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Fixed point strategies for mixed variational formulations of the stationary Boussinesq problem ☆

Stratégies de point fixe pour formulations variationnelles mixtes du problème stationnaire de Boussinesq

Eligio Colmenares [c,a], Gabriel N. Gatica [c,a], Ricardo Oyarzúa [b,c]

[a] *Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile*
[b] *GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile*
[c] *CI²MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile*

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ABSTRACT

In this paper, we report on the main results concerning the solvability analysis of two new mixed variational formulations for the stationary Boussinesq problem. More precisely, we introduce mixed-primal and fully-mixed approaches, both of them suitably augmented with Galerkin-type equations, and show that the resulting schemes can be rewritten, equivalently, as fixed-point operator equations. Then, classical arguments from linear and nonlinear functional analysis are employed to conclude that they are well-posed.

RÉSUMÉ

Dans cet article, on présente les principaux résultats concernant l'analyse de résolution de deux nouvelles formulations variationnelles mixtes pour le problème stationnaire de Boussinesq. Plus précisément, on introduit des approches mixtes-primal et entièrement mixtes, toute les deux convenablement augmentées avec des équations de type Galerkin, et l'on montre que les régimes qui en résultent peuvent être réécrits, de maniére équivalente, comme équations d'opérateur de point fixe. Ainsi, les arguments classiques de l'analyse fonctionnelle linéaires et non linéaires sont utilisés pour conclure qu'elles sont bien posées.

1. Introduction

We first let $\Omega \subseteq \mathrm{R}^n$, $n \in \{2,3\}$, be a given bounded domain with polyhedral boundary Γ, denote by $\boldsymbol{\nu}$ the outward unit normal vector on Γ, and consider a fluid occupying Ω. Throughout this work, a standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{H}^s(\Omega)$ with norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote

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E-mail addresses: eligio@ci2ma.udec.cl (E. Colmenares), ggatica@ci2ma.udec.cl (G.N. Gatica), royarzua@ubiobio.cl (R. Oyarzúa).

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the corresponding vectorial and tensorial counterparts of the generic scalar functional space M. Then, given a fluid viscosity $\mu > 0$, an external force per unit mass $\boldsymbol{g} \in \mathbf{L}^\infty(\Omega)$, a boundary velocity $\boldsymbol{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, a boundary temperature $\varphi_D \in \mathrm{H}^{1/2}(\Gamma)$, and a uniformly positive definite tensor $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ describing the thermal conductivity, the stationary Boussinesq problem reads: find the velocity $\boldsymbol{u}$, the pressure p, and the temperature φ of the fluid such that

$$\begin{aligned} -\mu\,\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p - \boldsymbol{g}\varphi \;=\; 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{u} \;=\; 0 \quad & \text{in} \quad \Omega\,, \\ -\operatorname{div}(\mathbb{K}\,\nabla\varphi) + \boldsymbol{u}\cdot\nabla\varphi \;=\; 0 \quad & \text{in} \quad \Omega\,, \\ \boldsymbol{u} = \boldsymbol{u}_D \quad \text{and} \quad \varphi = \varphi_D \quad & \text{on} \quad \Gamma\,. \end{aligned} \tag{1}$$

Note that $\boldsymbol{u}_D$ must satisfy the compatibility condition $\int_\Gamma \boldsymbol{u}_D \cdot \boldsymbol{\nu} = 0$, which comes from the incompressibility condition of the fluid. In turn, we also notice that the uniqueness of a pressure solution to (1) (see, e.g., [8]), is ensured in the space $\mathrm{L}_0^2(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q = 0 \right\}$. Next, introducing the auxiliary tensor unknown $\boldsymbol{\sigma} := \mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u}) - p\,\mathbb{I}$ in Ω, where $\mathbb{I}$ is the identity matrix of $\mathrm{R}^{n\times n}$, using the incompressibility condition to eliminate the pressure unknown by means of the formula $p = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u})$ in Ω, and denoting the deviatoric of a tensor $\boldsymbol{\tau}$ by $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbb{I}$, we arrive at the following system of equations with unknowns $\boldsymbol{u}$, $\boldsymbol{\sigma}$, and φ

$$\begin{aligned} \mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} \;=\; \boldsymbol{\sigma}^{\mathrm{d}} \quad \text{and} \quad -\,\mathbf{div}\,\boldsymbol{\sigma} - \boldsymbol{g}\varphi \;=\; 0 \quad & \text{in} \quad \Omega\,, \\ -\operatorname{div}(\mathbb{K}\,\nabla\varphi) + \boldsymbol{u}\cdot\nabla\varphi \;=\; 0 \quad & \text{in} \quad \Omega\,, \\ \boldsymbol{u} = \boldsymbol{u}_D \quad \text{and} \quad \varphi = \varphi_D \quad & \text{on} \quad \Gamma\,, \\ \int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u}) \;=\; 0\,. & \end{aligned} \tag{2}$$

