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Partial differential equations

Dynamics of time elapsed inhomogeneous neuron network model

Dynamique de réseaux de neurones inhomogènes structurés en âge

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ABSTRACT

Models for neural networks have been proposed, which describe the probability to find a neuron for which time s has elapsed since the last discharge. These are written under the form of a nonlinear age-structured equation where the total network activity modulates the firing rate. Here, we consider an inhomogeneous network with variability on the refractory period. We give conditions on the connectivity, leading to total desynchronization of the network.

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RÉSUMÉ

Pour décrire l'activité de réseaux de neurones, des modèles qui représentent la probabilité qu'un neurone ait passé le temps s depuis sa dernière décharge ont été proposés. Ce sont des équations structurées en âge, non linéaires, où l'activité totale du réseau contrôle le taux de décharge. Ici, nous considérons un réseau inhomogène prenant en compte la variabilité des périodes réfractaires. Nous donnons une condition sur la connectivité qui conduit à la désynchronisation totale.

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1. Introduction

A possible mathematical description of neural networks uses the time s past since the last discharge and leads to write a nonlinear age-structured equation for the probability of neurons in state $s \geq 0$ at time t. This age-structured type description of neural network has been studied by several authors [10–13], and the question on the link of this model with finite-size models has been studied in [6,15]. Here we consider an infinite inhomogeneous network parameterized by a real number σ, each network having a specific intrinsic dynamic and refractory period. We assume that each neural network itself is governed by an elapsed time model without delay [7,8,11–13] and that all the networks are coupled via their mean

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activity. For simplicity, we also assume that the synaptic weights are all the same, of intensity J. These assumptions lead to write infinitely many coupled neural networks on the probability density $n(s,\sigma,t)$ under the form

$$\begin{cases} \partial_t n + \partial_s n + p_\sigma(s, X(t))n = 0, \\ N(\sigma,t) := n(s=0,\sigma,t) = \int_0^\infty p_\sigma(s, X(t))n(s,\sigma,t)\,\mathrm{d}s, \\ X(t) = J\int_{\mathbb{R}} N(\sigma,t)\,\mathrm{d}\sigma. \end{cases} \tag{1}$$

We complete this equation with a Cauchy data that satisfies

$$n(s,\sigma,0) = n_0(s,\sigma) \geq 0, \qquad \int_0^\infty n_0 \mathrm{d}s = g(\sigma), \quad \int_{\mathbb{R}} g\mathrm{d}\sigma = 1, \tag{2}$$

where $g(\sigma)$ represents the probability density of neural networks parameterized by σ. Notice that the intrinsic dynamic of the neurons is entirely determined by the function p_σ.

For simplicity, we assume that p_σ is a piecewise constant function as follows:

$$p_\sigma(s,x) = \begin{cases} a_\sigma, \text{for } s < s^*(\sigma,x) \\ b_\sigma, \text{for } s > s^*(\sigma,x), \end{cases} \tag{3}$$

with $a_\sigma < b_\sigma$ and $0 < a := \min_\sigma a_\sigma$, $b := \max_\sigma b_\sigma < \infty$ and s^* a positive function satisfying $s^* \in L^\infty(\mathbb{R}; C_b^1(\mathbb{R}^+))$, and we express that the network activity increases the discharge rate with

$$\partial_x s^*(\sigma,x) \leq 0 \quad \text{for all } \sigma,\ x. \tag{4}$$

Here, $s^*(\sigma,x)$ represents the length of the refractory period and includes randomness due to external noise. Therefore, a_σ is a small rate of discharge and b_σ a large value depending on the intrinsic dynamic of the neurons under consideration. Finally, we assume that

$$0 \leq n_0(s,\sigma) \leq b_\sigma g(\sigma) \quad \text{for all } s > 0,\ \sigma \in \mathbb{R}. \tag{5}$$

