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Wave fluctuations near a de Sitter brane in an anti-de Sitter universe

Propagation des ondes près d'une brane de de Sitter dans un univers anti-de Sitter

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ABSTRACT

We investigate the Klein–Gordon equation in the past causal domain of a de Sitter brane imbedded in an anti-de Sitter bulk. We solve the global mixed hyperbolic problem. We prove that any finite energy solution can be expressed as a Kaluza–Klein tower that is a superposition of free fields in the Steady State Universe, of which we study the asymptotic behaviors. We show that the leading term of a gravitational fluctuation is a massless graviton, i.e. the de Sitter brane is linearly stable.

RÉSUMÉ

Nous étudions l'équation de Klein–Gordon dans le domaine causal passé d'une brane de de Sitter incluse dans un espace anti-de Sitter. Nous résolvons le problème mixte hyperbolique et prouvons que les solutions d'énergie finie s'expriment comme superposition de champs libres dans le demi-espace de de Sitter et nous en étudions les comportements asymptotiques. Nous montrons que le terme dominant des fluctuations gravitationnelles est un graviton sans masse, ce qui traduit la stabilité linéaire de la brane.

Version française abrégée

L'univers anti-de Sitter de dimension 5 est le cadre géométrique privilégié de la cosmologie branaire [5,7–10]. Sa carte de Poincaré est la variété lorentzienne

$$\mathcal{P} := \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z, \quad ds^2_{\text{AdS}} = z^{-2}\left(dt^2 - d\mathbf{x}^2 - dz^2\right).$$

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Une brane de de Sitter dans $\mathcal{P}$ est juste la sous-variété $\mathcal{B} := \{(t, \mathbf{x}, z) \in \mathcal{P};\ z = \alpha t\}$ où $\alpha \in]-1, 0[$, qui est le bord de genre temps de $\mathcal{M} := \{(t, \mathbf{x}, z) \in \mathcal{P};\ \alpha t < z < -t\}$. On considère l'équation de Klein–Gordon de masse $M \geq 0$ dans $\mathcal{M}$:

$$\left[\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + 3z^{-1}\partial_z + M^2 z^{-2}\right] u = 0,\ \ t < 0,\ \ \alpha t < z < -t,\ \ \mathbf{x} \in \mathbb{R},$$

complétée par la condition de Robin $\partial_\nu u = c\,u$, où ν est le quadrivecteur normal à $\mathcal{B}$, $c \in \mathbb{R}$, qui s'écrit :

$$\alpha\partial_t u + \partial_z u - c(\alpha t)^{-1} u\sqrt{1-\alpha^2} = 0,\ \ t < 0,\ \ \mathbf{x} \in \mathbb{R}^3,\ \ z = \alpha t.$$

On considère aussi la condition de Dirichlet $u = 0$. Pour éviter la dépendance temporelle du domaine et de la condition au bord, on introduit les coordonnées (τ, ρ) sur $\mathcal{M}$:

$$\tau := -2^{-1}\log\left(t^2 - z^2\right) \in \mathbb{R},\ \ \rho := \log\left(\sqrt{t^2 - z^2} - t\right) - \log z \in]0, \rho_0[,\ \ \rho_0 := \log\left(1 + \sqrt{1-\alpha^2}\right) - \log(-\alpha).$$

Dans ce système, on a $\mathcal{M} = \mathbb{R}_\tau \times \mathbb{R}^3_{\mathbf{x}} \times]0, \rho_0[_\rho$, $ds^2_{\mathrm{AdS}} = \sinh^2(\rho)\left(d\tau^2 - e^{2\tau} d\mathbf{x}^2\right) - d\rho^2$, $\mathcal{B} = \mathbb{R}_\tau \times \mathbb{R}^3_{\mathbf{x}} \times \{\rho = \rho_0\}$, $ds^2_{\mathcal{B}} = \left(\alpha^{-2} - 1\right)\left(d\tau^2 - e^{2\tau} d\mathbf{x}^2\right)$, et la dynamique prend la forme :

$$\left[\partial_\tau^2 + 3\partial_\tau - e^{-2\tau}\Delta_{\mathbf{x}} - (\sinh\rho)^{-2}\partial_\rho\left[(\sinh\rho)^4\partial_\rho\right] + M^2(\sinh\rho)^2\right] u = 0,\ \ \tau \in \mathbb{R},\ \ \mathbf{x} \in \mathbb{R}^3,\ \ 0 < \rho < \rho_0,$$

$$\partial_\rho u + c\,u = 0\ (ou\ u = 0),\ \ \tau \in \mathbb{R},\ \ \mathbf{x} \in \mathbb{R}^3,\ \ \rho = \rho_0.$$

Étant données des conditions initiales spécifiées à un temps τ_* fixé, on montre que le problème mixte hyperbolique est bien posé dans le cadre fonctionnel associé à l'énergie

$$\int_{\mathbb{R}^3}\int_0^{\rho_0} (\sinh\rho)^2\,|\partial_\tau u|^2 + e^{-2\tau}(\sinh\rho)^2\,|\nabla_{\mathbf{x}} u|^2 + (\sinh\rho)^4\left|\partial_\rho u\right|^2 + M^2(\sinh\rho)^4\,|\,u\,|^2\,d\mathbf{x}\,d\rho.$$