In the following sections we propose and analyze two new augmented mixed variational formulations for (2). For other approaches concerning this and related problems, we refer to [1,5,7,8], and the references therein.

2. The augmented mixed-primal formulation

In this section we consider an augmented mixed approach for the equations modeling $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, whereas a primal formulation is employed to deal with the main equation modeling the temperature φ.

2.1. The continuous formulation

In what follows we make use of the decomposition (see e.g. [2,6]) $\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus \mathrm{R}\,\mathbb{I}$, where

$$\mathbb{H}(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\,\boldsymbol{\zeta} \in \mathbf{L}^2(\Omega) \right\}, \quad \text{and} \quad \mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\zeta}) = 0 \right\}.$$

In particular, $\boldsymbol{\sigma}$ in (2) can be decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c\,\mathbb{I}$, where $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div};\Omega)$, and, thanks to the last equation in (2), c is given explicitly in terms of $\boldsymbol{u}$ as $c = -\frac{1}{n\,|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{u}\otimes\boldsymbol{u})$. Then, renaming $\boldsymbol{\sigma}_0$ as $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div};\Omega)$, noting that the first and second equations of (2) remain unchanged, multiplying all the equations of (2), except the last one, by suitable test functions, integrating by parts whenever it is necessary, incorporating the Dirichlet condition for $\boldsymbol{u}$ (which is a natural boundary condition in this case), introducing $\lambda := -\mathbb{K}\,\nabla\varphi\cdot\boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ as a new unknown, imposing the Dirichlet condition for φ weakly, and denoting by $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$) and $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$), we first obtain the following set of equations:

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \mu\int_\Omega \boldsymbol{u}\cdot\mathbf{div}\,\boldsymbol{\tau} + \int_\Omega (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \;=\; \mu\,\langle\boldsymbol{\tau}\boldsymbol{\nu}, \boldsymbol{u}_D\rangle_\Gamma \qquad & \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,, \\ -\,\mu\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \mu\int_\Omega \varphi\,\boldsymbol{g}\cdot\boldsymbol{v} \;=\; 0 \qquad & \forall\,\boldsymbol{v} \in \mathbf{L}^2(\Omega)\,, \\ \int_\Omega \mathbb{K}\,\nabla\varphi\cdot\nabla\psi + \langle\lambda,\psi\rangle_\Gamma + \int_\Omega (\boldsymbol{u}\cdot\nabla\varphi)\,\psi \;=\; 0 \qquad & \forall\,\psi \in \mathrm{H}^1(\Omega)\,, \\ \langle\xi,\varphi\rangle_\Gamma \;=\; \langle\xi,\varphi_D\rangle_\Gamma \qquad & \forall\,\xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{3}$$

However, since the trilinear terms in the first and third equations of (3) require the unknown $\boldsymbol{u}$ to live in a smaller space than $\mathbf{L}^2(\Omega)$ (which can be seen by applying Cauchy–Schwarz and Hölder inequalities, and the continuous injection $\mathbf{i}$ of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$), we propose here to augment the foregoing formulation with the following redundant Galerkin equations

$$\begin{aligned} \kappa_1 \int_\Omega \Big(\mu\,\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} - \boldsymbol{\sigma}^{\mathrm{d}}\Big) : \nabla\boldsymbol{v} &= 0 \qquad \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,,\\ \kappa_2 \int_\Omega \mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau} + \kappa_2\int_\Omega \varphi\,\mathbf{g}\cdot\mathbf{div}\,\boldsymbol{\tau} &= 0 \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div};\Omega)\,,\\ \kappa_3\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{v} &= \kappa_3\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{v} \quad \forall\,\boldsymbol{v}\in\mathbf{H}^1(\Omega)\,, \end{aligned} \tag{4}$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ are parameters to be specified later. Consequently, we arrive at the following augmented mixed-primal formulation: find $(\boldsymbol{\sigma}, \boldsymbol{u}, \varphi, \lambda) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) + \mathbf{B}_{\boldsymbol{u}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) &= \big(F_\varphi + F_D\big)(\boldsymbol{\tau},\boldsymbol{v}) \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\,,\\ \mathbf{a}(\varphi,\psi) + \mathbf{b}(\psi,\lambda) &= F_{\boldsymbol{u},\varphi}(\psi) \qquad \forall\,\psi\in\mathrm{H}^1(\Omega)\,,\\ \mathbf{b}(\varphi,\xi) &= G(\xi) \qquad \forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma)\,, \end{aligned} \tag{5}$$