With these assumptions, it is standard to prove the existence of a unique solution $n \in C(\mathbb{R}^+; L^1(\mathbb{R}))$, which satisfies for all $t > 0$, $s > 0$, $\sigma \in \mathbb{R}$,

$$0 \leq n(s,\sigma,t) \leq b_\sigma g(\sigma), \qquad \int_0^\infty n(t,s,\sigma)\,\mathrm{d}s = g(\sigma). \tag{6}$$

The aim of this note is to study the asymptotic behavior on $n(t)$ and, more precisely, to give conditions on the connectivity parameter J for desynchronization. We begin by studying the possible steady states and conclude by proving the exponential decay to the steady state for small connectivities J.

2. Steady states

For equation (1), we are going to prove that, for J small enough, there is a unique stationary state $\bar{n}_\sigma$, with activity X^*. Since, it has to satisfy

$$\partial_s \bar{n}_\sigma + p_\sigma(s, X^*)\bar{n}_\sigma = 0, \qquad \int_0^{+\infty} \bar{n}_\sigma(s)\,\mathrm{d}s = g(\sigma),$$

we obtain the semi-explicit formula $\bar{n}_\sigma(s) = \bar{n}_\sigma(0)\mathrm{e}^{-\int_0^s p_\sigma(\tau X^*)d\tau}$ with

$$\bar{n}_\sigma(0) = g(\sigma)\Big(\int_0^\infty \mathrm{e}^{-\int_0^s p_\sigma(\tau,X^*)\mathrm{d}\tau}\mathrm{d}s\Big)^{-1} = g(\sigma)\Big(\frac{1}{a_\sigma} - (\frac{1}{a_\sigma} - \frac{1}{b_\sigma})\mathrm{e}^{-a_\sigma s^*(\sigma,X^*)}\Big)^{-1}.$$

To investigate the existence and uniqueness of X^*, we use the function

$$F_J(x) = J\int_{\mathbb{R}} g(\sigma)\Big(\frac{1}{a_\sigma} - (\frac{1}{a_\sigma} - \frac{1}{b_\sigma})\mathrm{e}^{-a_\sigma s^*(\sigma,x)}\Big)^{-1}\mathrm{d}\sigma.$$

Notice that the relation for X^* satisfies

$$X^* = J \int_{\mathbb{R}} \bar{n}_\sigma(0)\,\mathrm{d}\sigma \iff X^* = F_J(X^*). \tag{7}$$

We have $F_J(0) > 0$ and $F_J(+\infty) \le J\int g(\sigma)b_\sigma \mathrm{d}\sigma < +\infty$; hence there exists at least one fixed point and thus, we obtain the existence of X^*. For uniqueness, we observe that, from (4), we can also compute

$$0 < F_J'(x) = J\int_{\mathbb{R}} g(\sigma)\frac{(\frac{a_\sigma}{b_\sigma}-1)\mathrm{e}^{-a_\sigma s^*(\sigma,x)}\partial_x s^*(\sigma,x)}{\left(\frac{1}{a_\sigma}-(\frac{1}{a_\sigma}-\frac{1}{b_\sigma})\mathrm{e}^{-a_\sigma s^*(\sigma,x)}\right)^2}\mathrm{d}\sigma.$$

From the inequality

$$\frac{1}{a_\sigma}-(\frac{1}{a_\sigma}-\frac{1}{b_\sigma})\mathrm{e}^{-a_\sigma s^*(\sigma,x)} \ge \frac{1}{b_\sigma},$$

we obtain that

$$F_J'(x) \le J\left\| \int_{-\infty}^{\infty} g(\sigma)\partial_x s^*(\sigma,x) b_\sigma(b_\sigma - a_\sigma)\mathrm{d}\sigma \right\|_\infty,$$

and we conclude the uniqueness under the (simple) condition

$$J\left\| \int_{-\infty}^{\infty} g(\sigma)\partial_x s^*(\sigma,\cdot) b_\sigma(b_\sigma - a_\sigma)\mathrm{d}\sigma \right\|_\infty < 1. \tag{8}$$

Remark 1. Obviously, if $J(b-a)$ is small enough, the assumption (8) is satisfied. It is also possible to give more general conditions for the uniqueness, but (8) turns out to be a good comprise for the study of the desynchronization rate.