Cette solution admet une décomposition en « tour de Kaluza–Klein » :

$$u(\tau, \mathbf{x}, \rho) = \sum_j u_{\lambda_j}(\tau, \mathbf{x}) w(\rho; \lambda_j) + \int_{\frac{3}{2}}^{\infty} u_{m^2}(\tau, \mathbf{x}) w(\rho; m^2)\,dm$$

où λ_j et m^2 décrivent respectivement le spectre ponctuel et le spectre absolument continu de l'opérateur de Sturm–Liouville $L_c := -\sinh^{-2}\rho\frac{d}{d\rho}\left(\sinh^4\rho\frac{d}{d\rho}\right) + M^2\sinh^2\rho$ sur $]0, \rho_0[$ avec la condition $w'(\rho_0) + cw(\rho_0) = 0$. De plus $L_c w(\rho; \kappa) = \kappa w(\rho; \kappa)$, et $u_\kappa(\tau, \mathbf{x})$ est une solution d'énergie finie de l'équation de Klein–Gordon de masse κ sur l'espace de de Sitter,

$$\left[\partial_\tau^2 + 3\partial_\tau - e^{-2\tau}\Delta_{\mathbf{x}} + \kappa\right] u_\kappa = 0.$$

L'étude du comportement asymptotique de u_κ montre que le graviton sans masse u_0 admet un profil asymptotique $\phi(\mathbf{x})$ quand $\tau \to \infty$, tandis que $u_\kappa \to 0$ si $\kappa > 0$. On en déduit la stabilité linéaire d'une brane de de Sitter : les fluctuations gravitationnelles $u(\tau, \mathbf{x}, \rho)$, qui sont solutions pour $M = c = 0$, sont asymptotes à un graviton sans masse $u_0(\tau, \mathbf{x})$. Le détail des démonstrations est présenté dans [4].

English version

The 5-dimensional anti-de Sitter universe plays an important role in brane cosmology and has been deeply investigated by the physicists (see, e.g., [5,7–10]). Its Poincaré chard is the Lorentzian manifold

$$\mathcal{P} := \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z,\ \ ds^2_{\mathrm{AdS}} = z^{-2}\left(dt^2 - d\mathbf{x}^2 - dz^2\right).$$

Given $\alpha \in]-1, 0[$, the de Sitter brane $\mathcal{B}_\alpha := \{(t, \mathbf{x}, z) \in \mathcal{P};\ z = \alpha t\}$ is just the time-like part of the boundary of the open set $\mathcal{M} := \{(t, \mathbf{x}, z) \in \mathcal{P};\ -\alpha t < z < -t\}$. The other part of $\partial\mathcal{M}$ is the light-like submanifold $\mathcal{N} := \{(t, \mathbf{x}, z) \in \mathcal{P};\ 0 < z = -t\}$ that is the Cauchy horizon of our setting. We consider the Klein–Gordon equation with mass $M \geq 0$ in $\mathcal{M}$:

$$\left[\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + 3z^{-1}\partial_z + M^2 z^{-2}\right] u = 0,\ \ t < 0,\ \ \alpha t < z < -t,\ \ \mathbf{x} \in \mathbb{R}, \tag{0.1}$$

with the Robin condition $\partial_\nu u = c\,u$, ν being the unit outgoing normal vector to $\mathcal{B}_\alpha$, and $c \in \mathbb{R}$:

$$\alpha\partial_t u + \partial_z u - c(\alpha t)^{-1} u\sqrt{1-\alpha^2} = 0\ \ t < 0,\ \ \mathbf{x} \in \mathbb{R}^3,\ \ z = \alpha\,t. \tag{0.2}$$

We also treat the Dirichlet condition in [4]. In this Note, we construct a rigorous mathematical framework to solve the mixed hyperbolic problem, and to perform the spectral analysis of the solution in the spirit of the articles [1,2] that were devoted to the Minkowski brane. The detail of the proofs is given in [4].

1. Klein–Gordon fields near a de Sitter brane

To avoid the time dependance of the domain and the boundary condition, we introduce new coordinates (τ,ρ) on $\mathcal{M}$:

$$\tau := -2^{-1}\log\left(t^2-z^2\right)\in\mathbb{R},\ \ \rho:=\log\left(\sqrt{t^2-z^2}-t\right)-\log z\in\,]0,\rho_0[,\ \ \rho_0:=\log\left(1+\sqrt{1-\alpha^2}\right)-\log(-\alpha).$$

In this frame, we have $\mathcal{M}=\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}}\times\,]0,\rho_0[_\rho$, $ds^2_{\mathrm{AdS}}=\sinh^2(\rho)\left(d\tau^2-e^{2\tau}d\mathbf{x}^2\right)-d\rho^2$, $\mathcal{B}_\alpha=\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}}\times\{\rho=\rho_0\}$, $ds^2_{\mathcal{B}_\alpha}=\left(\alpha^{-2}-1\right)\left(d\tau^2-e^{2\tau}d\mathbf{x}^2\right)$, and the dynamics has the form:

$$\left[\partial_\tau^2+3\partial_\tau-e^{-2\tau}\Delta_{\mathbf{x}}-(\sinh\rho)^{-2}\partial_\rho\left[(\sinh\rho)^4\partial_\rho\right]+M^2(\sinh\rho)^2\right]u=0,\ \ \tau\in\mathbb{R},\ \ \mathbf{x}\in\mathbb{R}^3,\ \ 0<\rho<\rho_0, \tag{1.1}$$

$$\partial_\rho u+c\,u=0,\ \ \tau\in\mathbb{R},\ \ \mathbf{x}\in\mathbb{R}^3,\ \ \rho=\rho_0. \tag{1.2}$$