where the forms $\mathbf{A}$, $\mathbf{B}_{\boldsymbol{w}}$, $\mathbf{a}$, and $\mathbf{b}$ are defined, respectively, as

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) := &\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : (\boldsymbol{\tau}^{\mathrm{d}} - \kappa_1\nabla\boldsymbol{v}) + \int_\Omega (\mu\,\boldsymbol{u} + \kappa_2\,\mathbf{div}\,\boldsymbol{\sigma})\cdot\mathbf{div}\,\boldsymbol{\tau}\\ &- \mu\int_\Omega \boldsymbol{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} + \mu\,\kappa_1\int_\Omega \nabla\boldsymbol{u}:\nabla\boldsymbol{v} + \kappa_3\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{v}\,, \end{aligned} \tag{6}$$

$$\mathbf{B}_{\boldsymbol{w}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) := -\int_\Omega (\boldsymbol{u}\otimes\boldsymbol{w})^{\mathrm{d}} : \big(\kappa_1\nabla\boldsymbol{v} - \boldsymbol{\tau}^{\mathrm{d}}\big)\,, \tag{7}$$

$$\mathbf{a}(\varphi,\psi) := \int_\Omega \mathbb{K}\,\nabla\varphi\cdot\nabla\psi\,, \qquad \text{and} \qquad \mathbf{b}(\psi,\xi) := \langle\xi,\psi\rangle_\Gamma\,, \tag{8}$$

and the functionals are given by

$$F_\varphi(\boldsymbol{\tau},\boldsymbol{v}) := \int_\Omega \varphi\,\mathbf{g}\cdot\big(\mu\,\boldsymbol{v} - \kappa_2\,\mathbf{div}\,\boldsymbol{\tau}\big)\,, \qquad F_D(\boldsymbol{\tau},\boldsymbol{v})) := \kappa_3\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{v} + \mu\,\langle\boldsymbol{\tau}\boldsymbol{\nu},\boldsymbol{u}_D\rangle_\Gamma\,, \tag{9}$$

$$F_{\boldsymbol{u},\varphi}(\psi) := -\int_\Omega (\boldsymbol{u}\cdot\nabla\varphi)\,\psi\,, \qquad \text{and} \qquad G(\xi) := \langle\xi,\varphi_D\rangle_\Gamma \quad \forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma)\,, \tag{10}$$

for all $(\boldsymbol{\sigma},\boldsymbol{u}), (\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$, for all $\boldsymbol{w}\in\mathbf{H}^1(\Omega)$, for all $\varphi, \psi \in \mathrm{H}^1(\Omega)$, and for all $\xi\in\mathrm{H}^{-1/2}(\Gamma)$. Note that when the Dirichlet datum $\boldsymbol{u}_D$ vanishes, the third equation in (4) is not needed, since in this case the unknown $\boldsymbol{u}$ and the associated test function $\boldsymbol{v}$ live in $\mathbf{H}_0^1(\Omega)$.

2.2. The fixed point approach

We begin by denoting $\mathbf{H} := \mathbf{H}^1(\Omega)\times\mathrm{H}^1(\Omega)$ and defining the operator $\mathbf{S} : \mathbf{H} \longrightarrow \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ by

$$\mathbf{S}(\boldsymbol{w},\phi) := (\mathbf{S}_1(\boldsymbol{w},\phi),\mathbf{S}_2(\boldsymbol{w},\phi)) = (\boldsymbol{\sigma},\boldsymbol{u}) \quad \forall\,(\boldsymbol{w},\phi)\in\mathbf{H}\,, \tag{11}$$

where $(\boldsymbol{\sigma},\boldsymbol{u})$ is the unique solution to the problem: find $(\boldsymbol{\sigma},\boldsymbol{u})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ such that

$$\mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) + \mathbf{B}_{\boldsymbol{w}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) = \big(F_\phi + F_D\big)(\boldsymbol{\tau},\boldsymbol{v})\,, \tag{12}$$

for all $(\boldsymbol{\tau},\boldsymbol{v})\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$. The following lemma guarantees that the operator $\mathbf{S}$ is well-defined.