3. Large time behavior

As for the uniqueness of the steady state, the desynchronization property can also be obtained when the connectivity J is small enough, since we have the following.

Theorem 3.1. *Assume (8) and that there exists $\beta > 0$ such that*

$$\beta_\sigma := a_\sigma - \frac{2Jb_\sigma \left\| \int_{-\infty}^{\infty} g(\sigma)b_\sigma(b_\sigma - a_\sigma)\partial_x s^*(\sigma,\cdot)\mathrm{d}\sigma \right\|_\infty}{1 - J\left\| \int_{-\infty}^{\infty} b_\sigma(b_\sigma - a_\sigma)g(\sigma)\partial_x s^*(\sigma,\cdot)\mathrm{d}\sigma \right\|_\infty} \ge \beta > 0.$$

Then the following exponential decay estimate holds

$$\int_{-\infty}^{\infty}\int_0^{\infty} |n - \bar{n}_\sigma|(t)\,\mathrm{d}s\,\mathrm{d}\sigma \le \mathrm{e}^{-\beta t}\int_{-\infty}^{\infty}\int_0^{\infty} |n_0 - \bar{n}_\sigma|\,\mathrm{d}s\,\mathrm{d}\sigma.$$

We believe that, for the model under consideration, a size condition on the connectivity J is necessary for desynchronization. Indeed, for the standard elapsed time model, periodic solutions (or self-sustained activity) occur for larger values of J, see [12]. This synchronization effect is desirable for several models of networks and has been widely studied, see [1,5,2–4,9,14].

Proof of Theorem 3.1. Let $\phi := n - \bar{n}_\sigma$. Since ϕ satisfies

$$\partial_t|\phi| + \partial_s|\phi| + p_\sigma(s, X^*)|\phi| \le |p_\sigma(s, X^*) - p_\sigma(s, X(t))|n,$$

we may integrate w.r.t. s and σ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\int_0^{\infty}|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma \le \int_{-\infty}^{\infty}|\phi(s=0)|\mathrm{d}\sigma + \int_{-\infty}^{\infty}\int_0^{\infty}|p_\sigma(s,X^*) - p_\sigma(s,X(t))|n\,\mathrm{d}s\,\mathrm{d}\sigma - \int_{-\infty}^{\infty}\int_0^{\infty}p_\sigma(s,X^*)|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma.$$

Since

$$\phi(s=0)=\int_0^\infty [p_\sigma(s,X(t))n-p_\sigma(s,X^*)\bar{n}_\sigma]\,\mathrm{d}s=\int_0^\infty (p_\sigma(s,X(t))-p_\sigma(s,X^*))n\,\mathrm{d}s+\int_0^\infty p_\sigma(s,X^*)\phi\,\mathrm{d}s,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^\infty\int_0^\infty |\phi|\,\mathrm{d}s\,\mathrm{d}\sigma\le 2\int_{-\infty}^\infty\int_0^\infty |p_\sigma(s,X^*)-p_\sigma(s,X(t))|n\,\mathrm{d}s\,\mathrm{d}\sigma+\int_{-\infty}^\infty\Big|\int_0^\infty p_\sigma(s,X^*)\phi\ \mathrm{d}s\Big|\mathrm{d}\sigma$$
$$-\int_{-\infty}^\infty\int_0^\infty p_\sigma(s,X^*)|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma.$$

