C. R. Acad. Sci. Paris, Ser. I 352 (2014) 1011–1016

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Mathematical economics

Asymptotic expansions for degenerate parabolic equations

Expansions asymptotiques pour équations paraboliques dégénérées

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ARTICLE INFO

Article history:
Received 14 May 2014
Accepted after revision 26 September 2014
Available online 11 October 2014

Presented by Olivier Pironneau

ABSTRACT

We prove asymptotic convergence results for some analytical expansions of solutions to degenerate PDEs with applications to financial mathematics. In particular, we combine short-time and global-in-space error estimates, previously obtained in the uniformly parabolic case, with some a priori bounds on “short cylinders”, and we achieve short-time asymptotic convergence of the approximate solution in the degenerate parabolic case.

RÉSUMÉ

On démontre des résultats de convergence asymptotique pour certaines expansions analytiques de solutions d’équations aux dérivés partielles dégénérées avec des applications aux mathématiques financières. En particulier, on combine des estimations d’erreur à temps petit, globales dans l’espace, obtenues précédemment dans le cas uniformément parabolique, avec quelques bornes a priori sur de « courts cylindres », et on attend la convergence asymptotique à temps petit de la solution approchée dans le cas parabolique dégénéré.

1. Introduction

Asymptotic expansions and perturbation methods for PDEs are widely used in the area of mathematical finance in order to restore the analytical tractability of sophisticated, today more than ever, financial models. Since the pioneering papers [9] and [10], several different approaches have been proposed in the literature (see, for instance, [8,11,1,15,16]). Only recently however, rigorous error bounds have been proved under the restrictive assumption of non-degeneracy of the generator of the underlying stochastic process. From the analytical point of view, this assumption is equivalent to a uniform parabolicity for the pricing PDE. In this regard, we refer to the results in [18,19,3], based on Malliavin calculus techniques, and [6,13] based on purely analytic arguments. Unfortunately, these assumptions are hardly ever satisfied by the financial models used in practice. To this extent, we register the result by Gobet et al. in [4] on asymptotic error estimates for the time-dependent Heston model, under the assumption of strictly positiveness (Feller-type condition) for the variance process.

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http://dx.doi.org/10.1016/j.crma.2014.09.024

The aim of this paper is to provide asymptotic convergence results for a quite general class of degenerate (i.e. non-uniformly parabolic) PDEs. This class includes well-known models such as the *constant elasticity of variance* (CEV) local volatility model, the Heston, and the SABR stochastic volatility models, and some hybrid credit-equity models such as the JDCEV model [5]. Indeed, our results provide, as a particular case, rigorous error estimates for the celebrated approximations of the CEV and SABR models in [9] and [10], where only heuristic arguments are presented. Our main result stems from the asymptotic estimates proved in [13] for uniformly parabolic PDEs, combined with some a priori bounds on "short cylinders" inspired by the work of Safonov [17].

We consider the backward Cauchy problem:

$$\begin{cases} (\partial_t + \mathcal{A})u(t,x) = 0, & t \in [0,T[,\ x \in D, \\ u(T,x) = \varphi(x), & x \in D, \end{cases} \tag{1}$$

with $\mathcal{A}$ being the second-order differential operator, defined on a domain D of $\mathbb{R}^d$,

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} a_i(t,x)\partial_{x_i} + a(t,x), \quad t \in [0,T],\ x \in D, \tag{2}$$

and with $(a_{ij}(t,x))_{1\le i,j\le d}$ being a symmetric matrix, positive semi-definite for any $(t,x) \in [0,T] \times D$.

In financial applications, typically the solution to (1) (if it exists and is unique within a certain class of functions) admits the stochastic representation

$$u(t,x) = E\left[e^{\int_t^T a(s,X_s)\,ds}\varphi(X_T) \mid X_t = x\right] \tag{3}$$

where X is the diffusion process with generator $\mathcal{A} - a$. In financial terms, (3) is usually interpreted as a risk-neutral pricing formula.

Example 1. In the classical CEV model, we have $d = 1$ and

$$\mathcal{A} = \frac{(\sigma x)^{2\gamma}}{2}\partial_{xx}, \quad x \in D = \mathbb{R}_+$$

is the generator of the diffusion

$$dX_t = \sigma X_t^{\gamma}\, dW_t,$$

where σ is a positive constant (the so-called volatility parameter), $\gamma \in]0,1]$ and W is a standard Brownian motion.