Lemma 2.1. *Assume that $\kappa_1 \in (0, 2\,\delta)$ with $\delta\in(0,2\,\mu)$, and $\kappa_2, \kappa_3 > 0$. Then, there exists $r_0 > 0$ such that for each $r\in(0,r_0)$, the problem (12) has a unique solution $(\boldsymbol{\sigma},\boldsymbol{u}) := \mathbf{S}(\boldsymbol{w},\phi)\in\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)$ for each $(\boldsymbol{w},\phi)\in\mathbf{H}$ such that $\|\boldsymbol{w}\|_{1,\Omega}\le r$. Moreover, there exists a constant $c_{\mathbf{S}} > 0$, independent of $(\boldsymbol{w},\phi)$, such that there holds*

$$\|\mathbf{S}(\boldsymbol{w},\phi)\| = \|(\boldsymbol{\sigma},\boldsymbol{u})\| \le c_{\mathbf{S}}\Big\{\|\mathbf{g}\|_{\infty,\Omega}\,\|\phi\|_{0,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{13}$$

Proof. See [4, Lemma 3.3]. □

Next, we introduce the operator $\widetilde{\mathbf{S}} : \mathbf{H} \longrightarrow \mathrm{H}^1(\Omega)$ defined as $\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi) := \varphi \ \forall (\boldsymbol{w}, \phi) \in \mathbf{H}$, where $\varphi \in \mathrm{H}^1(\Omega)$ is part of the unique solution to the problem: find $(\varphi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) = F_{\boldsymbol{w},\phi}(\psi) \quad \forall \psi \in \mathrm{H}^1(\Omega), \qquad \mathbf{b}(\varphi, \xi) = G(\xi) \quad \forall \xi \in \mathrm{H}^{-1/2}(\Gamma). \tag{14}$$

In this case, a straightforward application of the Babuška–Brezzi theory (cf. [6, Chapter II]) provides the well-posedness of (14).

Lemma 2.2. *For each $(\boldsymbol{w}, \phi) \in \mathbf{H} := \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$, there exists a unique pair $(\varphi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ solution to problem (14), and there holds*

$$\|\widetilde{\mathbf{S}}(\boldsymbol{w}, \phi)\| \le \|(\varphi, \lambda)\| \le c_{\widetilde{\mathbf{S}}} \left\{ \|\boldsymbol{w}\|_{1,\Omega} |\phi|_{1,\Omega} + \|\varphi_D\|_{1/2,\Gamma} \right\}, \tag{15}$$

where $c_{\widetilde{\mathbf{S}}}$ is a positive constant independent of $(\boldsymbol{w}, \phi)$.

Proof. See [4, Lemma 3.4]. □

Having established that the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are well-defined, we now introduce $\mathbf{T} : \mathbf{H} \longrightarrow \mathbf{H}$ as

$$\mathbf{T}(\boldsymbol{w}, \phi) := (\mathbf{S}_2(\boldsymbol{w}, \phi), \widetilde{\mathbf{S}}(\mathbf{S}_2(\boldsymbol{w}, \phi), \phi)) \quad \forall (\boldsymbol{w}, \phi) \in \mathbf{H},$$

and realize that (5) can be rewritten as the fixed-point problem: find $(\boldsymbol{u}, \varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\boldsymbol{u}, \varphi) = (\boldsymbol{u}, \varphi). \tag{16}$$

2.3. Solvability analysis

In this section we establish the existence of a unique fixed point of $\mathbf{T}$. We begin with the following lemma.

Lemma 2.3. *Let $r \in (0, r_0)$, and let $W_r := \left\{(\boldsymbol{w}, \phi) \in \mathbf{H} : \|(\boldsymbol{w}, \phi)\| \le r\right\}$. Assume that the data satisfy $c(r)\left\{\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\} + c_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2,\Gamma} \le r$, where $c(r) := \max\{r, 1\}\left(1 + c_{\widetilde{\mathbf{S}}} r\right) c_{\mathbf{S}}$, with $c_{\mathbf{S}}$ and $c_{\widetilde{\mathbf{S}}}$ as in (13) and (15), respectively. Then there holds $\mathbf{T}(W_r) \subseteq W_r$.*

Proof. It follows from the estimates provided by Lemmas 2.1 and 2.2. We refer to [4, Lemma 3.5] for details. □

Next, applying again the ellipticity of $\mathbf{A} + \mathbf{B}_{\boldsymbol{w}}$ and the Babuška–Brezzi theory, one can show that $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are Lipschitz-continuous with constants $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$, respectively (cf. [4, Lemmas 3.6 and 3.7]). As a consequence of the corresponding estimates, one can prove the following result.

