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Boundary regularity of weakly anchored harmonic maps

Régularité au bord des applications harmoniques avec ancrage faible

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A R T I C L E I N F O

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A B S T R A C T

In this note, we study the boundary regularity of the minimizers of a family of weak anchoring energies that model the states of liquid crystals. We establish optimal boundary regularity in all dimensions $n \geq 3$. In dimension $n = 3$, this yields full regularity at the boundary, which stands in sharp contrast with the observation of boundary defects in physics works. We also show that, in the cases of weak and strong anchoring, the regularity of the minimizers is inherited from that of their corresponding limit problems.

R É S U M É

Dans cette note, nous étudions la régularité au bord des minimiseurs d'une famille d'énergies avec ancrage faible utilisée dans la modélisation des cristaux liquides. Nous établissons la régularité au bord optimale en toute dimension supérieure à 3. En dimension $n = 3$, de tels minimiseurs sont lisses près du bord, ce qui va à l'encontre des observations de défauts sur le bord dans les travaux physiques. Nous montrons également que, dans les cas de faible et de fort ancrage, la régularité des minimiseurs est héritée de la régularité des minimiseurs des problèmes limites correspondants.

Version française abrégée

Soit $n \geq 3$, $\Omega \subset \mathbb{R}^n$ un domaine borné et régulier et $\mathcal{N}$ une variété lisse et compacte. On s'intéresse à la régularité au bord des minimiseurs de la famille d'énergies avec ancrage faible définies pour des applications $u \in H^1(\Omega; \mathcal{N})$ par

$$E_w(u) := \int_\Omega |\nabla u|^2 \mathrm{d}x + w \int_{\partial\Omega} g(x, u) \mathrm{d}\mathcal{H}^{n-1}.$$

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De telles énergies apparaissent dans la modélisation des cristaux liquides [3]. Ici $w \geq 0$ est un *coefficient d'ancrage*, et g est une fonction positive et bornée sur $\partial\Omega \times \mathcal{N}$, la *densité d'énergie d'ancrage*. Dans le contexte des cristaux liquides, on a $n = 3$ et $\mathcal{N} = \mathbb{S}^2$. La fonctionnelle E_w assouplit la contrainte d'ancrage fort

$$g(x, u(x)) = 0 \quad \text{pour presque tout } x \in \partial\Omega,$$

qui est physiquement irréaliste. Formellement, l'ancrage fort correspond à $w = \infty$. Par exemple, la densité d'ancrage « modèle »

$$g(x, u) = |u - u_0(x)|^2,$$

pour une certaine application $u_0 : \partial\Omega \to \mathcal{N}$, correspond, dans la limite d'ancrage fort, à la condition de Dirichlet $u_{|\partial\Omega} = u_0$. D'autres formes de densités d'énergie d'ancrage sont également intéressantes physiquement [3].

La régularité intérieure pour les minimiseurs de E_w est une conséquence de [6] : l'ensemble singulier est de dimension de Hausdorff au plus $n - 3$ et est discret lorsque $n = 3$. En revanche, la régularité au bord ne semble pas avoir été considérée, même si, dans le contexte des cristaux liquides, la présence de défauts sur le bord est envisagée [3]. Le but principal de cette note est d'établir la régularité optimale au bord des minimiseurs de E_w. Nous adoptons deux points de vues différents : nous obtenons d'abord une borne optimale sur la dimension de l'ensemble singulier, valide pour toutes valeurs de $w \in [0, \infty)$; ensuite, nous considérons des perturbations de $w \in [0, \infty]$ et observons que l'absence de défauts au bord est un phenomène stable.

Dans ce qui suit, $\mathrm{Sing}(u) \subset \overline{\Omega}$ désigne l'ensemble des points où u est discontinue, et $\dim A$ la dimension de Hausdorff d'un ensemble $A \subset \mathbb{R}^n$. De manière naturelle, on étend la définition de E_w à $w = \infty$, en posant $E_w(u) = \infty$ si u ne satisfait pas la contrainte d'ancrage fort (3). Notre résultat principal est le suivant :

Théorème 0.1.