Example 2. In the classical Heston model, we have $d = 2$ and

$$\mathcal{A} = \frac{yx^2}{2}\partial_{xx} + \frac{\delta^2 y}{2}\partial_{yy} + \rho\delta yx\,\partial_{xy} + \kappa(\theta - y)\partial_y, \quad (x,y) \in D = \mathbb{R}_+ \times \mathbb{R}_+$$

is the generator of the diffusion

$$dX_t = \sqrt{Y_t}X_t\, dW_t, \qquad dY_t = \kappa(\theta - Y_t)\, dt + \delta\sqrt{Y_t}\, dB_t, \tag{4}$$

where δ is a positive constant (the so-called vol-of-vol parameter), $\kappa, \theta > 0$ are the drift-mean and the mean-reverting term of the variance process, respectively, and where (W, B) is a two-dimensional Brownian motion with correlation $\rho \in]-1,1[$. The asymptotic formulas given in Definition 2.1 below provide an accurate approximation of option prices and implied volatilities, even for long maturities. As an example, Fig. 1 shows the implied volatilities in the Heston model for the realistic set of parameters given in [7] and for maturities from one month to ten years. The exact implied volatility is obtained by inverting the Fourier representation in [12]; the dashed line represents the implied volatility corresponding to our third approximation; the dotted line corresponds to the recent implied volatility expansion proposed in [7].

In Section 2, we briefly review the construction of asymptotic expansions of solutions to parabolic PDEs and we also recall an asymptotic convergence result for short times. The main result of the paper is Theorem 3.1 in Section 3.

2. Previous results: asymptotic convergence for uniformly parabolic equations

For our analysis, we will make use of the following notations. For any $N \in \mathbb{N}_0$, we denote by:

- $C_b^N(D)$: the class of the functions $f \in C^N(D)$ with bounded derivatives up to order N;
- $C_b^{0,N}([0,T] \times D)$: the class of the functions $f = f(t,x)$ such that $\partial_x^{\beta} f \in C_b([0,T] \times D)$ for any $\beta \in \mathbb{N}_0^d$ with $\beta_1 + \cdots + \beta_d =: |\beta| \le N$.