Lemma 2.4. *Let $r \in (0, r_0)$, and let $W_r := \left\{(\boldsymbol{w}, \phi) \in \mathbf{H} : \|(\boldsymbol{w}, \phi)\| \le r\right\}$. Then, there exists $C_{\mathbf{T}} > 0$, depending on r and the constants $c_{\mathbf{S}}$, $C_{\mathbf{S}}$, and $C_{\widetilde{\mathbf{S}}}$, such that*

$$\|\mathbf{T}(\boldsymbol{w}, \phi) - \mathbf{T}(\widetilde{\boldsymbol{w}}, \widetilde{\phi})\| \le C_{\mathbf{T}} \left\{\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\} \|(\boldsymbol{w}, \phi) - (\widetilde{\boldsymbol{w}}, \tilde{\phi})\| \quad \forall (\boldsymbol{w}, \phi), (\widetilde{\boldsymbol{w}}, \widetilde{\phi}) \in W_r.$$

Proof. See [4, Lemma 3.8]. □

Hence, our main result concerning the solvability of (5) (equivalently (16)) is established as follows.

Theorem 2.5. *Let $\kappa_1 \in (0, 2\delta)$, with $\delta \in (0, 2\mu)$, and $\kappa_2, \kappa_3 > 0$, and given $r \in (0, r_0)$, define the ball $W_r := \left\{(\boldsymbol{w}, \phi) \in \mathbf{H} : \|(\boldsymbol{w}, \phi)\| \le r\right\}$. Assume that the data satisfy $c(r)\left\{\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\} + c_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2,\Gamma} \le r$ and $C_{\mathbf{T}}\left\{\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\} < 1$. Then, problem (5) has a unique solution $(\boldsymbol{\sigma}, \boldsymbol{u}, \varphi, \lambda) \in \mathbb{H}_0(\mathbf{div}; \Omega) \times \mathbf{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^{1/2}(\Gamma)$, with $(\boldsymbol{u}, \varphi) \in W_r$. Moreover, there hold*

$$\|(\boldsymbol{\sigma}, \boldsymbol{u})\| \le c_{\mathbf{S}} \left\{ r\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\} \quad \textit{and} \quad \|(\varphi, \lambda)\| \le c_{\widetilde{\mathbf{S}}} \left\{ r\|\boldsymbol{u}\|_{1,\Omega} + \|\varphi_D\|_{1/2,\Gamma}\right\}.$$

Proof. It follows from Lemmas 2.3 and 2.4, and from a straightforward application of the Banach fixed-point Theorem. □

For further details on the present augmented mixed-primal formulation of the Boussinesq problem, including Galerkin approximations, a priori error analysis, and corresponding numerical experiments, we refer the reader to [4].

3. The augmented fully-mixed formulation

In this section, we apply an augmented mixed approach, not only to the equations modeling $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, but also to those modeling the temperature φ.

3.1. The continuous formulation

The use of a mixed formulation for the heat equation means that we now introduce the auxiliary unknown $\mathbf{p} := \mathbb{K}\nabla\varphi - \varphi\boldsymbol{u}$ in Ω, so that, thanks to the incompressibility condition, and integrating by parts the equation $\mathbb{K}^{-1}\mathbf{p} - \nabla\varphi + \mathbb{K}^{-1}\varphi\boldsymbol{u} = 0$ in Ω, we arrive now at the following system of equations

$$\begin{aligned} \mu\nabla\boldsymbol{u} - (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{d}} &= \boldsymbol{\sigma}^{\mathrm{d}} \quad \text{and} \quad -\,\mathbf{div}\,\boldsymbol{\sigma} - \boldsymbol{g}\varphi = 0 \quad \text{in} \quad \Omega, \\ \mathbb{K}^{-1}\mathbf{p} - \nabla\varphi + \mathbb{K}^{-1}\varphi\,\boldsymbol{u} &= 0 \quad \text{and} \quad \operatorname{div}\mathbf{p} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{u}_D \quad \text{and} \quad \varphi = \varphi_D \quad \text{on} \quad \Gamma, \\ \int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + \boldsymbol{u}\otimes\boldsymbol{u}) &= 0. \end{aligned} \tag{17}$$