Given a Cauchy data at time τ_*,

$$u(\tau_*,.)=u_0(.),\ \ \partial_\tau u(\tau_*,.)=u_1(.), \tag{1.3}$$

we solve the global mixed hyperbolic problem in the functional framework associated with the energy

$$\int_{\mathbb{R}^3}\int_0^{\rho_0}(\sinh\rho)^2\,|\partial_\tau u|^2+e^{-2\tau}(\sinh\rho)^2\,|\nabla_{\mathbf{x}}u|^2+(\sinh\rho)^4\left|\partial_\rho u\right|^2+M^2(\sinh\rho)^4\,|\,u\,|^2\,d\mathbf{x}\,d\rho.$$

We introduce the following Hilbert spaces endowed with the natural norms:

$$X^0:=L^2\left(\mathbb{R}^3_{\mathbf{x}}\times\,]0,\rho_0[_\rho,\sinh^2(\rho)\,d\rho\,d\mathbf{x}\right),\ \ X^1:=\left\{u\in X^0;\ \nabla_{\mathbf{x}}u,\ \sinh(\rho)\partial_\rho u\in X^0\right\},$$

and to take into account the boundary condition:

$$X^2:=\left\{u\in X^1;\ \Delta_{\mathbf{x}}u,\ (\sinh\rho)^{-2}\partial_\rho\left[(\sinh\rho)^4\partial_\rho u\right]\in X^0\right\},\ \ X^2_c:=\left\{u\in X^2;\ \partial_\rho u(\rho_0)+c\,u(\rho_0)=0\right\}.$$

We introduce also $H:=L^2\left(]0,\rho_0[,\sinh^2(\rho)d\rho\right)$ and the Sturm–Liouville operator

$$L_c:=-\sinh^{-2}\rho\frac{d}{d\rho}\left(\sinh^4\rho\frac{d}{d\rho}\right)+M^2\sinh^2\rho$$

that is self-adjoint when it is endowed with the domain $D_c:=\left\{w\in H;\ L_cu\in H,\ w'(\rho_0)+cw(\rho_0)=0\right\}$. To define the weak solutions, we denote $\mathcal{D}'\left(\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}};D_c\right)$ the space of the vector distributions on $\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}}$, which are D_c-valued, and we call *finite energy solutions* to (1.1) and (1.2), the distributions that are solutions to (1.1) and belong to $C^1\left(\mathbb{R}_\tau;X^0\right)\cap C^0\left(\mathbb{R}_\tau;X^1\right)\cap\mathcal{D}'\left(\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}};D_c\right)$. Using the famous result of T. Kato on the propagator of the time-dependent hyperbolic evolution equations, we can establish the following theorem.

Theorem 1.1. *Given $M\geq 0$, $\alpha\in]-1,0[$, $c\in\mathbb{R}$, $\tau_*\in\mathbb{R}$, $u_0\in X^1$ and $u_1\in X^0$, there exists a unique finite energy solution u of (1.1), (1.3). There exists $f\in C^0(\mathbb{R}^+;\mathbb{R}^+)$ such that for all $\tau\in\mathbb{R}$ we have*

$$\|u(\tau)\|_{X^1}+\|\partial_\tau u(\tau)\|_{X^0}\leq f(|\,d\tau-\tau_*\,|)\left(\|u_0\|_{X^1}+\|u_1\|_{X^0}\right),$$

and there exists $C\in C^0\left(\mathbb{R}_{\tau_*}\times C^\infty_0(\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}});\mathbb{R}^+\right)$ such that for all $\Theta\in C^\infty_0(\mathbb{R}_\tau\times\mathbb{R}^3_{\mathbf{x}})$,

$$\|\int\Theta(\tau,\mathbf{x})u(\tau,\mathbf{x},.)d\tau\,d\mathbf{x}\|_{D_c}\leq C(\tau_*,\Theta)\left(\|u_0\|_{X^1}+\|u_1\|_{X^0}\right).$$

Here, f and C are independent of u_0, u_1, and only depend on M, α and c.

If $u_0\in X^2_c$ and $u_1\in X^1$, then u is a strong solution, i.e. $u\in C^2\left(\mathbb{R}_\tau;X^0\right)\cap C^1\left(\mathbb{R}_\tau;X^1\right)\cap C^0\left(\mathbb{R}_\tau;X^2_c\right)$.

2. Kaluza–Klein tower

In this section, we show that any finite energy solution to (1.1) and (1.2) can be expressed as a *Kaluza–Klein tower*, which is a superposition of fields u_κ solution to the Klein–Gordon equation:

$$\left[\partial_\tau^2+3\partial_\tau-e^{-2\tau}\Delta_{\mathbf{x}}+\kappa\right]u_\kappa=0, \tag{2.1}$$

on the Steady State universe $dS_{\frac{1}{2}}$ that is isometric to the de Sitter brane $\mathcal{B}_\alpha$:

$$dS_{\frac{1}{2}} := \mathbb{R}_\tau \times \mathbb{R}^3_{\mathbf{x}},\ \ ds^2_{dS} = d\tau^2 - e^{2\tau} d\mathbf{x}^2. \tag{2.2}$$

The key of the proof is a complete spectral analysis of the operator L_c that we present now. In the sequel, $P_\nu^{-\mu}$ and $\mathbf{Q}_\nu^{\mu}$ are the associated Legendre functions of first and second kinds.