Since $\int_0^\infty \phi\,\mathrm{d}s=0$, for all σ, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^\infty\int_0^\infty |\phi|\,\mathrm{d}s\,\mathrm{d}\sigma\le 2\int_{-\infty}^\infty\int_0^\infty |p_\sigma(s,X^*)-p_\sigma(s,X(t))|n\,\mathrm{d}s\,\mathrm{d}\sigma$$
$$+\int_{-\infty}^\infty\Big|\int_0^\infty (p_\sigma(s,X^*)-a_\sigma)\phi\mathrm{d}s\Big|\mathrm{d}\sigma-\int_{-\infty}^\infty\int_0^\infty (p_\sigma(s,X^*)-a_\sigma)|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma-\int_{-\infty}^\infty\int_0^\infty a_\sigma|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^\infty\int_0^\infty |\phi|\,\mathrm{d}s\,\mathrm{d}\sigma\le \underbrace{2\int_{-\infty}^\infty\int_0^\infty |p_\sigma(s,X^*)-p_\sigma(s,X(t))|n\,\mathrm{d}s\,\mathrm{d}\sigma}_{I}-\int_{-\infty}^\infty\int_0^\infty a_\sigma|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma. \tag{9}$$

To control the term I, we use (3) to get, with the χ the indicator function,

$$|p_\sigma(s,X^*)-p_\sigma(s,X(t))|\le (b_\sigma-a_\sigma)\chi_{s\in[\min(s^*(\sigma,X(t)),s^*(\sigma,X^*)),\max(s^*(\sigma,X(t)),s^*(\sigma,X^*))]}.$$

From this inequality, and using (6), we conclude that

$$\begin{aligned}I&\le 2\int_{-\infty}^\infty\int_0^\infty (b_\sigma-a_\sigma)n\chi_{s\in[\min(s^*(\sigma,X(t)),s^*(\sigma,X^*)),\max(s^*(\sigma,X(t)),s^*(\sigma,X^*))]}\,\mathrm{d}s\,\mathrm{d}\sigma\\&\le 2\int_{-\infty}^\infty b_\sigma g(\sigma)(b_\sigma-a_\sigma)|s^*(\sigma,X(t))-s^*(\sigma,X^*)|\,\mathrm{d}\sigma\\&\le 2\Big\|\int_{-\infty}^\infty g(\sigma)b_\sigma(b_\sigma-a_\sigma)\partial_x s^*(\sigma,\cdot)\,\mathrm{d}\sigma\Big\|_\infty|X(t)-X^*|.\end{aligned}$$

The last inequality follows from the sign condition on $\partial_x s^*(\sigma,\cdot)$ in assumption (4). Similarly, we can upper bound the activities difference as

$$\begin{aligned}|X(t)-X^*|&\le J\int_{-\infty}^\infty\int_0^\infty p_\sigma(X(t))|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma+J\int_{-\infty}^\infty\int_0^\infty |p_\sigma(X(t))-p_\sigma(X^*)|\bar{n}_\sigma\,\mathrm{d}s\,\mathrm{d}\sigma\\&\le J\int_{-\infty}^\infty\int_0^\infty b_\sigma|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma+J\Big\|\int_{-\infty}^\infty b_\sigma(b_\sigma-a_\sigma)g(\sigma)\partial_x s^*(\sigma,\cdot)\mathrm{d}\sigma\Big\|_\infty|X(t)-X^*|.\end{aligned}$$

Thanks to the condition (8), we can conclude our estimate on I with

$$|X(t)-X^*|\le\frac{J}{1-J\Big\|\partial_x\int_{-\infty}^\infty b_\sigma(b_\sigma-a_\sigma)g(\sigma)s^*(\sigma,\cdot)\mathrm{d}\sigma\Big\|_\infty}\int_0^\infty b_\sigma|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma.$$

Including the estimate on I in (9), and recalling the definition of β_σ in Theorem 3.1, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\int_{0}^{\infty}|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma \le -\int_{-\infty}^{\infty}\int_{0}^{\infty}\beta_\sigma|\phi|\,\mathrm{d}s\,\mathrm{d}\sigma$$

which ends the proof of Theorem 3.1. □

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