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Complex analysis

Coefficient estimates for a certain class of analytic and bi-univalent functions defined by fractional derivative

Estimation des coefficients d'une classe de fonctions analytiques, bi-univalentes, définie par dérivation fractionnaire

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ABSTRACT

We introduce and investigate a subclass of analytic and bi-univalent functions defined by a fractional derivative operator in the open unit disk. Using the Faber polynomial expansions, we obtain upper bounds for the coefficients of functions belonging to this class.

RÉSUMÉ

Nous introduisons et étudions une classe de fonctions analytiques, bi-univalentes, dans le disque unité, définie par une condition sur des dérivées fractionnaires. En utilisant les développements en termes de polynômes de Faber, nous obtenons des majorations des coefficients des fonctions de cette classe.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ that are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ that are univalent in $\mathbb{U}$ (for details, see [6]).

The Koebe One Quarter Theorem (e.g., see [6]) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U})$$

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$$f\big(f^{-1}(w)\big)=w \quad \left(|w|<r_0(f),\ r_0(f)\geq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w)=f^{-1}(w)=w-a_2w^2+\big(2a_2^2-a_3\big)w^3-\big(5a_2^3-5a_2a_3+a_4\big)w^4+\cdots$$
$$=w+\sum_{n=2}^{\infty}b_nw^n.$$

A function $f\in\mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor–Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see [18] (see also [4,19,9,10]).

In fact, the aforecited work of Srivastava et al. [18] essentially revived the investigation of various sublasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Ali et al. [3], Srivastava et al. [17], Jahangiri and Hamidi [8] (see also [7,14,5,13] and the references cited in each of them).

The following definition of the fractional derivative will be required in our investigation (see, for details, [11,12,15,16]).

Definition 1. The fractional integral of order δ is defined, for a function f, by

$$D_z^{-\delta}f(z)=\frac{1}{\Gamma(\delta)}\int_0^z\frac{f(\xi)}{(z-\xi)^{1-\delta}}\,\mathrm{d}\xi;\quad (\delta>0),$$

where f is an analytic function in a simply-connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{\delta-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi>0$.

Definition 2. The fractional derivative of order δ is defined, for a function f, by

$$D_z^{\delta}f(z)=\frac{1}{\Gamma(1-\delta)}\frac{\mathrm{d}}{\mathrm{d}z}\int_0^z\frac{f(\xi)}{(z-\xi)^{\delta}}\,\mathrm{d}\xi\quad (0\leq\delta<1),$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\delta}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta}f(z)=\frac{\mathrm{d}^n}{\mathrm{d}z^n}D_z^{\delta}f(z)\quad \big(0\leq\delta<1, n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\big).$$

By virtue of Definitions 1, 2 and 3, we have:

$$D_z^{-\delta}z^n=\frac{\Gamma(n+1)}{\Gamma(n+\delta+1)}z^{n+\delta}\quad (n\in\mathbb{N},\delta>0)$$

and

$$D_z^{\delta}z^n=\frac{\Gamma(n+1)}{\Gamma(n-\delta+1)}z^{n-\delta}\quad (n\in\mathbb{N},0\leq\delta<1).$$

The object of the present paper is to introduce a new subclass of the function class Σ and use Faber polynomial coefficient techniques to provide bounds for the general coefficients $|a_n|$ for the functions in this class. We also obtain estimates for the first two coefficients $|a_2|$ and $|a_3|$ of these functions.

We begin by introducing the function class $\mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ by means of the following definition.

Definition 4. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ $(0\leq\alpha<1,\ \lambda\geq 0,\ 0\leq\delta<1)$ if the following conditions are satisfied:

$$f\in\Sigma\quad\text{and}\quad \mathrm{Re}\big\{(1-\lambda)\Gamma(2-\delta)z^{\delta-1}D_z^{\delta}f(z)+\lambda\Gamma(1-\delta)z^{\delta}D_z^{\delta+1}f(z)\big\}>\alpha\quad (z\in\mathbb{U}).\tag{1.2}$$

In order to prove our main results, we need the following theorem due to Airault and Bouali [1].

Theorem A. *(See Airault and Bouali [1].) Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. The inverse function of f, $f^{-1}(f(z)) = z$ is given in terms of the Faber polynomials of $f(z)$, with*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n) w^n. \tag{1.3}$$

In [1], Airault and Bouali obtained general expansion of K_n^p as follows:

$$K_n^p = pa_n + \frac{p(p-1)}{2} D_n^2 + \frac{p!}{(p-3)!3!} D_n^3 + \cdots + \frac{p!}{(p-n)!n!} D_n^n \quad (p \in \mathbb{Z}),$$

where $\mathbb{Z} = \{0, \mp 1, \mp 2, \cdots\}$ and $D_n^p = D_n^p(a_2, a_3, \ldots)$ and by [20],

$$D_n^m = D_n^m(a_1, a_2, \ldots, a_n) = \sum \frac{m!(a_1)^{\mu_1} \ldots (a_n)^{\mu_n}}{\mu_1! \ldots \mu_n!},$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, ..., \mu_n$ satisfying the following conditions:

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_n = m, \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n = n \end{cases}$$

(see, for details, [1] and [2]; see also [17,8]).