Proposition 2.1. *For all $c \in \mathbb{R}$ the spectrum of the self-adjoint operator L_c has the following form:*

$$\sigma_{ac}(L_c) = [9/4, \infty[,\ \ \sigma_{sc}(L_c) = \emptyset,$$

$\sigma_p(L_c)$ is a finite subset included in $]-\infty, 9/4[$ and when $c = M = 0$, $\sigma_p(L_0) = \{0\}$.

The eigenvalues are the solutions $\lambda < \frac{9}{4}$ of the transcendental equation

$$\left(c\sqrt{1-\alpha^2} - 2 + \sqrt{M^2+4}\right) P^{-\sqrt{\frac{9}{4}-\lambda}}_{-\frac{1}{2}+\sqrt{M^2+4}}(-\alpha^{-1}) + \alpha\left(-\frac{1}{2} + \sqrt{M^2+4} - \sqrt{\frac{9}{4}-\lambda}\right) P^{-\sqrt{\frac{9}{4}-\lambda}}_{-\frac{3}{2}+\sqrt{M^2+4}}(-\alpha^{-1}) = 0,$$

and the corresponding eigenfunctions are given by:

$$w(\rho;\lambda) = \gamma(\sinh\rho)^{-\frac{3}{2}} P^{-\sqrt{\frac{9}{4}-\lambda}}_{-\frac{1}{2}+\sqrt{M^2+4}}(\cosh\rho),\ \ \gamma \in \mathbb{C}.$$

For all $m > \frac{3}{2}$, the generalized eigenfunction, solution to $L_c w = m^2 w$, $\partial_\rho w(\rho_0) = c w(\rho_0)$, is given by:

$$\begin{aligned} w\left(\rho; m^2\right) := {} & \frac{\sqrt{2m}}{\pi}(4m^2-9)^{\frac{1}{4}}\Gamma\left(\frac{1}{2} + \sqrt{M^2+4} - i\sqrt{m^2-\frac{9}{4}}\right)(\sinh\rho)^{-\frac{3}{2}} \\ & \times \left|\left(P^{i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}\right)'(-\alpha^{-1}) - \frac{\alpha}{\sqrt{1-\alpha^2}}\left(c - \frac{3}{2\sqrt{1-\alpha^2}}\right) P^{i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}(-\alpha^{-1})\right|^{-1} \\ & \times \left\{\left[\left(P^{i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}\right)'(-\alpha^{-1}) - \frac{\alpha}{\sqrt{1-\alpha^2}}\left(c - \frac{3}{2\sqrt{1-\alpha^2}}\right) P^{i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}(-\alpha^{-1})\right]\right. \\ & \times \mathbf{Q}^{-i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}(\cosh\rho) \\ & - \left[\left(\mathbf{Q}^{-i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}\right)'(-\alpha^{-1}) - \frac{\alpha}{\sqrt{1-\alpha^2}}\left(c - \frac{3}{2\sqrt{1-\alpha^2}}\right) \mathbf{Q}^{-i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}(-\alpha^{-1})\right] \\ & \left.\times P^{i\sqrt{m^2-\frac{9}{4}}}_{-\frac{1}{2}+\sqrt{M^2+4}}(\cosh\rho)\right\}. \end{aligned}$$

The main result of this section states the existence of Kaluza–Klein towers:

Theorem 2.2. *The finite-energy solutions u to (1.1) and (1.2) can be expressed as*

$$u(\tau, \mathbf{x}, \rho) = \sum_{\lambda_j \in \sigma_p(L_c)} u_{\lambda_j}(\tau, \mathbf{x}) w(\rho; \lambda_j) + \lim_{R\to\infty} \int_{\frac{3}{2}}^{R} u_{m^2}(\tau, \mathbf{x}) w(\rho; m^2)\, dm \ \ in\ \ C^0\left(\mathbb{R}_\tau; X^1\right) \cap C^1\left(\mathbb{R}_\tau; X^0\right),$$

where u_κ is solution to (2.1), and

$$u_{\lambda_j} \in C^0\left(\mathbb{R}_\tau; H^1\left(\mathbb{R}^3_{\mathbf{x}}\right)\right) \cap C^1\left(\mathbb{R}_\tau; L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)\right),$$

and for any $T > 0$

$$u_{m^2} \in L^2\left(\left]\frac{3}{2}, \infty\right[_m ; C^0\left([-T,T]_\tau; H^1\left(\mathbb{R}^3_{\mathbf{x}}\right)\right) \cap C^1\left([-T,T]_\tau; L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)\right)\right),$$

$$m u_{m^2} \in L^2\left(\left]\frac{3}{2}, \infty\right[_m ; C^0\left([-T,T]_\tau; L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)\right)\right).$$

Since $w(\rho;\kappa)$ is explicit and u_κ is given by the formula (3.1) below, we have obtained an analytic expression of the finite-energy solutions to the mixed hyperbolic problem in term of special functions of Bessel and Legendre that could be useful to prove some decay estimates like in [1] and [2] in the case of the Minkowski branes.