It is important to remark here that, similarly as for $\boldsymbol{u}$ in Section 2.1, and as it is usual for mixed variational formulations, the Dirichlet boundary condition for φ on Γ becomes also natural in this case, and hence there is no need for introducing the Lagrange multiplier given by the boundary unknown $\lambda \in \mathrm{H}^{-1/2}(\Gamma)$, as we did in that section. Furthermore, and because of similar reasons to those mentioned there, we additionally need to augment the formulation that arises from (17) with the following redundant Galerkin equations

$$\begin{aligned} \kappa_4\int_\Omega \Big(\mathbb{K}^{-1}\mathbf{p} - \nabla\varphi + \mathbb{K}^{-1}\varphi\,\boldsymbol{u}\Big)\cdot\nabla\psi &= 0 \qquad \forall\,\psi \in \mathrm{H}^1(\Omega), \\ \kappa_5\int_\Omega \operatorname{div}\mathbf{p}\operatorname{div}\mathbf{q} &= 0 \qquad \forall\,\mathbf{q} \in \mathbf{H}(\mathrm{div};\Omega), \\ \kappa_6\int_\Gamma \varphi\,\psi &= \kappa_6\int_\Gamma \varphi_D\,\psi \quad \forall\,\psi \in \mathrm{H}^1(\Omega), \end{aligned} \tag{18}$$

where $\kappa_4,\ \kappa_5,\ \kappa_6 > 0$ are parameters to be specified later. In this way, we obtain the following augmented fully-mixed formulation for the stationary Boussinesq problem: find $(\boldsymbol{\sigma},\boldsymbol{u},\mathbf{p},\varphi) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}(\mathrm{div};\Omega)\times\mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathbf{A}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) + \mathbf{B}_{\boldsymbol{u}}((\boldsymbol{\sigma},\boldsymbol{u}),(\boldsymbol{\tau},\boldsymbol{v})) &= \big(F_\varphi + F_D\big)(\boldsymbol{\tau},\boldsymbol{v}) \quad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega), \\ \widehat{\mathbf{A}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) + \widehat{\mathbf{B}}_{\boldsymbol{u}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) &= \widehat{F}_D(\mathbf{q},\psi) \qquad \forall(\mathbf{q},\psi) \in \mathbf{H}(\mathrm{div};\Omega)\times\mathrm{H}^1(\Omega), \end{aligned} \tag{19}$$

where $\mathbf{A}$, $\mathbf{B}_{\boldsymbol{w}}$, F_φ, and F_D are given by (6), (7), and (9), respectively, and $\widehat{\mathbf{A}}$, $\widehat{\mathbf{B}}_{\boldsymbol{w}}$, and $\widehat{F}_D$ are defined as

$$\widehat{\mathbf{A}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) := \int_\Omega \mathbb{K}^{-1}\mathbf{p}\cdot(\mathbf{q} - \kappa_4\nabla\psi) + \int_\Omega (\varphi + \kappa_5\operatorname{div}\mathbf{p})\operatorname{div}\mathbf{q} - \int_\Omega \psi\operatorname{div}\mathbf{p} + \kappa_4\int_\Omega \nabla\varphi\cdot\nabla\psi + \kappa_6\int_\Gamma \varphi\,\psi,$$

$$\widehat{\mathbf{B}}_{\boldsymbol{w}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) := \int_\Omega \mathbb{K}^{-1}\varphi\,\boldsymbol{w}\cdot(\mathbf{q} - \kappa_4\nabla\psi) \qquad \text{and} \qquad \widehat{F}_D(\mathbf{q},\psi) := \kappa_6\int_\Gamma \varphi_D\,\psi + \langle \mathbf{q}\cdot\boldsymbol{\nu},\varphi_D\rangle,$$

for all $(\mathbf{p},\varphi)$, $(\mathbf{q},\psi) \in \mathbf{H}(\mathrm{div},\Omega)\times\mathrm{H}^1(\Omega)$, and for all $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$. Analogously as observed in Section 2.1, we notice that when the Dirichlet datum φ_D vanishes, the third equation in (18) is not needed, since in this case the unknown φ and the associated test function ψ live in $\mathrm{H}^1_0(\Omega)$.