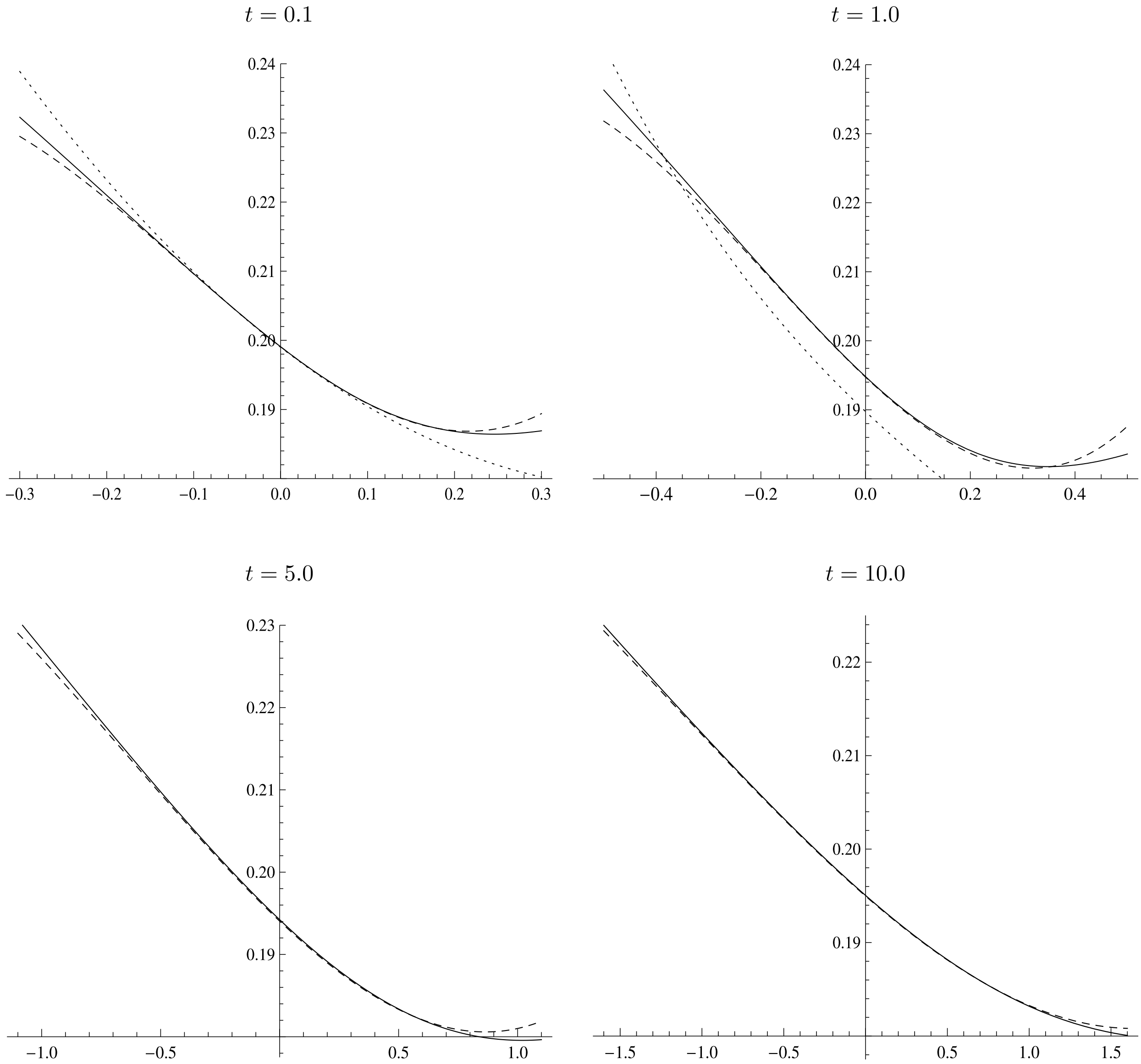

Fig. 1. The implied volatility for the Heston model in (4) is plotted as a function of log-moneyness $(k-x)$ for four different maturities t. The solid line corresponds to the implied volatility obtained by inverting the Fourier representation in [12] of the exact price. The dashed line represents the implied volatility corresponding to our third approximation. The dotted line (which only appears for the shortest two maturities) corresponds to the implied volatility expansion of [7]. We use the parameters given in [7]: $\kappa = 1.15, \theta = 0.04, \delta = 0.2, \rho = -0.40, x = 0.0, y = \log\theta$.

For simplicity, throughout the paper we shall assume the following hypothesis.

H.1 The final datum $\varphi \in C_b^k(D)$ for some $0 \le k \le 2$.

Remark 1. Assumption H.1 can be considerably relaxed to include unbounded or discontinuous payoff functions typically encountered in financial applications: for more details, we refer to [13].

Next we recall the construction of the N-th-order asymptotic expansion of the solution to (1) proposed in [13] under the condition that $D = \mathbb{R}^d$ and $\mathcal{A}$ is uniformly elliptic with coefficients $a_{ij}, a_i, a \in C_b^{0,N}([0,T] \times \mathbb{R}^d)$.

Definition 2.1. For any $\bar{x} \in \mathbb{R}^d$ and $n \le N$, let

$$\mathcal{A}_n^{\bar{x}} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}^{\bar{x},n}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} a_i^{\bar{x},n}(t,x)\partial_{x_i} + a^{\bar{x},n}(t,x), \quad t\in\mathbb{R},\ x\in\mathbb{R}^d, \tag{5}$$

where

$$f^{\bar{x},n}(x) = \sum_{|\beta|=n} \frac{\partial_x^\beta f(t,\bar{x})}{\beta!}(x-\bar{x})^\beta,$$

is the n-th-order term of the Taylor expansion of the function f. We denote by $(u_n^{\bar{x}})_{0\le n\le N}$ the solutions to the Cauchy problems defined recursively as follows:

$$\begin{cases} (\partial_t + \mathcal{A}_0^{\bar{x}})u_0^{\bar{x}}(t,x) = 0, & t\in[0,T[,\ x\in\mathbb{R}^d, \\ u_0^{\bar{x}}(T,x) = \varphi(x), & x\in\mathbb{R}^d, \end{cases} \tag{6}$$