3. Asymptotics in the Steady-State Universe

The Cauchy problem for the Klein–Gordon equation (2.1) in the Steady State Universe $dS_{\frac{1}{2}}$ is easily solved in the scale of the Sobolev spaces by using Fourier analysis. For any $\kappa \in \mathbb{C}$, $\tau_* \in \mathbb{R}$, $u_0 \in H^s(\mathbb{R}^3)$, $u_1 \in H^{s-1}(\mathbb{R}^3)$, there exists a unique $u \in C^0\left(\mathbb{R}_\tau; H^s(\mathbb{R}^3)\right) \cap C^1\left(\mathbb{R}_\tau; H^{s-1}(\mathbb{R}^3)\right)$ solution of (2.1) satisfying $u(\tau_*,.) = u_0(.)$, $\partial_\tau u(\tau_*,.) = u_1(.)$. The partial Fourier transform with respect to $\mathbf{x}$ of the solution, $\hat{u}(\tau,\boldsymbol{\xi}) = \mathcal{F}_{\mathbf{x}}(u(\tau,.))(\boldsymbol{\xi})$ is given by

$$\begin{aligned}\hat{u}(\tau,\boldsymbol{\xi}) = \frac{\pi}{2}e^{-\frac{3}{2}\tau}\Big\{&\left[Y'_\nu\left(|\boldsymbol{\xi}|e^{-\tau_*}\right)J_\nu\left(|\boldsymbol{\xi}|e^{-\tau}\right) - J'_\nu\left(|\boldsymbol{\xi}|e^{-\tau_*}\right)Y_\nu\left(|\boldsymbol{\xi}|e^{-\tau}\right)\right]|\boldsymbol{\xi}|e^{\frac{1}{2}\tau_*}\hat{u}_0(\boldsymbol{\xi})\\ &+\left[Y_\nu\left(|\boldsymbol{\xi}|e^{-\tau_*}\right)J_\nu\left(|\boldsymbol{\xi}|e^{-\tau}\right) - J_\nu\left(|\boldsymbol{\xi}|e^{-\tau_*}\right)Y_\nu\left(|\boldsymbol{\xi}|e^{-\tau}\right)\right]e^{\frac{3}{2}\tau_*}\left(\hat{u}_1(\boldsymbol{\xi}) + \frac{3}{2}\hat{u}_0(\boldsymbol{\xi})\right)\Big\},\end{aligned} \tag{3.1}$$

where $\nu \in \mathbb{C}$ satisfies $\Re\nu \geq 0$, $\nu^2 = \frac{9}{4} - \kappa$ (for another representation with the fundamental solution, see [6]). When $\kappa \in \mathbb{R}$, $s \geq 1$ the natural energy defined as

$$\mathcal{E}_\kappa(u,\tau) := \int |\partial_\tau u(\tau,\mathbf{x})|^2 + e^{-2\tau}|\nabla_{\mathbf{x}}u(\tau,\mathbf{x})|^2 + \kappa|u(\tau,\mathbf{x})|^2\,d\mathbf{x}$$

is a decreasing function of τ and when $\kappa \geq 9/4$, we have for all $\tau \geq \tau_*$:

$$\int\left|\partial_\tau u(\tau,\mathbf{x}) + \frac{3}{2}u(\tau,\mathbf{x})\right|^2 d\mathbf{x} \leq 2e^{3(|\tau_*|-\tau)}\int|u_1|^2 + |\nabla_{\mathbf{x}}u_0|^2 + \kappa|u_0|^2\,d\mathbf{x},$$

$$\int|\partial_\tau u(\tau,\mathbf{x})|^2 + \kappa|u(\tau,\mathbf{x})|^2\,d\mathbf{x} \leq \min\left(\frac{3\kappa e^{-3\tau}}{\kappa - \frac{9}{4}}, 1\right)e^{3|\tau_*|}\int|u_1|^2 + |\nabla_{\mathbf{x}}u_0|^2 + \kappa|u_0|^2\,d\mathbf{x},$$

$$\int|\nabla_{\mathbf{x}}u(\tau,\mathbf{x})|^2 \leq 2e^{3|\tau_*|-\tau}\int|u_1|^2 + |\nabla_{\mathbf{x}}u_0|^2 + \kappa|u_0|^2\,d\mathbf{x}.$$

An important point related to exponential expansion is the existence of a horizon: when u_0 and u_1 are compactly supported in $|\mathbf{x}| \leq R$, then for all $\tau \geq \tau_*$, $u(\tau,.)$ is supported in $|\mathbf{x}| \leq R + e^{-\tau_*} - e^{-\tau}$. Another crucial consequence is the asymptotic behavior of the field as $\tau \to \infty$:

