S is a homotopy, and thus $\gamma=0$. First we have to show that, for fixed $b\in S$, $t\to\pi_t^S$ is continuous. For that, as $\|\pi_t^S(s)\|\le\|s\|$, we only have to see it on generators $g\in S$, which is obvious.

Besides, we need to show that, for $b\in S$ and $t\in[0,1]$ fixed, we have:

$$\pi_t^S(b)-\pi_0^S(b)\in\mathcal{K}_S(\mathcal{F}(E)\otimes_{i_A}S).$$

We only need to check it for $g\in S$ generator with $g\overset{\text{def}}{=}\sum_{i=1}^{n}\lambda_iT_{\xi_i}+\sum_{i=1}^{m}\mu_iT^*_{\zeta_i}$. The projection P, introduced at the beginning, is clearly a compact operator of $\mathcal{F}(E)$, so $P\otimes 1_S$ is a compact operator of $\mathcal{F}(E)\otimes_{i_A}S$. We can see that $\pi_t^S(g)-\pi_0^S(g)=U+V\in\mathcal{K}_S(\mathcal{F}(E)\otimes_{i_A}S)$ where

$$U=\sum_{i=1}^{n}\lambda_i(\pi_t^S(T_{\xi_i})-\pi_0^S(T_{\xi_i}))(P\otimes 1_S)$$
$$V=\sum_{i=1}^{m}\mu_i(P\otimes 1_S)(\pi_t^S(T^*_{\zeta_i})-\pi_0^S(T^*_{\zeta_i})).$$

Thus we have the relation $\alpha\otimes_A i_A=1_S$.

Corollary 0.7. *We have $K_0(S)=K_0(A)$. Particularly, we have:*

$$K_0(\mathcal{S}_E)=K_0(A)$$

And thus a different proof of the result of [4]:

Corollary 0.8. *Let $\varphi: f\in\mathcal{C}([0,1])\mapsto\int_0^1 f(t)\,\mathrm{d}t$. φ is a state of the C^*-algebra $\mathcal{C}([0,1])$.*

We have $K_0((\mathcal{C}([0,1]),\varphi)*_r(\mathcal{C}([0,1]),\varphi)=K_0(\mathbb{C})$.

Indeed, for $A=\mathbb{C}$ and $E=\mathbb{C}^2$, if S_1 and S_2 are the creation operators associated with the vectors $(1,0)$ and $(0,1)$, we consider $C^*(1,S_1+S_1^*,S_2+S_2^*)$. It is well known that $C^*(1,S_1+S_1^*,S_2+S_2^*)\backsimeq(\mathcal{C}([-2,2]),\psi)*_r(\mathcal{C}([-2,2]),\psi)$, where $\psi: f\in\mathcal{C}([-2,2])\mapsto\frac{1}{2\pi}\int_{-2}^{2}f(t)\sqrt{4-t^2}\,\mathrm{d}t$ (see [5]), and that there is an homeomorphism on $[-2,2]$ onto $[0,1]$ that sends the semi-circular measure to the Lebesgue one. That gives rise to an *-isomorphism:

$$(\mathcal{C}([0,1]),\varphi)*_r(\mathcal{C}([0,1]),\varphi)\backsimeq(\mathcal{C}([-2,2]),\psi)*_r(\mathcal{C}([-2,2]),\psi)$$

so $K_0((\mathcal{C}([0,1]),\varphi)*_r(\mathcal{C}([0,1]),\varphi)=K_0(C^*(1,S_1+S_1^*,S_2+S_2^*))=K_0(\mathbb{C})$.

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