for $n=0$, and

$$\begin{cases} (\partial_t + \mathcal{A}_0^{\bar{x}})u_n^{\bar{x}}(t,x) = -\displaystyle\sum_{h=1}^{n} \mathcal{A}_h^{\bar{x}} u_{n-h}^{\bar{x}}(t,x), & t\in[0,T[,\ x\in\mathbb{R}^d, \\ u_n^{\bar{x}}(T,x) = 0, & x\in\mathbb{R}^d, \end{cases} \tag{7}$$

for $1\le n\le N$. Finally, we call

$$\bar{u}_N(t,x) := \sum_{n=0}^{N} u_n^x(t,x) \tag{8}$$

an N-th-order asymptotic approximation of the solution u to (1).

For practical purposes, it is remarkable that the approximation $\bar{u}_N$ can be computed explicitly for any N. The numerical results in [14] show that the approximation is very accurate for several models; recently, the codes of the third-order approximation of the Heston model have been included in the QuantLib library. Moreover, the following global-in-space error estimates are proved in [13].

Theorem 2.1. *Assume **H.1** and that for some $N\in\mathbb{N}_0$ and $M>0$ we have:*

i) *$\mathcal{A}$ is uniformly elliptic on $\mathbb{R}^d$, that is*

$$M^{-1}|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(t,x)\xi_i\xi_j \le M|\xi|^2, \quad t\in[0,T],\ x,\xi\in\mathbb{R}^d; \tag{9}$$

ii) *the coefficients $a_{ij}, a_i, a \in C_b^{0,N}([0,T]\times\mathbb{R}^d)$ with $\|a_{ij}, a_i, a\|_{C_b^{0,N}([0,T]\times\mathbb{R}^d)} \le M$.*

Let u be the bounded classical solution to problem (1). Then we have:

$$\left|u(t,x) - \bar{u}_N(t,x)\right| \le C(T-t)^{\frac{N+k+1}{2}}, \quad 0\le t\le T,\ x\in\mathbb{R}^d, \tag{10}$$

where C is a positive constant that depends only on M, N, T and $\|\varphi\|_{C_b^k(\mathbb{R}^d)}$.

3. Asymptotic convergence for locally elliptic equations

In this section, we assume that the operator $\mathcal{A}$ in (2) is elliptic on a compact subset of its domain D. More precisely, for $t<T$, $x_0\in\mathbb{R}^d$ and $r>0$, let us consider the cylinder

$$H(t,T,x_0,r) =]t,T[\times D(x_0,r), \qquad D(x_0,r) = \left\{x\in\mathbb{R}^d \;\middle|\; |x-x_0|<r\right\},$$

and denote by

$$\Sigma(t,T,x_0,r) := [t,T]\times\partial D(x_0,r)$$

the lateral boundary of $H(t,T,x_0,r)$. Throughout this section we assume the following hypothesis.

H.2 $\mathcal{A}$ is uniformly elliptic on some cylinder $H(t,T,x_0,r)$, compactly contained in $[0,T]\times D$. Precisely, $\mathcal{A}\equiv\widetilde{\mathcal{A}}$ in $H(t,T,x_0,r)$, where $\widetilde{\mathcal{A}}$ is a uniformly elliptic operator on $[0,T]\times\mathbb{R}^d$, satisfying assumptions i) and ii) of Theorem 2.1, for some $N\in\mathbb{N}_0$ and $M>0$.

Now let us consider the classical bounded solution $\widetilde{u}$ to the Cauchy problem:

$$\begin{cases} (\partial_t + \widetilde{\mathcal{A}})\widetilde{u}(t,x) = 0, & t \in [0,T[,\ x \in \mathbb{R}^d, \\ \widetilde{u}(T,x) = \widetilde{\varphi}(x), & x \in \mathbb{R}^d, \end{cases} \tag{11}$$

where $\widetilde{\varphi} \in C_b^k(\mathbb{R}^d)$ is such that $\widetilde{\varphi} = \varphi$ in $D(x_0,r)$ and $\|\widetilde{\varphi}\|_{C_b^k(\mathbb{R}^d)} \le \|\varphi\|_{C_b^k(D)}$. In the next theorem, $\bar{u}_N$ denotes the N-th-order approximation of $\widetilde{u}$, as in Definition 2.1.