Theorem 3.1. *When $\kappa = 0$, u and $e^{2\tau}\partial_\tau u$ have an asymptotic profile at the time infinity: there exists $\phi \in H^{s+1}(\mathbb{R}^3_{\mathbf{x}})$ such that*

$$\|u(\tau,.) - \phi\|_{H^s} + \|e^{2\tau}\partial_\tau u(\tau,.) - \Delta\phi\|_{H^{s-1}} \longrightarrow 0,\ \tau \to +\infty, \tag{3.2}$$

and ϕ is given by:

$$\hat{\phi}(\boldsymbol{\xi}) = \sqrt{\frac{\pi}{2}}\,e^{\frac{1}{2}\tau_*}|\boldsymbol{\xi}|^{-\frac{1}{2}}\left\{J_{\frac{1}{2}}\left(e^{-\tau_*}|\boldsymbol{\xi}|\right)\hat{u}_0(\boldsymbol{\xi}) + J_{\frac{3}{2}}\left(e^{-\tau_*}|\boldsymbol{\xi}|\right)e^{\tau_*}|\boldsymbol{\xi}|^{-1}\hat{u}_1(\boldsymbol{\xi})\right\}. \tag{3.3}$$

When $\kappa > 0$, the solution is vanishing as $\tau \to +\infty$: for almost all $\boldsymbol{\xi} \in \mathbb{R}^3$, when $0 < \kappa < 9/4$, $\hat{u}(\tau,\boldsymbol{\xi}) = O\left(e^{(\sqrt{\frac{9}{4}-\kappa}-\frac{3}{2})\tau}\right)$, for $\kappa = 9/4$, $\hat{u}(\tau,\boldsymbol{\xi}) = O\left(\tau e^{-\frac{3}{2}\tau}\right)$, and when $\kappa > 9/4$, $\hat{u}(\tau,\boldsymbol{\xi}) = O\left(e^{-\frac{3}{2}\tau}\right)$. Moreover, given τ_*, there exists $C > 0$ independent of u_0 and u_1 such that for all $\tau > \tau_*$ we have:

$$0 < \kappa < \frac{9}{4},\ \begin{cases}\|u(\tau,.)\|_{H^s} + \|\partial_\tau u(\tau,.)\|_{H^{s-1}} \leq C\,e^{\max(\sqrt{\frac{9}{4}-\kappa}-\frac{3}{2},-1)\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\\ \|u(\tau,.)\|_{H^{s-\frac{1}{2}}} + \|\partial_\tau u(\tau,.)\|_{H^{s-\frac{3}{2}}} \leq C\,e^{(\sqrt{\frac{9}{4}-\kappa}-\frac{3}{2})\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\end{cases}$$

$$\kappa = \frac{9}{4},\ \begin{cases}\|u(\tau,.)\|_{H^s} + \|\partial_\tau u(\tau,.)\|_{H^{s-1}} \leq C\,\tau\,e^{-\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\\ \|u(\tau,.)\|_{H^{s-\frac{1}{2}}} + \|\partial_\tau u(\tau,.)\|_{H^{s-\frac{3}{2}}} \leq C\,\tau\,e^{-\frac{3}{2}\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\end{cases}$$

$$\kappa > \frac{9}{4},\ \begin{cases}\|u(\tau,.)\|_{H^s} + \|\partial_\tau u(\tau,.)\|_{H^{s-1}} \leq C\,e^{-\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\\ \|u(\tau,.)\|_{H^{s-\frac{1}{2}}} + \|\partial_\tau u(\tau,.)\|_{H^{s-\frac{3}{2}}} \leq C\,e^{-\frac{3}{2}\tau}\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}\right),\end{cases}$$

When $\kappa < 0$, the solution can blow up at the time infinity: there exists Schwartz functions u_0, u_1 such that the solution satisfies

$$\forall s \in \mathbb{R},\ \|u(\tau,.)\|_{H^s} \sim e^{\left(\sqrt{\frac{9}{4}-\kappa}-\frac{3}{2}\right)\tau},\ \tau \to +\infty.$$

We make some comments on this result. A consequence of this theorem is that when $u_0, u_1 \in C_0^\infty$, we have in C_0^∞

$$u(\tau,.) = O\left(e^{-\left(\frac{3}{2}-\sqrt{\frac{9}{4}-\kappa}\right)\tau}\right) \text{ if } 0 \le \kappa \ne \frac{9}{4},\ u(\tau,.) = O\left(\tau\, e^{-\frac{3}{2}\tau}\right) \text{ if } \kappa = \frac{9}{4}.$$

This is a consequence of the estimates of decay in $H^{s-\frac{1}{2}} \times H^{s-\frac{3}{2}}$, the Sobolev embedding, and the horizon of radius $R + e^{-\tau_*}$ when u_j are supported in $|\mathbf{x}| \le R$. A. Vasy has established in [11] precise asymptotics in the much more large class of the asymptotically de Sitter space-times and we can deduce from his work that when $u_0, u_1 \in C_0^\infty$, there exists $v \in C_0^\infty(\mathbb{R}^3)$ such that as $\tau \to \infty$ we have:

$$u(\tau,\mathbf{x}) = e^{-\left(\frac{3}{2}-\sqrt{\frac{9}{4}-\kappa}\right)\tau} v(\mathbf{x}) + o\left(e^{-\left(\frac{3}{2}-\sqrt{\frac{9}{4}-\kappa}\right)\tau}\right) \text{ if } 0 \le \kappa \ne \frac{9}{4},\ u(\tau,\mathbf{x}) = \tau\, e^{-\frac{3}{2}\tau} v(\mathbf{x}) + O\left(e^{-\frac{3}{2}\tau}\right) \text{ if } \kappa = \frac{9}{4}.$$