The first and the last polynomials are:

$$D_n^1 = a_n \qquad D_n^n = a_1^n = 1.$$

A simple calculation reveals that the first three terms of K_{n-1}^{-n} are:

$$K_1^{-2} = -2a_2,$$
$$K_2^{-3} = 3\big(2a_2^2 - a_3\big),$$
$$K_3^{-4} = -4\big(5a_2^3 - 5a_2a_3 + a_4\big).$$

2. Main results

Theorem 1. *Let $f \in \mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ $(0 \le \alpha < 1,\ \lambda \ge 0,\ 0 \le \delta < 1)$ be given by (1.1). If $a_k = 0$ for $2 \le k \le n-1$, then*

$$|a_n| \le \frac{2(1-\alpha)\Gamma(n+1-\delta)}{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]} \quad n \in \mathbb{N} \setminus \{1, 2\}. \tag{2.1}$$

Proof. Using the fact that, for analytic functions f of the form (1.1),

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta},$$

we have:

$$(1-\lambda)\Gamma(2-\delta)z^{\delta-1} D_z^{\delta} f(z) + \lambda \Gamma(1-\delta) z^{\delta} D_z^{\delta+1} f(z)$$
$$= 1 + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} a_n z^{n-1} \tag{2.2}$$

and for its inverse map, $g = f^{-1}$, we have:

$$(1-\lambda)\Gamma(2-\delta)w^{\delta-1} D_w^{\delta} g(w) + \lambda \Gamma(1-\delta) w^{\delta} D_w^{\delta+1} g(w)$$
$$= 1 + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} b_n w^{n-1}$$
$$= 1 + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} \frac{1}{n} K_{n-1}^{-n}(a_1, a_2, ..., a_n) w^{n-1}. \tag{2.3}$$

On the other hand, since $f \in \mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ and $g = f^{-1} \in \mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$, by definition, there exist two positive real part functions

$$\mathrm{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$\mathrm{q}(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$$

where $\mathrm{Re}(\mathrm{p}(z)) > 0$ and $\mathrm{Re}(\mathrm{q}(w)) > 0$ in $\mathbb{U}$ so that

$$\begin{aligned}(1-\lambda)\Gamma(2-\delta)z^{\delta-1}D_z^{\delta} f(z) + \lambda\Gamma(1-\delta)z^{\delta}D_z^{\delta+1} f(z) &= \alpha + (1-\alpha)\mathrm{p}(z)\\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n,\end{aligned} \tag{2.4}$$

$$\begin{aligned}(1-\lambda)\Gamma(2-\delta)w^{\delta-1}D_w^{\delta} g(w) + \lambda\Gamma(1-\delta)w^{\delta}D_w^{\delta+1} g(w) &= \alpha + (1-\alpha)\mathrm{q}(w)\\ &= 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n.\end{aligned} \tag{2.5}$$

Comparing the corresponding coefficients of (2.2) and (2.4) yields:

$$\frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} a_n = (1-\alpha)c_{n-1} \tag{2.6}$$

and similarly (2.3) and (2.5)

$$\frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} \frac{1}{n} K_{n-1}^{-n}(a_1, a_2, \ldots, a_n) = (1-\alpha)d_{n-1}. \tag{2.7}$$

Note that for $a_k = 0$, $2 \le k \le n-1$, we have $b_n = -a_n$ and so

$$\frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} a_n = (1-\alpha)c_{n-1}, \tag{2.8}$$

$$-\frac{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}{\Gamma(n+1-\delta)} a_n = (1-\alpha)d_{n-1}. \tag{2.9}$$

Note that, according to the Caratheodory lemma (see [6]), $|c_n| \le 2$ and $|d_n| \le 2$ for $n \in \mathbb{N}$. Now taking the absolute values of (2.8) and (2.9) and applying the Caratheodory lemma, we obtain:

$$\begin{aligned}|a_n| &\le \frac{(1-\alpha)\Gamma(n+1-\delta)|c_{n-1}|}{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}\\ &= \frac{(1-\alpha)\Gamma(n+1-\delta)|d_{n-1}|}{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]}\\ &\le \frac{2(1-\alpha)\Gamma(n+1-\delta)}{\Gamma(n+1)\Gamma(1-\delta)[1-\delta+\lambda(n-1)]},\end{aligned}$$

which evidently completes the proof of Theorem 1. □

Relaxing the coefficient restrictions imposed in Theorem 1, we obtain early coefficients of functions $f \in \mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ given in the following theorem.