3.2. The fixed-point approach

We now proceed analogously to Section 2.2, and apply a fixed-point strategy to analyze the solvability of (19). Indeed, since the first equations of (5) and (19) are exactly the same, we recall that $\mathbf{H} := \mathbf{H}^1(\Omega)\times\mathrm{H}^1(\Omega)$, and consider again the operator $\mathbf{S} : \mathbf{H} \longrightarrow \mathbb{H}_0(\mathbf{div},\Omega)\times\mathbf{H}^1(\Omega)$ defined by (11). Similarly, we introduce the operator $\widehat{\mathbf{S}} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}(\mathrm{div};\Omega)\times\mathrm{H}^1(\Omega)$ defined as $\widehat{\mathbf{S}}(\boldsymbol{w}) := (\widehat{\mathbf{S}}_1(\boldsymbol{w}),\widehat{\mathbf{S}}_2(\boldsymbol{w})) = (\mathbf{p},\varphi)\ \forall\,\boldsymbol{w} \in \mathbf{H}^1(\Omega)$, where $(\mathbf{p},\varphi)$ is the unique solution to the problem: find $(\mathbf{p},\varphi) \in \mathbf{H}(\mathrm{div};\Omega)\times\mathrm{H}^1(\Omega)$ such that

$$\widehat{\mathbf{A}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) + \widehat{\mathbf{B}}_{\boldsymbol{w}}((\mathbf{p},\varphi),(\mathbf{q},\psi)) = \widehat{F}_D(\mathbf{q},\psi) \quad \forall\,(\mathbf{q},\psi) \in \mathbf{H}(\mathrm{div};\Omega)\times\mathrm{H}^1(\Omega). \tag{20}$$

In this way, we realize that (19) can be rewritten, equivalently, as: find $(\boldsymbol{u},\varphi) \in \mathbf{H}$ such that

$$\widehat{\mathbf{T}}(\boldsymbol{u},\varphi) = (\boldsymbol{u},\varphi), \tag{21}$$

where $\widehat{\mathbf{T}} : \mathbf{H} \longrightarrow \mathbf{H}$ is the operator defined by $\widehat{\mathbf{T}}(\boldsymbol{w},\phi) := \big(\mathbf{S}_2(\boldsymbol{w},\phi),\widehat{\mathbf{S}}_2(\mathbf{S}_2(\boldsymbol{w},\phi))\big)\ \forall\,(\boldsymbol{w},\phi) \in \mathbf{H}$.

The fact that $\mathbf{S}$ is well-defined is already established by Lemma 2.1. Similarly, applying again the Lax–Milgram Lemma, one can show that $\widehat{\mathbf{S}}$ is also well-defined. For this purpose, we now let κ_0 be a positive constant such that $\mathbb{K}^{-1}\mathbf{x}\cdot\mathbf{x} \geq \kappa_0 |\mathbf{x}|^2$ $\forall \mathbf{x} \in \mathrm{R}^n$. Then, we have the following result.

Lemma 3.1. *Assume that* $\kappa_4 \in \left(0, \frac{2\kappa_0\widehat{\delta}}{\|\mathbb{K}^{-1}\|_{\infty,\Omega}}\right)$ *with* $\widehat{\delta} \in \left(0, \frac{2}{\|\mathbb{K}^{-1}\|_{\infty,\Omega}}\right)$, *and* $\kappa_5, \kappa_6 > 0$. *Then, there exists* $\widehat{r}_0 > 0$ *such that for each* $r \in (0, \widehat{r}_0)$, *the problem* (20) *has a unique solution* $(\mathbf{p},\mathbf{q}) := \widehat{\mathbf{S}}(\boldsymbol{w}) \in \mathbf{H}(\mathrm{div};\Omega) \times \mathrm{H}^1(\Omega)$ *for each* $\boldsymbol{w} \in \mathbf{H}^1(\Omega)$ *such that* $\|\boldsymbol{w}\|_{1,\Omega} \leq r$. *Moreover, there exists* $c_{\widehat{\mathbf{S}}} > 0$, *independent of* $\boldsymbol{w}$, *such that*

$$\|\widehat{\mathbf{S}}(\boldsymbol{w})\| = \|(\mathbf{p},\varphi)\| \leq c_{\widehat{\mathbf{S}}}\,\|\varphi_D\|_{1/2,\Gamma}\,. \tag{22}$$

Proof. See [3, Lemma 3.3]. □

3.3. Solvability analysis

We begin with the following result showing that $\mathbf{T}$ maps a ball into itself.