1. *(Régularité au bord optimale à coefficient d'ancrage fixé) Pour tout $w \in [0, \infty)$ et u un minimiseur de E_w,*

$$\begin{aligned} &\dim(\mathrm{Sing}(u) \cap \partial\Omega) \leq n - 4 && \text{si } n \geq 4,\\ &\mathrm{Sing}(u) \cap \partial\Omega \text{ est discret} && \text{si } n = 4,\\ &\mathrm{Sing}(u) \cap \partial\Omega = \emptyset && \text{si } n = 3.\end{aligned}$$

2. *(Stabilité par rapport au coefficient d'ancrage) Supposons que pour un certain $w_0 \in [0, \infty]$, les minimiseurs de E_{w_0} n'aient pas de singularité au bord. Dans le cas $w_0 = \infty$, on suppose de plus que $\inf E_{w_0} < \infty$. Alors, pour w dans un voisinage de w_0, les minimiseurs de E_w n'ont pas non plus de singularité au bord.*

Remarquons que, de manière un peu surprenante, dans le cas de la dimension physique $n = 3$, la première partie du Théorème 0.1 fournit la régularité complète au bord, ce qui contraste fortement avec l'observation physique de défauts de surface [3]. En outre, la seconde partie du théorème implique en particulier que les minimiseurs de E_w n'ont pas de singularités au bord pour w proche de zéro (cas de l'ancrage faible). En effet, les minimiseurs de l'énergie de Dirichlet avec condition au bord de Neumann sont constants (donc lisses). Dans le cas d'un ancrage fort (c'est-à-dire lorsque w est grand) et pour g de la forme « modèle » (4) avec u_0 lisse, les minimiseurs de E_w ne possèdent pas de singularités au bord. Ceci provient encore de la seconde partie du Théorème 0.1, car les applications harmoniques minimisantes avec des conditions de Dirichlet lisses sont lisses près du bord [7].

La preuve de la première partie du Théorème 0.1 suit le schéma classique de la régularité des applications harmoniques [6], qui repose sur l'étude des applications tangentes (limites de *blow-up*). Un ingrédient crucial dans [6] est la monotonie de l'énergie renormalisée. L'observation clé dans notre cas est que le terme d'énergie d'ancrage au bord réagit différemment aux changements d'échelle de l'énergie de Dirichlet, ce qui permet de démontrer une formule de monotonie approchée ; de plus, le terme d'ancrage au bord disparaît après le *blow-up*. On peut alors voir que les applications tangentes sont les *applications tangentes minimisantes à bord libre* étudiées dans [2], où l'équivalent de la première partie du Théorème 0.1 est démontré.

La seconde partie du Théorème 0.1 est une conséquence de la convergence forte H^1 des minimiseurs de E_w vers les minimiseurs de E_{w_0} lorsque $w \to w_0$, et de la semicontinuité supérieure de la fonction de densité d'énergie (cf. (8)).

1. Introduction

Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and $\mathcal{N}$ a smooth compact manifold. We are interested in the boundary regularity of the minimizers of the family of weak anchoring energies defined for maps $u \in H^1(\Omega; \mathcal{N})$,

$$E_w(u) := \int_\Omega |\nabla u|^2 \mathrm{d}x + w \int_{\partial\Omega} g(x, u) \mathrm{d}\mathcal{H}^{n-1}, \tag{1}$$

that arise in the study of liquid crystals [3]. Above, $w \geq 0$ is referred to as the *anchoring strength*, while g, the *anchoring energy density*, is a non-negative bounded function on $\partial\Omega \times \mathcal{N}$. The Euler–Lagrange equations satisfied by a minimizer u of E_w are

$$\begin{cases} -\Delta u = A_{\mathcal{N}}(u)(\nabla u, \nabla u) & \text{in } \Omega, \\ \dfrac{1}{w}\dfrac{\partial u}{\partial \nu} = \pi_{\mathcal{N}}(u)\nabla_u g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where ν is the outward unit normal to $\partial\Omega$, $A_{\mathcal{N}}$ is the second fundamental form of $\mathcal{N}$ and $\pi_{\mathcal{N}}$ is the projection on the tangent space. In the context of liquid crystals, $n = 3$ and the target manifold is $\mathcal{N} = \mathbb{S}^2$. The functional E_w relaxes the physically unrealistic strong anchoring constraint

$$g(x, u(x)) = 0 \quad \text{for a.e. } x \in \partial\Omega, \tag{3}$$

which formally corresponds to $w = \infty$. A model case of anchoring density, though not the only one of physical interest, is given by

$$g(x,u) = |u - u_0(x)|^2, \tag{4}$$

for some $u_0 \colon \partial\Omega \to \mathcal{N}$, which corresponds to Dirichlet boundary conditions in the strong anchoring limit.

Interior regularity for the minimizers of E_w follows directly from [6]: the singular set has Hausdorff dimension at most $n-3$ and is discrete when $n = 3$. On the other hand, boundary regularity does not seem to have been considered. In the liquid crystal setting, however, boundary defects have been discussed in [3]. The chief goal of this note is to address the question of the optimal boundary regularity of the minimizers of E_w. We tackle this question from two different perspectives: first we obtain an optimal bound on the dimension of the singular set of such maps valid for all values of w, and then we take on a perturbation point of view to observe that boundary smoothness is a stable condition in w.