Theorem 3.1. *Assume **H.1**, **H.2** and that problem (1) admits a classical solution u. Then for any $\delta \in]0,1[$, we have*

$$\left|u(t,x) - \bar{u}_N(t,x)\right| \le C(T-t)^{\frac{N+1+k}{2}}, \quad (t,x) \in H(0,T,x_0,\delta r), \tag{12}$$

where the constant C depends only on δ, d, M, N, T and on the norms $\|\varphi\|_{C_b^k(D)}$ and $\|u\|_{L^\infty(\Sigma(0,T,x_0,r))}$. If u admits the Feynman–Kac representation (3), then C in (12) depends only on δ, d, M, N, T and $\|\varphi\|_{C_b^k(D)}$.

Proof. First we note that it is not restrictive to assume $x_0 = 0$ because $w(t,x) := u(t,x+x_0)$ is a solution to $(\partial_t + \mathcal{A})w = 0$ on $H(0,T,0,r)$, where the coefficients of $\mathcal{A}$ are computed in $(t,x+x_0)$ instead of (t,x). Moreover it suffices to prove the estimate (12) for $T-t$ small, say $T-t < \varepsilon r^2$ for some positive ε.

By assumption there exists an operator $\widetilde{\mathcal{A}}$ satisfying the hypotheses of Theorem 2.1 and such that $\mathcal{A} = \widetilde{\mathcal{A}}$ on $H(0,T,0,r)$. We denote by $\widetilde{\Gamma}(t,x;T,y)$ the fundamental solution to the uniformly parabolic operator $\partial_t + \widetilde{\mathcal{A}}$ and recall the following classical Gaussian upper and lower bounds for $\widetilde{\Gamma}$ proved in [2]: let $\Gamma^{\pm}$ be the fundamental solutions to the heat operators

$$K^- = \partial_t + \frac{1}{2M}\triangle_{\mathbb{R}^d}, \qquad K^+ = \partial_t + \frac{M}{2}\triangle_{\mathbb{R}^d},$$

respectively. Then there exist two positive constants c^- and c^+, dependent only on M and T, such that

$$c^-\Gamma^-(t,x;T,y) \le \widetilde{\Gamma}(t,x;T,y) \le c^+\Gamma^+(t,x;T,y), \quad t \in [0,T[,\ x,y \in \mathbb{R}^d. \tag{13}$$

Next we split the proof in two main steps.

Step 1. We prove the following preliminary result: let $\psi_{r,\delta} \in C^\infty(\mathbb{R}^d;[0,1])$ be a function such that $\psi_{r,\delta}(x) = 0$ for $|x| < \frac{(1+\delta)r}{2}$ and $\psi_{r,\delta}(x) = 1$ for $|x| > \frac{(2+\delta)r}{3}$. There exist positive constants C_0 and ε, only dependent on δ, M, T, d and $c^{\pm}$ in (13), such that the function

$$v(t,x) := \frac{2}{c^-}\int_{\mathbb{R}^d} \widetilde{\Gamma}(t,x;T,y)\psi_{r,\delta}(y)\,\mathrm{d}y, \quad t < T,\ x \in \mathbb{R}^d, \tag{14}$$

satisfies

$$v(t,x) \ge 1, \quad (t,x) \in \Sigma\left(T-\varepsilon r^2, T, 0, r\right), \tag{15}$$

$$v(t,x) \le C_0\,\mathrm{e}^{-\frac{r^2}{C_0\sqrt{T-t}}}, \quad (t,x) \in H(0,T,0,\delta r). \tag{16}$$

Indeed, let $x \in \partial D(0,r)$: by (13) we have:

$$v(t,x) \ge 2\int_{\mathbb{R}^d} \Gamma^-(t,x;T,y)\psi_{r,\delta}(y)\,\mathrm{d}y \ge 2\int_{|y|>\frac{(2+\delta)r}{3}} \left(\frac{M}{2\pi(T-t)}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{M|x-y|^2}{2(T-t)}}\,\mathrm{d}y$$