Theorem 2. *Let $f \in \mathcal{A}_{\Sigma}^{(\alpha,\lambda,\delta)}$ $(0 \le \alpha < 1,\ \lambda \ge 0,\ 0 \le \delta < 1)$ be given by (1.1). Then*

(i) $$|a_2| \le \begin{cases} \sqrt{\frac{(1-\alpha)\Gamma(4-\delta)}{3\Gamma(1-\delta)(1-\delta+2\lambda)}}, & 0 \le \alpha < 1 - \frac{(3-\delta)\Gamma(1-\delta)(1-\delta+\lambda)^2}{3\Gamma(3-\delta)(1-\delta+2\lambda)},\\ \frac{(1-\alpha)\Gamma(3-\delta)}{\Gamma(1-\delta)(1-\delta+\lambda)}, & 1 - \frac{(3-\delta)\Gamma(1-\delta)(1-\delta+\lambda)^2}{3\Gamma(3-\delta)(1-\delta+2\lambda)} \le \alpha < 1.\end{cases}$$

(ii) $$|a_3| \le \frac{(1-\alpha)\Gamma(4-\delta)}{3\Gamma(1-\delta)(1-\delta+2\lambda)}.$$

Proof. Replacing n with 2 and 3 in (2.6) and (2.7), respectively, we obtain:

$$\frac{2\Gamma(1-\delta)(1-\delta+\lambda)}{\Gamma(3-\delta)}a_2 = (1-\alpha)c_1, \tag{2.10}$$

$$\frac{6\Gamma(1-\delta)(1-\delta+2\lambda)}{\Gamma(4-\delta)}a_3 = (1-\alpha)c_2, \tag{2.11}$$

$$-\frac{2\Gamma(1-\delta)(1-\delta+\lambda)}{\Gamma(3-\delta)}a_2 = (1-\alpha)d_1, \tag{2.12}$$

$$\frac{6\Gamma(1-\delta)(1-\delta+2\lambda)}{\Gamma(4-\delta)}\left(2a_2^2-a_3\right) = (1-\alpha)d_2. \tag{2.13}$$

Dividing (2.10) or (2.12) by $\frac{2\Gamma(1-\delta)(1-\delta+\lambda)}{\Gamma(3-\delta)}$, taking their absolute values, and applying the Caratheodory lemma, we obtain:

$$|a_2| \le \frac{(1-\alpha)\Gamma(3-\delta)|c_1|}{2\Gamma(1-\delta)(1-\delta+\lambda)} = \frac{(1-\alpha)\Gamma(3-\delta)|d_1|}{2\Gamma(1-\delta)(1-\delta+\lambda)} \le \frac{(1-\alpha)\Gamma(3-\delta)}{\Gamma(1-\delta)(1-\delta+\lambda)}.$$

Adding (2.11) to (2.13) implies:

$$\frac{12\Gamma(1-\delta)(1-\delta+2\lambda)}{\Gamma(4-\delta)}a_2^2 = (1-\alpha)(c_2+d_2) \tag{2.14}$$

or

$$a_2^2 = \frac{(1-\alpha)\Gamma(4-\delta)(c_2+d_2)}{12\Gamma(1-\delta)(1-\delta+2\lambda)}. \tag{2.15}$$

An application of the Caratheodory lemma followed by taking the square roots yields:

$$|a_2| \le \sqrt{\frac{(1-\alpha)\Gamma(4-\delta)}{3\Gamma(1-\delta)(1-\delta+2\lambda)}}. \tag{2.16}$$

We note that for $1-\frac{(3-\delta)\Gamma(1-\delta)(1-\delta+\lambda)^2}{3\Gamma(3-\delta)(1-\delta+2\lambda)} \le \alpha < 1$,

$$\frac{(1-\alpha)\Gamma(3-\delta)}{\Gamma(1-\delta)(1-\delta+\lambda)} \le \sqrt{\frac{(1-\alpha)\Gamma(4-\delta)}{3\Gamma(1-\delta)(1-\delta+2\lambda)}}.$$

Dividing (2.11) by $\frac{6\Gamma(1-\delta)(1-\delta+2\lambda)}{\Gamma(4-\delta)}$, taking the absolute values of both sides, and applying the Caratheodory lemma yield:

$$|a_3| \le \frac{(1-\alpha)\Gamma(4-\delta)}{3\Gamma(1-\delta)(1-\delta+2\lambda)}. \tag{2.17}$$

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