Lemma 3.2. *Let* r *be such that* $0 < r \leq \min\{r_0, \widehat{r}_0\}$, *and let* $W_r := \left\{(\boldsymbol{w},\phi) \in \mathbf{H} : \ \|(\boldsymbol{w},\phi)\| \leq r\right\}$. *In addition, assume that the data* $\boldsymbol{g}$, $\boldsymbol{u}_D$, *and* φ_D *satisfy*

$$r\,\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \leq \frac{r}{2\,c_{\mathbf{S}}} \qquad \text{and} \qquad \|\varphi_D\|_{1/2,\Gamma} \leq \frac{r}{2\,c_{\widehat{\mathbf{S}}}}\,,$$

where $c_{\mathbf{S}}$ *and* $c_{\widehat{\mathbf{S}}}$ *are given by* (13) *and* (22), *respectively. Then* $\mathbf{T}(W_r) \subseteq W_r$.

Next, applying now the ellipticity of $\widehat{\mathbf{A}} + \widehat{\mathbf{B}}_{\boldsymbol{w}}$, one can show that $\widehat{\mathbf{S}}$ is Lipschitz-continuous with constant $C_{\widehat{\mathbf{S}}}$ (cf. [3, Lemma 3.7]), which together with the Lipschitz-continuity of $\mathbf{S}$ (cf. [4, Lemma 3.6]), imply that our fixed point operator $\widehat{\mathbf{T}}$ is Lipschitz-continuous as well, with a constant $C_{\widehat{\mathbf{T}}}$ (cf. [3, Lemma 3.8]). Consequently, we are able to provide the following main result concerning the solvability of (19) (equivalently (21)).

Theorem 3.3. *Let* $\kappa_1 \in (0, 2\,\delta)$, *with* $\delta \in (0, 2\,\mu)$, $\kappa_4 \in \left(0, \frac{2\,k_0\,\widehat{\delta}}{\|\mathbb{K}^{-1}\|_{\infty,\Omega}}\right)$, *with* $\widehat{\delta} \in \left(0, \frac{2}{\|\mathbb{K}^{-1}\|_{\infty,\Omega}}\right)$, *and* $\kappa_2, \kappa_3, \kappa_5, \kappa_6 > 0$, *and given* $0 < r \leq \min\{r_0, \widehat{r}_0\}$, *let* $W_r := \left\{(\boldsymbol{w},\phi) \in \mathbf{H} : \ \|(\boldsymbol{w},\phi)\| \leq r\right\}$. *Assume that* $\boldsymbol{g}$, $\boldsymbol{u}_D$, *and* φ_D *satisfy*

$$r\,\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma} \leq \frac{r}{2\,c_{\mathbf{S}}}, \qquad \|\varphi_D\|_{1/2,\Gamma} \leq \frac{r}{2\,c_{\widehat{\mathbf{S}}}}, \quad \text{and} \quad C_{\widehat{\mathbf{T}}}\left\{\|\boldsymbol{g}\|_{\infty,\Omega} + \frac{r}{2}\right\} < 1\,.$$

Then, the problem (5) *has a unique solution* $(\boldsymbol{\sigma}, \boldsymbol{u}, \mathbf{p}, \varphi) \in \mathbb{H}_0(\mathbf{div};\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}(\mathrm{div};\Omega) \times \mathrm{H}^1(\Omega)$, *with* $(\boldsymbol{u},\varphi) \in W_r$. *Moreover, there hold* $\|(\boldsymbol{\sigma},\boldsymbol{u})\| \leq c_{\mathbf{S}}\left\{r\,\|\boldsymbol{g}\|_{\infty,\Omega} + \|\boldsymbol{u}_D\|_{1/2,\Gamma}\right\}$ *and* $\|(\mathbf{p},\varphi)\| \leq c_{\widehat{\mathbf{S}}}\,\|\varphi_D\|_{1/2,\Gamma}$.

Proof. It follows from the previous estimates and a direct application of the Banach fixed point Theorem. We refer to [3, Theorem 3.9] for further details. □

Similarly to the case of the mixed-primal approach, we now end the paper by mentioning that further details on the present augmented fully-mixed formulation of the Boussinesq problem, including Galerkin approximations, a priori error analysis, and several numerical experiments, are available in [3].

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