(setting $z = \frac{\sqrt{M}(x-y)}{\sqrt{2(T-t)}}$ and $x = r\eta$ with $|\eta| = 1$)

$$= 2\pi^{-\frac{d}{2}} \int_{|\eta - z\frac{\sqrt{2M(T-t)}}{r}| > \frac{2+\delta}{3}} \mathrm{e}^{-|z|^2}\,\mathrm{d}z =: w(t,\eta).$$

Now by the dominated convergence theorem $w(t,\eta)$ tends to 2 as $t \to T^-$, uniformly in $\eta \in \partial D(0,1)$, and therefore $w(t,\eta) \ge 1$ if $\frac{T-t}{r^2} < \varepsilon$ for a suitably small ε that depends only on M and δ: this proves (15).

Next we prove (16). We first note that, if $|x| < \delta r$ and $|y| \ge \frac{(1+\delta)r}{2}$ then we have:

$$|x-y| \ge |y| - |x| \ge \frac{(1-\delta)r}{2}. \tag{17}$$

Thus, for any $(t,x) \in H(0,T,0,\delta r)$, by the second inequality in (13) we have:

$$v(t,x) \le \frac{2c^+}{c^-}\int_{\mathbb{R}^d} \Gamma^+(t,x;T,y)\psi_{r,\delta}(y)\,\mathrm{d}y \le \frac{2c^+}{c^-}\int_{|y|\ge\frac{(1+\delta)r}{2}} \Gamma^+(t,x;T,y)\,\mathrm{d}y$$

(by (17))

$$\le \frac{2c^+}{c^-}\int_{|x-y|\ge\frac{(1-\delta)r}{2}} \Gamma^+(t,x;T,y)\,\mathrm{d}y = \frac{2c^+}{c^-}\int_{|x-y|\ge\frac{(1-\delta)r}{2}} \left(\frac{1}{2\pi M(T-t)}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{|x-y|^2}{2M(T-t)}}\,\mathrm{d}y$$

(setting $z = \frac{x-y}{\sqrt{2M(T-t)}}$)

$$= \frac{2c^+}{c^-\pi^{\frac{d}{2}}}\int_{|z|\ge\frac{(1-\delta)r}{2\sqrt{2M(T-t)}}} \mathrm{e}^{-|z|^2}\mathrm{d}z \le \frac{2c^+}{c^-\pi^{\frac{d}{2}}}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{d}z \max_{|z|\ge\frac{(1-\delta)r}{2\sqrt{2M(T-t)}}} \mathrm{e}^{-\frac{|z|^2}{2}},$$

from which (16) follows.

Step 2. Recall that $\widetilde{u}$ is the classical bounded solution to (11). Notice that both $u-\widetilde{u}$ and v in (14) solve equation $(\partial_t+\widetilde{\mathcal{A}})w=0$ in $H(T-\varepsilon r^2,T,0,r)$ and also $(u-\widetilde{u})(T,x)=0$ for $x\in D(0,r)$. Thus, if we set

$$C_1 = \max_{\Sigma(T-\varepsilon r^2,T,0,r)} |u-\widetilde{u}|$$

then by (15) and the maximum principle, we have:

$$|u-\widetilde{u}| \le C_1 v, \quad \text{in } H\big(T-\varepsilon r^2,T,0,r\big)$$

and also

$$|u-\bar{u}_N| \le |u-\widetilde{u}| + |\widetilde{u}-\bar{u}_N| \le C_1 v + |\widetilde{u}-\bar{u}_N|, \quad \text{in } H\big(T-\varepsilon r^2,T,0,\delta r\big).$$

Therefore the thesis follows from (16) and the asymptotic estimate of Theorem 2.1 applied to $\partial_t+\widetilde{\mathcal{A}}$. Finally, if u admits the Feynman–Kac representation (3), then

$$\|u\|_{L^\infty([0,T]\times D)} \le \mathrm{e}^{T\|a\|_{L^\infty([0,T]\times D)}}\|\varphi\|_{L^\infty(D)},$$

and a similar estimate holds for $\widetilde{u}$. Therefore it is clear that the constant C in (12) depends only on δ, d, M, N, T and $\|\varphi\|_{C_b^k(D)}$. □

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