The main novelty of Theorem 3.1 is the explicit formula (3.3) for the trace at $\tau = +\infty$ when $\kappa = 0$. We remark also that, in this case, there exists "disappearing solutions" that tends to zero as $\tau \to \infty$ since ϕ can be equal to zero. We note also a somewhat unexpected loss of rate of decay in $H^s \times H^{s-1}$ when $\kappa > 2$. $\kappa = 2$ is the critical mass for which the Klein–Gordon equation is conformal invariant, hence the solution can be expressed in this case from a free field on the static Einstein universe $\mathbb{R} \times S^3$ by using the conformal method and the decay estimates simply follow from the conformal factor. Besides, since the proof is based on the Fourier analysis and the investigation of the differential equation

$$\left[\frac{d^2}{d\tau^2} + 3\frac{d}{d\tau} + e^{-2\tau}\lambda^2 + \kappa\right] u = 0, \quad \tau \in \mathbb{R},\ \ \lambda \in \mathbb{R},$$

we could obtain a similar theorem for the Klein–Gordon equation on the exponentially expanding Friedmann–Robertson–Walker universe

$$\mathbb{R}_\tau \times K_x,\ \ ds^2 = d\tau^2 - e^{2\tau} g_{ij}\, dx^i\, dx^j$$

where (K, g) is any Riemannian manifold. Parenthetically, we also could obtain several formula of products of Bessel functions, by using the propagator property of the map $(u_0, u_1) \mapsto (u(\tau), \partial_\tau u(\tau))$ and formulas (3.1) and (3.3).

4. Gravitational fluctuations

The gravitational waves are described by equation (0.1) with $M = 0$ and the Neumann condition on the brane, i.e. (0.2) with $c = 0$. The crucial point is the existence of the sector of the massless graviton that is just the set of the solutions

$$u_\varphi(t,\mathbf{x},z) := \varphi\left(-2^{-1}\log\left(t^2 - z^2\right), \mathbf{x}\right),$$

where $\varphi = \varphi(\tau,\mathbf{x})$ is any solution to (2.1) with $\kappa = 0$. Then u_φ is solution to (0.1) with $M = 0$ in $\mathcal{O} = \Omega \times \mathbb{R}^3_{\mathbf{x}}$, $\Omega := \{(t,z); t < -z < 0\}$, and satisfies the Neumann condition $\alpha\partial_t u_\varphi + \partial_z u_\varphi = 0$ on $z = \alpha t$, $t < 0$, for any $\alpha \in]-1, 0[$. In the τ, ρ coordinates, it is just a solution independent of ρ, i.e. $u_\varphi(\tau,\mathbf{x},\rho) = \varphi(\tau,\mathbf{x})$ (by abuse of notation we write $u_\varphi(\tau,\mathbf{x},\rho) := u_\varphi(t,\mathbf{x},z)$). When $\varphi \in C^0\left(\mathbb{R}_\tau; H^1(\mathbb{R}^3_{\mathbf{x}})\right) \cap C^1\left(\mathbb{R}_\tau; L^2(\mathbb{R}^3_{\mathbf{x}})\right)$, then $u_\varphi \in C^0\left(\mathbb{R}_\tau; X^1\right) \cap C^1(\mathbb{R}_\tau; X^0) \cap \mathcal{D}'\left(\mathbb{R}_\tau \times \mathbb{R}^3_{\mathbf{x}}; D_c\right)$. Moreover, we have $u_\varphi \in C^0\left(\Omega; H^1(\mathbb{R}^3_{\mathbf{x}})\right)$, $\nabla_{t,\mathbf{x},z} u_\varphi \in C^0\left(\Omega; L^2(\mathbb{R}^3_{\mathbf{x}})\right)$. A direct application of Theorem 3.1 and formula (3.3) allows us to investigate the behavior of this graviton along the de Sitter brane $\mathcal{B}_\alpha$ as $t \to 0^-$, and when we approach its Cauchy horizon $\mathcal{N}$. We introduce $\gamma := \sqrt{2/(\sinh\rho_0 \cosh\rho_0 - \rho_0)}$.

Proposition 4.1. *For any finite energy massless graviton u_φ, there exists $\phi \in H^2(\mathbb{R}^3_{\mathbf{x}})$ such that when $e^{-2\tau} = t^2 - z^2 \to 0$ with $t < -z < 0$, we have:*

$$\gamma \|u_\varphi(\tau,.) - \phi(.)\|_{X^1} = \|u_\varphi(t,.,z) - \phi\|_{H^1(\mathbb{R}^3_{\mathbf{x}})} \to 0,$$

$$\gamma \left\|e^{2\tau}\partial_\tau u_\varphi(\tau,.) - \Delta\phi\right\|_{X^0} = \left\|t^{-1}\partial_t u_\varphi(t,.,z) + \Delta\phi\right\|_{L^2(\mathbb{R}^3_{\mathbf{x}})} = \left\|z^{-1}\partial_z u_\varphi(t,.,z) - \Delta\phi\right\|_{L^2(\mathbb{R}^3_{\mathbf{x}})} \to 0,$$

$$\left\|(z \mp t)^{-1}\left(\partial_t \pm \partial_z\right) u_\varphi(t,.,z) \mp \Delta\phi\right\|_{L^2(\mathbb{R}^3_{\mathbf{x}})} \to 0,$$

and the Fourier transform of ϕ is given by

$$\hat{\phi}(\xi) = \sqrt{\frac{\pi}{2}}\, |\xi|^{-\frac{1}{2}} \left\{ J_{\frac{1}{2}}(|\xi|)\hat{\varphi}(0,\xi) + J_{\frac{3}{2}}(|\xi|)\, |\xi|^{-1}\, \partial_\tau\hat{\varphi}(0,\xi)\right\}. \tag{4.1}$$