In what follows, $\mathrm{Sing}(u) \subset \overline{\Omega}$ denotes the set of points where u is not continuous, while $\dim A$ corresponds to the Hausdorff dimension of a set $A \subset \mathbb{R}^n$. As is natural, we extend the definition of E_w to $w = \infty$ by setting $E_\infty(u) = +\infty$ if u does not satisfy the strong anchoring constraint (3). Our main result is the following.

Theorem 1.1. *Let E_w be as in (1). The following holds about the minimizers of E_w in $H^1(\Omega; \mathcal{N})$.*

1. *(Optimal boundary regularity for fixed anchoring strength) For any $w \in [0, \infty)$ and u a minimizer of E_w,*

$$\begin{aligned} &\dim(\mathrm{Sing}(u) \cap \partial\Omega) \leq n-4 && \text{if } n \geq 4, \\ &\mathrm{Sing}(u) \cap \partial\Omega \text{ is discrete} && \text{if } n = 4, \\ &\mathrm{Sing}(u) \cap \partial\Omega = \emptyset && \text{if } n = 3. \end{aligned}$$

2. *(Stability with respect to the anchoring strength) Assume that for some $w_0 \in [0, \infty]$, the minimizers of E_{w_0} have no boundary singularities, and in the case $w_0 = \infty$ assume in addition that $\inf E_{w_0} < \infty$. Then, for w in a neighborhood of w_0, the minimizers of E_w have no boundary singularities.*

Let us note that, somewhat surprisingly, in the case of the physical dimension $n = 3$, the first part of Theorem 1.1 gives full regularity at the boundary, which is in strong contrast with physical observations [3]. At the same time, the second part of the theorem implies in particular that the minimizers of E_w have no boundary singularities for w close to zero (weak anchoring case), since the minimizers of the Dirichlet energy with Neumann boundary conditions are constants. In the case of extreme anchoring (that is when w is large) and for g of the form (4) with a smooth u_0, the minimizers of E_w have no boundary singularities, again as a consequence of the second part of Theorem 1.1, because minimizing harmonic maps with smooth Dirichlet conditions are smooth near the boundary [7].

2. Proof of Theorem 1.1

The proof of the first part of Theorem 1.1 follows the classical scheme for the regularity of harmonic maps [6], which relies on the study of tangent maps. Let $x_0 \in \partial\Omega$ and $r > 0$. Defining $\hat{u}(x) := u(x_0 + rx)$, we have

$$\begin{cases} -\Delta \hat{u} = A_{\mathcal{N}}(\hat{u})(\nabla \hat{u}, \nabla \hat{u}) & \text{in } \dfrac{1}{r}(\Omega \setminus x_0), \\ \dfrac{1}{w}\dfrac{\partial \hat{u}}{\partial \nu} = r\pi_{\mathcal{N}}(\hat{u})\nabla_{\hat{u}} g(x,\hat{u}) & \text{on } \partial\left[\dfrac{1}{r}(\Omega \setminus x_0)\right]. \end{cases}$$

Since $\pi_{\mathcal{N}}(\hat{u})\nabla_{\hat{u}} g(x,\hat{u})$ is bounded, taking the formal limit $r \to 0$ yields a map ϕ satisfying

$$\begin{cases} -\Delta \phi = A_{\mathcal{N}}(\phi)(\nabla\phi, \nabla\phi) & \text{in } \mathbb{R}^n_+, \\ \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \mathbb{R}^{n-1} \times \{0\}. \end{cases}$$

Such maps, when they are 0-homogeneous and locally minimizing, are called *free-boundary minimizing tangent maps* and have been studied by Hardt and Lin in [2]. They discovered that their singular set has Hausdorff dimension at most $n-4$ at the boundary. This result allows us to conclude, provided we adapt the techniques developed in [6] to our case. An essential ingredient in [6] is the energy monotonicity formula, which – together with a technical extension lemma – ensures convergence of blow-up sequences to tangent maps. The key observation in our case is that the surface anchoring term in the energy (1) scales differently from the Dirichlet energy whence an approximate monotonicity formula is still valid; moreover, the surface anchoring term disappears after blow-up and thus our tangent maps are precisely the ones studied in [2], where the equivalent of Theorem 1.1, part 1, is proven.