The following consequence of Theorem 2.2 states that the usual de Sitter gravity is recovered on the brane: the leading term of any gravitational fluctuation in the *ADS* bulk $\mathcal{M}$ is a massless graviton propagating on the brane. The physical meaning of this result is that the brane is stable under the scalar perturbations when we omit the back-reaction to the metric (linearized gravity). This conclusion matches with numerous analyses of the linear stability of the de Sitter brane in the physical literature, e.g., [5,7,9].

Theorem 4.2. *We assume $M = 0$, $c = 0$. Then given $u_0 \in X^1$, $u_1 \in X^0$, there exists a massless graviton u_φ such that the solution $u \in C^1\left(\mathbb{R}_\tau; X^0\right) \cap C^0\left(\mathbb{R}_\tau; X^1\right) \cap \mathcal{D}'\left(\mathbb{R}_\tau \times \mathbb{R}^3_{\mathbf{x}}; D_c\right)$ of (1.1) and (1.3), satisfies*

$$\left\| (u - u_\varphi)(\tau) \right\|_{X^1} + \left\| \partial_\tau (u - u_\varphi)(\tau) \right\|_{X^0} \longrightarrow 0, \quad \tau \to +\infty.$$

The initial data of φ are given by

$$\varphi(\tau_*, \mathbf{x}) = \gamma^2 \int_0^{\rho_0} u_0(\mathbf{x}, \rho) \sinh^2 \rho \, \mathrm{d}\rho \in H^1(\mathbb{R}^3_{\mathbf{x}}), \;\; \partial_\tau \varphi(\tau_*, \mathbf{x}) = \gamma^2 \int_0^{\rho_0} u_1(\mathbf{x}, \rho) \sinh^2 \rho \, \mathrm{d}\rho \in L^2(\mathbb{R}^3_{\mathbf{x}}). \tag{4.2}$$

In terms of the coordinates (t, z), we have $\|u(t, ., z) - \phi\|_{H^1(\mathbb{R}^3)} + \|(t\partial_t + z\partial_z)u(t, ., z)\|_{L^2(\mathbb{R}^3)} \to 0$ as $t + z \to 0$, $t < 0 < z$. We end this work by investigating the gravitational fluctuations beyond the Cauchy horizon $\mathcal{N} = \{t + z = 0\}$. We obviously have to add some constraint when $z > -t$. Roughly speaking, we suppose that there is no field incoming from the past null infinity, i.e. $u(t, \mathbf{x}, z) \to 0$ as $t \to -\infty$, $z + t = Cst > 0$. More precisely, we consider initial data $u_0, u_1 \in C_0^\infty\left(\mathbb{R}^3_{\mathbf{x}} \times (\epsilon, \rho_0)_\rho\right)$, $0 < \epsilon < \rho_0$ and $u(\tau, \mathbf{x}, \rho)$ the solution to (1.1), (1.2) and (1.3) in $\mathcal{M}$ ($M = c = 0$). Then, if $\tilde{\mathcal{M}} := \left\{(t, \mathbf{x}, z) \in \mathbb{R} \times \mathbb{R}^3 \times]0, \infty[; \; \max(\alpha t, t) < z\right\}$, u can be uniquely extended into a solution $\tilde{u} \in C^\infty(\tilde{\mathcal{M}} \cup \mathcal{B}_\alpha)$ of $\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{3}{z}\partial_z\right)\tilde{u} = 0$ in $\tilde{\mathcal{M}}$, satisfying the Neumann condition $\alpha \partial_t \tilde{u} + \partial_z \tilde{u} = 0$ on $\mathcal{B}_\alpha$, and $\tilde{u}(t, \mathbf{x}, z) = 0$ for $z + t > 0$, $t << 0$. We study the energy of $\tilde{u}$ on the hypersurface $t = Cst$.

Theorem 4.3. *We assume that ϕ defined by (4.1) and (4.2) is nonzero. Then we have*

$$\int_{\alpha t}^{\infty} \int_{\mathbb{R}^3} | \nabla_{t,\mathbf{x},z} \tilde{u}(t, \mathbf{x}, z) |^2 z^{-3} \, \mathrm{d}\mathbf{x} \, \mathrm{d}z \gtrsim \frac{1}{t^2}, \;\; t \to 0^-. \tag{4.3}$$

We conclude that the energy of $\tilde{u}(t = 0, .)$ is infinite, hence the finite energy spaces used in [2] cannot be used if we want to continue the solution in $[0, \infty[_t \times \mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z$. Beyond the Cauchy horizon $\left\{(t, \mathbf{x}, z) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+, \;\; t = z\right\}$, it would be natural to consider a boundary constraint on the time-like conformal infinity $z = 0$ in the new functional framework introduced in [3].

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