Proof of Theorem 1.1 part 1. We denote by B_r^+ the half ball

$$B_r^+ = \left\{x \in \mathbb{R}^n : |x| < 1,\, x_n > 0\right\},$$

by Σ_r the "flat" part of its boundary $\Sigma_r = B_r^+ \cap \{x_n = 0\}$, and by Γ_r the "round" part of its boundary $\Gamma_r = \partial B_r^+ \cap \{x_n > 0\}$. By locally flattening the boundary of Ω, our problem can be reduced to studying maps minimizing an energy functional of the form

$$\mathcal{E}_w(u) = \int_{B_1^+} |\nabla u|^2 + w \int_{\Sigma_1} g(x', u),$$

among maps $u \in H^1(B_1^+; \mathcal{N})$ with fixed boundary values on Γ_r. Here g is a bounded non-negative function on $\Sigma_1 \times \mathcal{N}$, and we study the regularity of the minimizers on Σ_1. It is important to remark that the two terms in the above energy scale differently: setting $u_r(x) = u(rx)$, it holds

$$r^{n-2}\mathcal{E}_w(u_r) = \int_{B_r^+} |\nabla u|^2 + r \int_{\Sigma_r} g(x'/r, u). \tag{5}$$

A first consequence of (5) is that "small energy regularity" holds for the minimizers of $\mathcal{E}_w$: there exist r_0 and ε_0 (depending on n, w and $\sup g$) such that for $r < r_0$,

$$r^{2-n} \int_{B_r^+} |\nabla u|^2 < \varepsilon_0^2 \quad \Longrightarrow \quad u \text{ is continuous in } \overline{B_{r/2}^+}. \tag{6}$$

This can be proved arguing by contradiction as in [4, Proposition 1]. An essential step there is to construct good comparison maps, which is done with the help of an important extension lemma. Our setting is slightly different, since we are dealing with maps defined on half balls, but after extending by reflection, the proof carries over.

Comparison with rescaled homogeneous maps as in [6, §2] implies the following monotonicity formula: for some $c > 0$ depending only on n, w and $\sup g$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{2-n} \int_{B_r^+} |\nabla u|^2 \right] \geq -c + r^{2-n} \int_{\Gamma_r} \left| \frac{\partial u}{\partial r} \right|^2. \tag{7}$$

Together with the construction of good comparison maps, this monotonicity formula implies, following [4, Proposition 2] and taking (5) into account, the strong H^1 convergence of blow-up subsequences $u_{x_0, r_i}(x) = u(x_0 + r_i x)$ for any $x_0 \in \Sigma_1$. The limits, called *tangent maps*, are homogeneous $\mathcal{N}$-valued maps defined in the half-space $\{x_n > 0\}$. Moreover, tangent maps minimize the Dirichlet energy $\mathcal{E}_0$ with *free boundary conditions* on Σ_1 (and fixed boundary values on Γ_1). Therefore the proof can be continued exactly as in [2, Theorem 2.8]. □

Remark 1. The Dirichlet energy of a tangent map at $x_0 \in \overline{\Omega}$ equals the *density function*

$$\Theta(u, x_0) = \lim_{r \to 0} \left[r^{2-n} \int_{B_r(x_0) \cap \Omega} |\nabla u|^2 \right]. \tag{8}$$

The limit exists thanks to the monotonicity formula (7). The small energy regularity property (6) amounts to

$$\Theta(u, x_0) < \varepsilon_0 \quad \Longrightarrow \quad u \text{ is continuous at } x_0, \tag{9}$$

and ε_0 can be *a posteriori* taken as the infimum of the Dirichlet energy over all non-constant tangent maps. In particular, ε_0 in (9) is *independent* of w and $\sup g$, which was *a priori* not obvious.

Theorem 1.1, part 2, follows from the strong H^1 convergence of the minimizers of E_w to the minimizers of E_{w_0} as $w \to w_0$.

Proof of Theorem 1.1 part 2. Were the result not true, there would exist a sequence $w_k \to w_0$ and maps u_k minimizing E_{w_k}, with singularities at $x_k \to x_0 \in \partial\Omega$. By Remark 1 above, this implies in particular $\Theta(u_k, x_k) \geq \varepsilon_0$.

We may also assume that u_k converges weakly in H^1 to a $\mathcal{N}$-valued map u_0. The convergence is in fact strong, and u_0 minimizes E_{w_0}: this follows from the inequalities

$$E_{w_0}(u_0) \leq \liminf E_{w_k}(u_k) \leq \liminf E_{w_k}(u) = E_{w_0}(u), \qquad \forall u \in H^1(\Omega; \mathcal{N}).$$

By assumption, u_0 has no boundary singularities. On the other hand, it holds $\Theta(u_0, x_0) \geq \varepsilon_0$ since the density function is upper-semicontinuous [1, Proposition 10.26]. This contradiction completes the proof. □

3. Future directions

The proof of Theorem 1.1 is really specific to minimizing maps. A natural question is then if it can be extended to also consider stationary harmonic maps. Another line of investigation is more directly linked to the harmonic map depiction of liquid crystals, which can be seen as the London limit of a more general model based on Q-tensors [5]: in the case of weak anchoring, does the convergence of minimizing Q-tensors hold up to the boundary? Finally, the upper bound in Theorem 1.1, part 1, is very general and valid for any w and any function g. It would be interesting to see if this bound can be improved, incorporating the dependence on the anchoring strength and the map g. This would require much finer analysis.

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